TOPOLOGY I

Exercise sheet no.5

23.05.2005

Exercise 1: Let X be a locally compact Hausdorff space, and $K \subset X$ compact. Show that there exists an open set O in X, and a compact set K', with $K \subset O \subset K'$.

Exercise 2: Show the Lindelöf theorem: Every open covering of a second countable topological space contains a countable subcovering.

Exercise 3: Let X be a second countable locally compact Hausdorff space. Show that X is σ-compact.

Exercise 4: Let

$$\ell^2 := \left\{ x = (x_i)_{i \in \mathbb{N}} | x_i \in \mathbb{R} \text{ for } i \in \mathbb{N}; \sum_{i=0}^{\infty} x_i^2 < \infty \right\} \quad \text{("little ell two")}$$

and

$$d_2(x,y) := \left(\sum_{i=0}^{\infty} (x_i - y_i)^2 \right)^{\frac{1}{2}}; \quad x, y \in \ell^2.$$

(a) Show that d_2 is a metric on ℓ^2.
(b) Is (ℓ^2, d_2) locally compact? Hint: Consider first $\overline{B(x,\varepsilon)} = \{y \in \ell^2 | d_2(x,y) \leq \varepsilon\}$. Is it compact?

Exercise 5: Show that the one-point compactification of $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ is homeomorphic to $\{\frac{1}{n} | n \in \mathbb{N}^*\} \cup \{0\}$.

Exercise 6: Let $n \geq 1$. Show that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the n-sphere S^n.

Exercise 7: Let X be a locally compact Hausdorff space, and X_Δ its one-point compactification. A real valued function on X is said to *vanish at infinity* provided that for each $\varepsilon > 0$ there is a compact subset K of X such that $|f(x)| < \varepsilon$ for every $x \in X \setminus K$.
(a) In case $X = \mathbb{R}$ exhibit a continuous function $f : \mathbb{R} \to \mathbb{R}$ that vanishes at infinity such that $f(x) \neq 0$ for all $x \in \mathbb{R}$.
(b) Let X be non-compact. Prove that $f : X \to \mathbb{R}$ vanishes at infinity precisely when its extension $\overline{f} : X_\Delta \to \mathbb{R}$ with $\overline{f}(\Delta) = 0$ is continuous at Δ.
(c) Show that a continuous function that vanishes at infinity is necessarily bounded. Must such a function attain its maximum or minimum on X?

Exercise 8: (Pasting lemma) Let $X = A \cup B$, where A, B, are closed in X. Let $f : A \to Y$, $g : B \to Y$, be continuous and $f(x) = g(x)$ for any $x \in A \cap B$. Then $h : X \to Y$, defined by $h|_A := f$, $h|_B := g$, is continuous.