TOPOLOGY I

Exercise sheet no.6

06.06.2005

Exercise 1: (Lebesgue covering lemma) Let $(U_i)_{i\in\mathbb{N}}$ be an open covering of a compact metric space (X, d). Then there exists a real number $\delta > 0$ (the *Lebesgue number* of the covering $(U_i)_{i\in\mathbb{N}}$) such that for each $B(x, \frac{\delta}{2})$ there exists some U_{i_0} with $B(x, \frac{\delta}{2}) \subset U_{i_0}$.

Exercise 2: Given metric spaces (X, d), (Y, d'), a map $f : X \to Y$ is said to be (d, d')-uniformly continuous if for each $\varepsilon > 0$ there is some $\delta > 0$ such that

$$d(x, y) < \delta \quad \Rightarrow \quad d'(f(x), f(y)) < \varepsilon$$

for all $x, y \in X$. Show that if $f : X \to Y$ is a continuous map from a compact metric space (X, d) into a metric space (Y, d') then f is (d, d')-uniformly continuous. Hint: Use the Lebesgue covering lemma.

Exercise 3: Let $H : I \times I \to S^1$ be continuous such that $H(0, 0) = (1, 0)$. Then there exists a unique continuous $H' : I \times I \to \mathbb{R}$ (the lifting of H) with $H'(0, 0) = 1$ and $\Phi \circ H' = H$.

Exercise 4: Show that a retract of a contractible space is itself contractible.

Exercise 5: $A \subset X$ is said to be a *strong deformation retract* of X, if there is a continuous map $H : I \times X \to X$ such that

(i) $H(0, \cdot) = id_X, \quad H(1, x) \in A$ for $x \in X$
(ii) $H(s, a) = a$ for $s \in I$, $a \in A$.

Show that:
(a) If $A \subset X$ strong deformation retract of X, then A has same homotopy type as X.
(b) $\pi_1(S^{n-1}) \cong \pi_1(\mathbb{R}^n \setminus \{0\})$, $n \geq 2$.