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Topics in Mathematics 1 - Spring term 2017

Exercise sheet no.4 (30.3.2017)

Exercise 1: Let $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$ be a measurable spaces, $T : \Omega \to \Omega'$ an $\mathcal{A}/\mathcal{A}'$-measurable map, and $\sigma(T)$ the σ-algebra on Ω that is generated by T. Show that any $\sigma(T)$-measurable real valued random variable $X : \Omega \to \mathbb{R}$ has the form

$$X = f(T)$$

where f is some measurable function on Ω'.

(*Hint: Use "measure theoretic induction", i.e. show the statement first for simple functions, then for r.v.'s that are positive by using a monotone approximation with simple functions, and finally for general* $X = X^+ - X^-$).

Exercise 2:(i) Let $(X_i)_{i\in I}$, $(Y_i)_{i\in I}$, be two uniformly integrable families of random variables on a probability space $(\Omega, \mathcal{A}, P)$ and $\alpha, \beta \in \mathbb{R}$. Show that $(\alpha X_i + \beta Y_i)_{i\in I}$ is uniformly integrable.
(ii) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables on $(\Omega, \mathcal{A}, P)$ such that $X_n \to X$ in $\mathcal{L}^1$. Show that then $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable.

Exercise 3: Let $\Omega = (0, 1]$ and P be the Lebesgue measure on $\mathcal{B}((0, 1])$. Define

$$A_{2^i+k} := \left(\frac{k}{2^i}, \frac{k+1}{2^i}\right], \quad 0 \le k < 2^i, \quad k, i \in \mathbb{N} \cup \{0\},$$

$$Y_n := 1_{A_n}, \quad \widetilde{Y}_n := n^{\frac{1}{p}} \cdot 1_{(0,\frac{1}{n})}, \quad n \in \mathbb{N}.$$

Show that:

(i) $Y_n \to 0$ in $\mathcal{L}^p$ for $p > 0$ ($\Longrightarrow Y_n \to 0$ in probability).

(ii) $Y_n \nrightarrow 0$ P-a.s.

(iii) $\widetilde{Y}_n \to 0$ P-a.s. ($\Longrightarrow \widetilde{Y}_n \to 0$ in probability).

(iv) $\widetilde{Y}_n \nrightarrow 0$ in $\mathcal{L}^p$ for $p > 0$.

Please drop the solutions into the homework box for the lecture at the basement of building no. 25 until 6.4.2017, 6 pm