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Topics in Mathematics 1 - Spring term 2017

Exercise sheet no.6 (20.4.2017)

Exercise 1: Let P be the uniform distribution on $\Omega := \{0,1\}^N$. Consider the events

$$A_j := \{(\omega_1, ..., \omega_N) \in \Omega \,|\, \omega_j = 1\}, \quad j = 1, ..., N$$

and

$$A_{N+1} := \{(\omega_1, ..., \omega_N) \in \Omega \,|\, \omega_1 + ... + \omega_N \text{ is even}\}.$$

Show that $A_1, ..., A_{N+1}$ are dependent but that any N of these $N+1$ events are independent.

Exercise 2: Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of independent events and $p_n := P(A_n)$, $n \in \mathbb{N}$. Under which assumptions on $(p_n)_{n\in\mathbb{N}}$ does it follow that $\lim_{n\to\infty} 1_{A_n} = 0$

(i) in probablility,

(ii) P-a.s ?

Hint: Use the Lemma of Borel-Cantelli.

Exercise 3: As in Assignment no.2 Exercise 2 we consider the model for ∞-many coin tosses and

$$\ell_n\big((x_n)_{n\in\mathbb{N}}\big) := \max\{k \geq 1 | x_n = ... = x_{n+k-1} = 1\}.$$

For $r \geq 0$ let $E_n(r) := \{\ell_n \geq r\}$.

(i) Show with the help of the Lemma of Borel-Cantelli that for an increasing sequence of positive real numbers $r_1, r_2, ...$ with $\sum_{n\geq 1} \frac{2^{-r_n}}{r_n} = \infty$, we have

$$P(\limsup_{n\to\infty} E_n(r_n)) = 1$$

(ii) Conclude from (i), that $P(\limsup_{n\to\infty} E_n(\log_2 n)) = 1$ and then use Assignment no.2 Exercise 2 to conclude

$$P\Big(\limsup_{n\to\infty} \frac{\ell_n}{\log_2 n} = 1\Big) = 1.$$

(*Hint for (i): Define a sequence* (n_k) *inductively through* $n_1 = 1$ *and* $n_{k+1} = n_k + r_{n_k}$. *Then the events* $E_{n_k}(r_{n_k})$, $k \geq 1$, *are independent.*)

Please drop the solutions into the homework box for the lecture at the basement of building no. 25 until 27.4.2017, 6 pm