# TOPOLOGY I

Exercise sheet no.2

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**Exercise 1**: Let $(X, d)$ be a metric space. For $0 < r \in \mathbb{R}$, $x \in X$ let $B(x, r) := \{y \in X | d(x, y) < r\}$. Let $D \subset X$ be dense. Let

$$\mathcal{B} := \{B(x, q) \,|\, x \in D, 0 < q \in \mathbb{Q}\}.$$

Let $y \in X$, $0 < r \in \mathbb{R}$. Show that $B(y, r) = \bigcup\{B \in \mathcal{B} | B \subset B(y, r)\}$.

**Exercise 2**: Let $X = (-\infty, \pi + 1)$, $f : X \to \mathbb{R}^2$ such that

$$f(x) := \begin{cases} (x, 0) & \text{for } x \leq 1 \\ (\cos(x-1), \sin(x-1)) & \text{for } 1 \leq x < \pi + 1. \end{cases}$$

Is $f$ an embedding?

**Exercise 3**: a) Consider the following equivalence relation on $\mathbb{R}$: $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$. Show that $\mathbb{R}/\!\sim\, \cong S^1$. b) Let $T$ be the (2-dimensional) torus. Show that $T \cong S^1 \times S^1$.

**Exercise 4**: Let $p : X \to X/\!\sim$ be the quotient map induced by an equivalence relation $\sim$ on $X$. Let $f : X \to Y$ be continuous such that

$$x \sim y \Rightarrow f(x) = f(y).$$

Show that there exists a unique continuous map

$$f^* : X/\!\sim\, \to Y$$

with $f^* \circ p = f$.

**Exercise 5**: Let $X = \mathbb{R}^{n+1} \setminus \{0\}$, $n \geq 1$, $n \in \mathbb{N}$. For $x, y \in X$ let

$$x \sim y \Leftrightarrow \exists \lambda > 0 \; x = \lambda y.$$

Equip $X/\!\sim$ with the quotient topology. Show that $X/\!\sim\, \cong \mathbb{R}P^n$.

**Exercise 6**: a) Let $S^2_- := \{x \in S^2 | x_3 \leq 0\}$. For $x, y \in S^2_-$ let: $x \sim y \Leftrightarrow x = y$ or $x = -y$. Consider $S^2_-/\!\sim$ with the quotient topology. Show that $S^2_-/\!\sim\, \cong \mathbb{R}P^2$. Use this to indicate an embedding of the Moebius strip into $\mathbb{R}P^2$.
b) Show that $\mathbb{R}P^1 \cong S^1$.

**Exercise 7**: a) Show that $[0, 1] \subset \mathbb{R}$ is connected.
b) Let $f : [0, 1] \to [0, 1]$ be a continuous map. Show that $f$ has a fixed point, i.e. there exists an $x \in [0, 1]$ with $f(x) = x$.