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Topics in Mathematics 1 - Spring term 2017

Exercise sheet no.2 (16.3.2017)

Exercise 1: Show that the hypergeometric distribution converges against the binomial distribution with parameters n, p as $N, K \to \infty$ with $p = \frac{K}{N}$ constant (cf. lecture Example 2.3 (iii)).

Exercise 2: For $i = 1, 2$ let $(\Omega_i, \mathcal{A}_i, P_i)$ be probability spaces and P_2 be the distribution of (the measurable map) $X_1 : \Omega_1 \to \Omega_2$ under P_1, i.e. $P_2 = P_1 \circ X_1^{-1}$. Let $h_2 \geq 0$ be a random variable on $(\Omega_2, \mathcal{A}_2)$. Show the *transformation theorem*

$$E_2[h_2] = E_1[h_2 \circ X_1].$$

(*Hint*: Start with simple functions and then use that any measurable $h_2 \geq 0$ can be approximated be an increasing sequence of simple functions).

Exercise 3: Consider the model for ∞-many coin tosses from Example 3.6 of the lecture. For $n \in \mathbb{N}$ let

$$\ell_n\big((x_n)_{n\in\mathbb{N}}\big) := \max\{k \geq 1 | x_n = ... = x_{n+k-1} = 1\}$$

be the number of consecutive ones from the n^{th} coin toss on (a "run"). Here $\max \emptyset := 0$.

(i) Show that ℓ_n, $n \in \mathbb{N}$, is measurable.

(ii) For a sequence $(r_n)_{n\in\mathbb{N}} \subset \mathbb{N} \cup \{0\}$ consider the events $E_n := \{\ell_n \geq r_n\}$. Show with the help of the Lemma of Borel-Cantelli that

$$P(\ell_n \geq r_n \ \text{ for infinitely many n}) = 0$$

if $\sum_{n=1}^{\infty} 2^{-r_n} < +\infty$.

(iii) It follows in particular from *(ii)* with $r_n := (1+\varepsilon)\log_2 n$, $\varepsilon > 0$, that $P(\ell_n \geq (1+\varepsilon)\log_2 n \ \text{ for infinitely many n}) = 0$. Conclude from this that

$$P\Big(\limsup_{n\to\infty} \frac{\ell_n}{\log_2 n} > 1\Big) = 0.$$

Please drop the solutions into the homework box for the lecture at the basement of building no. 25 until 23.3.2017, 6 pm