TOPOLOGY I

Exercise sheet no.4

13.05.2005

Exercise 1: (Heine-Borel) Show that a subset of $\mathbb{R}^d$ is compact, iff it is closed and bounded.

Exercise 2: Let X be an infinite set. Consider the family $\mathcal{T}$ of those sets, which consist of $\emptyset$ and all complements of finite subsets in X (cofinite Topology). Show that X is compact.

Exercise 3: Let $\mathcal{M}$ be the set of all bounded sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Show that:
(a) $d\left((x_n),(y_n)\right) := \sup_{n\in\mathbb{N}} |x_n - y_n|$ is a metric on $\mathcal{M}$.
(b) $(\mathcal{M}, d)$ is not compact.

Exercise 4: Let $K, L \subset \mathbb{R}^d$ be compact. Show that also $K + L := \{x + y | x \in K, y \in L\}$ is compact.

Exercise 5: Let (X, d) be a metric space, $A \subset X$, $x \in X \backslash A$ with $d(x, A) = \inf\{d(x, y) | y \in A\} = 0$. Show that $x \in \partial A$.

Exercise 6: (Banach fixed-point theorem) Let (X, d) be a compact metric space, $f : X \to X$ a contraction, i.e. there is $\gamma \in (0, 1)$ with $d(f(x), f(y)) \leq \gamma d(x, y)$ for every $x, y \in X$. Show that f admits excactly one fixed point. Hint: Start with $x_0 \in X$ arbitrary and consider the sequence $x_i = f(x_{i-1})$.