# TOPOLOGY I

Exercise sheet no.1

15.04.2005

**Exercise 1**: Let $S$ denote the class of all the topological spaces with three elements. Decompose $S$ into its homeomorphism classes.

**Exercise 2**: Let $f : \mathbb{R} \to \mathbb{R}$ strictly monotone and surjective. Show that f is a homeomorphism.

**Exercise 3**: A topological space $(X, \mathcal{T})$ is said to be *metrizable* if $\mathcal{T}$ is induced by some metric $d$.
(i) Show that a metrizable space is first countable.
(ii) Let $(X, \mathcal{T})$ be a discrete topological space, i.e. $\mathcal{T}$ is the discrete topology. Show that $X$ is metrizable. Show further that: $X$ is countable $\iff$ $(X, \mathcal{T})$ is second countable
(iii) Use (i) and (ii) to give an example of a first countable space which is not second countable.

**Exercise 4**: Let $X = \{f : \mathbb{R} \to \mathbb{R} \,|\, f \text{ continuous and bounded}\}$ with the usual sup norm, i.e. $||f|| := \sup_{x \in X} |f(x)|$. Show that $X$ is not second countable but first countable as a metric space.
Hint: It is enough to find an uncountable discrete subset of $X$. (A discrete subset $A$ of a topological space $X$ is a set with the following property: $\forall x, y \in A$, $x \neq y$ there exist neighborhoods $U_x$ of $x$, and $U_y$ of $y$ with $U_x \cap U_y = \emptyset$.)

**Exercise 5**: (*French railway metric*) Let $P \in \mathbb{R}^2$ be an arbitrary point (P=Paris). For $Q_1$, $Q_2 \in \mathbb{R}^2$, define $d(Q_1, Q_2)$ as follows: If $P$, $Q_1$, and $Q_2$ lie on the same line, then $d(Q_1, Q_2) = |Q_1 - Q_2|$, otherwise $d(Q_1, Q_2) = |Q_1 - P| + |Q_2 - P|$ (here $|\cdot|$ is the euclidean norm). Show that $(\mathbb{R}^2, d)$ is a metric space. Is $(\mathbb{R}^2, d)$ complete? Describe the open sets.