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Topics in Mathematics 1 - Spring term 2017

Exercise sheet no.5 (6.4.2017)

Exercise 1: Let $\sum_{n=1}^{\infty} \frac{d_n(x)}{2^n}$ be the binary expansion of $x \in [0,1]$. x is called *normal*, if the sequence of relative frequencies of "ones" in the binary expansion converges to $\frac{1}{2}$, i.e.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d_i(x) = \frac{1}{2}.$$

Show with the help of the strong law of large numbers that the set N of normal numbers in $[0,1]$ has Lebesgue measure one.

Exercise 2: Let $\Omega = \{\omega = (x_1, ..., x_N) \,|\, x_i \in \{-1, +1\}\}$, P the uniform distribution on Ω, and $X_i(\omega) = x_i$. We interprete

$$S_n = X_1 + ... + X_n \qquad (n = 0, 1, ..., N)$$

as the random motion of a particle on $\mathbb{Z}$ with start in 0 (*random walk*). For $0 < a \in \mathbb{N}$ let

$$T_a = \min\{n > 0 \,|\, S_n = a\}$$

the time of the first visit in a. Show:

(i) (*Reflection principle*) For any $c > 0$ we have:

$$P(S_n = a - c, T \leq n) = P(S_n = a + c)$$

(ii) For the distribution of T we have:

$$\begin{aligned} P(T_a \leq n) &= P(S_n \notin [-a, a-1]) \\ P(T_a = n) &= \frac{P(S_{n-1} = a-1) - P(S_{n-1} = a+1)}{2} = \frac{a}{n} \cdot P(S_n = a). \end{aligned}$$

Please turn the page !

Exercise 3: Let S_n $(n = 1, 2, ..., 2N)$ be the *random walk* from Exercise 2. Let

$$T_0(\omega) = \min\{n > 0 \mid S_n(\omega) = 0\}$$

be the time of the first return to 0 and

$$L(\omega) = \max\{0 \le n \le 2N \mid S_n(\omega) = 0\}$$

be the time of the last visit in 0.

(i) Show:

$$P(T_0 > 2n) = P(S_{2n} = 0)$$

and

$$P(L = 2n) = P(S_{2n} = 0) \cdot P(S_{2N-2n} = 0) = 2^{-2N} \binom{2n}{n} \binom{2N-2n}{N-n}$$

(*"discrete arcsine distribution"*)

(ii) Show that the distribution μ_N of $\frac{L}{2N}$ converges weakly as $N \uparrow \infty$ towards the distribution with density

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}} \quad (0 < x < 1)$$

and distribution function

$$F(x) = \frac{2}{\pi} \arcsin \sqrt{x}$$

(*"arcsine distribution"*).

Exercise 4: Let μ_n, $n \ge 1$, μ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Suppose that the sequence $(\mu_n)_{n\ge1}$ converges vaguely to μ. Show that then $(\mu_n)_{n\ge1}$ also converges weakly to μ (cf. lecture I. Corollary 10.5 (i)$\Rightarrow$(ii)).

Please drop the solutions into the homework box for the lecture at the basement of building no. 25 until 20.4.2017, 6 pm