LOGARITHMIC TREE-NUMBERS FOR ACYCLIC COMPLEXES

H. KIM* AND W. KOOK

Abstract. For a d-dimensional cell complex Γ with $\tilde{H}_i(\Gamma) = 0$ for $-1 \le i < d$, an i-dimensional tree is a non-empty collection B of i-dimensional cells in Γ such that $\tilde{H}_i(B \cup \Gamma^{(i-1)}) = 0$ and $w(B) := |\tilde{H}_{i-1}(B \cup \Gamma^{(i-1)})|$ is finite, where $\Gamma^{(i)}$ is the i-skeleton of Γ. Define the i-th tree-number to be $k_i := \sum_B w(B)^2$, where the sum is over all i-dimensional trees. In this paper, we will show that if Γ is acyclic and $k_i > 0$ for $-1 \le i \le d$, then k_i and the combinatorial Laplace operators Δ_i are related by $\sum_{i=-1}^{d} \omega_i \, x^{i+1} = (1+x)^2 \sum_{i=0}^{d-1} \kappa_i x^i$, where $\omega_i = \log\det\Delta_i$ and $\kappa_i = \log k_i$. We will discuss various applications of this equation.

1. Introduction

In this paper, we will extend Temperley's tree-number formula for finite graphs [10] to a class of cell complexes, called γ-complexes, and show interesting applications to acyclic complexes. We will review this formula shortly.

A *γ-complex* is a non-empty finite cell complex Γ whose integral cellular chain complex $\{C_i, \partial_i\}_{i \ge -1}$ with $C_{-1} = \mathbb{Z}$ satisfies the following conditions:

(γ1) $\operatorname{rk}\partial_i > 0$ (equivalently, $\partial_i \ne 0$) for $0 \le i \le \dim\Gamma$, and

(γ2) the reduced integral homology $\tilde{H}_i(\Gamma) = 0$ for $i < \dim\Gamma$.

This definition is intended as a generalization of connected finite graphs. Other examples of γ-complexes are matroid complexes, standard simplexes, and cubical complexes [3] with the latter two being acyclic. However, the n-dimensional sphere S^n made of one 0-cell and one n-cell is not a γ-complex for $n \ge 2$ because it violates condition (γ1).

We will define high-dimensional spanning trees for a γ-complex extending the ideas in [1]. Given a γ-complex Γ, let Γ_i be the set of all i-dimensional cells, and $\Gamma^{(i)}$ the i-skeleton of Γ. Given a subset $S \subset \Gamma_i$, define $\Gamma_S = S \cup \Gamma^{(i-1)}$ as a subcomplex of Γ. An *i-dimensional spanning tree* of Γ (or simply, an *i-tree*) is a non-empty subset $B \subset \Gamma_i$ such that $\tilde{H}_i(\Gamma_B) = 0$ and $w(B) := |\tilde{H}_{i-1}(\Gamma_B)|$ is finite. Define the *i-th tree-number* of Γ by

$$k_i(\Gamma) = k_i = \sum_B w(B)^2 \,,$$

where the sum is over all i-trees in Γ. We will see that $k_i > 0$ for all $-1 \le i \le \dim\Gamma$ where we define $k_{-1} = 1$. If Γ is a connected graph, then k_0 is the number of vertices and k_1 is the number of graph theoretic spanning trees in Γ. Note that S^n with one 0-cell and one n-cell has no i-tree for $i \ge 1$: it is an n-dimensional "loop".

Key words and phrases. cell complexes, high-dimensional trees, Combinatorial Laplacians.

Mathematics Subject Classification: primary 05E99 secondary 05C05.

* Supported by the Korea Research Foundation (KRF) grant funded by the Korea government (MEST)(No. 2011-0011223).

An important method for computing the tree-numbers for Γ is given by the combinatorial Laplacians Δ_i ([1], [3], and [10]). For example, let $\Delta_0 = L + J$, where L is the Laplacian matrix of a finite graph G of order n, and J is the all 1's matrix. Temperley [10] showed that $\det \Delta_0 = n^2 k_1$ for G (refer to Corollary 7). This method is more efficient than the matrix-tree theorem for certain graphs. Indeed, for $\Gamma = K_n$ the complete graph on n vertices, we have $\Delta_0 = nI$ and $\det \Delta_0 = n^n$, from which the Cayley's Theorem $k_1(K_n) = n^{n-2}$ is immediate.

We will show that Temperley's formula can be extended to any γ-complex Γ (refer to Theorem 6). Also, if Γ is acyclic of dimension d, then each Δ_i is positive-definite, and the following polynomials are well-defined: $D(x) = \sum_{i=-1}^{d} (\log \det \Delta_i) x^{i+1}$ and $K(x) = \sum_{i=0}^{d-1} (\log k_i) x^i$. The main result of the paper will show that

$$D(x) = (1 + x)^2 K(x) . \tag{1.1}$$

This paper is organized as follows. Section 2 is a review of useful facts from matrix theory and combinatorial Laplacians for γ-complexes. In section 3, we will describe high-dimensional spanning trees for a γ-complex via the boundary operators of its chain complex. In section 4, we will prove the main results of the paper which consist of a generalization of Temperley's tree-number formula and a logarithmic version (1.1) of this result for acyclic γ-complexes. In section 5, we will discuss a new method for computing tree-numbers for certain planar graphs. We will also discuss applications of the main results to standard simplexes [6] and the cubical complexes [3].

2. Preliminaries

2.1. **Matrix Theory.** We will review several important facts about symmetric matrices. For all definitions and basic facts from matrix theory, one may refer to [5]. All matrices are assumed to have real entries. For a square matrix M, let P_M denote the multiset of all non-zero eigenvalues of M, and let $\pi_M = \prod_{\lambda \in P_M} \lambda$.

Lemma 1. *Let A and B be $n \times n$ symmetric matrices such that $AB = BA = 0$. Then, we have $P_{A+B} = P_A \cup P_B$ as multisets. In particular, if $A+B$ is non-singular,*

$$\det(A + B) = \pi_A \pi_B . \tag{2.1}$$

Proof. Since A and B commute, there is a basis of common eigenvectors $\{v_1, \ldots, v_n\}$. For each $1 \leq i \leq n$, let λ_i and μ_i be the eigenvalues of A and B corresponding to v_i so that the collection $\{\lambda_i + \mu_i \,|\, 1 \leq i \leq n\}$ is the multiset of all eigenvalues of $A + B$. Since $AB = 0$, we have $\lambda_i \mu_i = 0$, i.e., either $\lambda_i = 0$ or $\mu_i = 0$ for each i. Therefore $\alpha = \lambda_i + \mu_i \in P_{A+B}$ if and only if $\alpha = \lambda_i \in P_A$ or $\alpha = \mu_i \in P_B$. □

Lemma 2. *Let M be a rectangular matrix of rank r ($r > 0$). Let $\mathcal{B}(M)$ be the collection of all non-singular $r \times r$ submatrices of M. If $A = MM^t$, or $M^t M$, then*

$$\pi_A = \sum_{B \in \mathcal{B}(M)} (\det B)^2 . \tag{2.2}$$

Proof. This result follows from Binet-Cauchy theorem and the fact that the product of all non-zero eigenvalues of a diagonalizable matrix of rank r equals the sum of all principal minors of order r. Equation (2.2) holds for both MM^t and $M^t M$ because they have the same multiset of non-zero eigenvalues. Details will be omitted. □

2.2. **Combinatorial Laplacians for γ-complexes.** We will assume familiarity with basic definitions concerning finite cell complexes and reduced homology groups. Refer to [8] for details. Let X be a finite cell complex of dimension d. For $-1 \le i \le d$, we will let X_i denote the set of all i-dimensional cells where we define $X_{-1} = \{\emptyset\}$, and $X^{(i)}$ the i-skeleton $X_{-1} \cup X_0 \cup \cdots \cup X_i$ as a subcomplex of X. Also we define X_{-2} and $X^{(-2)}$ to be the void set.

Let $\{C_i, \partial_i\}$ $(-1 \le i \le d)$ be the integral cellular chain complex of X where $C_{-1} = \mathbb{Z}$ and ∂_0 the usual augmentation. Define $\partial_{-1} = 0$, and we have $\partial_{i-1}\partial_i = 0$. The i-th reduced homology group of X is defined by $\tilde{H}_i(X) = \operatorname{Ker}\partial_i / \operatorname{Im}\partial_{i+1}$. Define $\tilde{H}_i(X) = 0$ for $i \le -1$. Note that $\tilde{H}_d(X) = \operatorname{Ker}\partial_d$ is free abelian. Recall that $\operatorname{rk}\tilde{H}_i(X) = 0$ iff $\tilde{H}_i(X)$ is finite. X is acyclic if $\tilde{H}_i(X) = 0$ for all i.

Suppose that $\partial_i \ne 0$ for $0 \le i \le d$. Then, we have $X_i \ne \emptyset$ and $C_i \cong \mathbb{Z}^{|X_i|}$. Regard the boundary map $\partial_i : C_i \to C_{i-1}$ as a $|X_{i-1}| \times |X_i|$ matrix whose rows and columns are indexed by X_{i-1} and X_i . The coboundary map $\partial_i^t : C_{i-1} \to C_i$ is the transpose of ∂_i. For $-1 \le i \le d$, the combinatorial Laplacian $\Delta_i : C_i \to C_i$ for X is defined by

$$\Delta_i = \partial_{i+1}\partial_{i+1}^t + \partial_i^t\partial_i \,,$$

where we define ∂_{d+1} to be the zero map. Denote $L_i = \partial_{i+1}\partial_{i+1}^t$ and $J_i = \partial_i^t\partial_i$. Then, L_i and J_i are non-zero, symmetric, non-negative definite, and $L_iJ_i = J_iL_i = 0$. Hence Δ_i is non-zero, symmetric, and non-negative definite by Lemma 1.

An important property of the combinatorial Laplacians is that the 0-eigenspace of Δ_i regarded as a matrix over $\mathbb{Q}$ is isomorphic to the rational homology group $\tilde{H}_i(X;\mathbb{Q})$ [4, Proposition 2.1]. Therefore, if $\partial_i \ne 0$ for $0 \le i \le d$, then we have

$$\det\Delta_i > 0 \quad \text{if and only if} \quad \operatorname{rk}\tilde{H}_i(X) = 0\,. \tag{2.3}$$

Note that $\Delta_{-1} = L_{-1} : \mathbb{Z} \to \mathbb{Z}$ is a multiplication by $|\Gamma_0|$. Now the following lemma is immediate from the definition of γ-complex and (2.3).

Lemma 3. *If Γ is a γ-complex of dimension d, then we have* $\det\Delta_i > 0$ *for* $-1 \le i < d$. *Moreover, if Γ is acyclic, then we also have* $\det\Delta_d > 0$. □

3. High-dimensional trees for γ-complexes

In this section, Γ will denote a γ-complex of dimension d. For a non-empty subset $S \subset \Gamma_i$, define $\Gamma_S := S \cup \Gamma^{(i-1)}$ as an i-dimensional subcomplex of Γ. For $-1 \le i \le d$, a non-empty subset $B \subset \Gamma_i$ is an *i-dimensional spanning tree* (or simply, *i-tree*) if

(1) $\tilde{H}_i(\Gamma_B) = 0$,
(2) $w(B) := |\tilde{H}_{i-1}(\Gamma_B)|$ is finite, and
(3) $\tilde{H}_j(\Gamma_B) = 0$ for $j \le i-2$.

Note that (3) is a consequence of the fact $\Gamma_B^{(i-1)} = \Gamma^{(i-1)}$. We see that $B \subset \Gamma_i$ is an i-tree iff Γ_B is $\mathbb{Q}$-acyclic. We will denote the set of all i-trees in Γ by $\mathcal{B}_i = \mathcal{B}_i(\Gamma)$ with $\mathcal{B}_{-1} = \{\emptyset\}$. It is clear that $\mathcal{B}_0$ is the set of all single 0-cells in Γ and $\mathcal{B}_1$ is the set of all graph theoretic spanning trees of $\Gamma^{(1)}$ as a finite graph.

Define the *i-th tree-number* of Γ to be

$$k_i = k_i(\Gamma) = \sum_{B \in \mathcal{B}_i} w(B)^2 \,.$$

We have $k_{-1} = 1$ by definition, and $k_0 = |\Gamma_0|$. If Γ is a connected graph, then k_1 is the number of spanning trees in Γ because $w(B) = 1$ for $B \in \mathcal{B}_1$. However, $w(B)$ may not equal 1 for $B \in \mathcal{B}_i$ when $i > 1$. (See [6].)

Next, we will describe i-trees via the boundary operator ∂_i of Γ, which will show that $k_i > 0$ for $i \geq 0$. Denote $r_i = \operatorname{rk} \partial_i$. Recall that $r_i > 0$ for $0 \leq i \leq d$, which also implies that both Γ_{i-1} and Γ_i are non-empty. Given a non-empty subset $T \subset \Gamma_i$, define ∂_T to be the $|\Gamma_{i-1}| \times |T|$ submatrix of ∂_i consisting of the columns of ∂_i indexed by T. Recall that if Γ is a connected finite graph of order n with the incidence matrix ∂_1, then $T \subset \Gamma_1$ is a spanning tree of Γ iff $|T| = \operatorname{rk} \partial_T = \operatorname{rk} \partial_1 = n-1$. (Refer to [2] for details.) More generally, we have the following useful fact.

Proposition 4. *Let Γ be a γ-complex of dimension d. Let $r_i = \operatorname{rk} \partial_i$ for $0 \leq i \leq d$. Then $\mathcal{B}_i$ is non-empty, and it is given by*

$$\mathcal{B}_i = \{\, B \subset \Gamma_i \mid |B| = \operatorname{rk} \partial_B = r_i \,\}. \tag{3.1}$$

Moreover, we have $r_i = |\Gamma_{i-1}| - r_{i-1}$, where $r_{-1} = 0$.

Proof. Suppose $B \in \mathcal{B}_i$. Since $\operatorname{Ker} \partial_B = \tilde{H}_i(\Gamma_B) = 0$, we have $\operatorname{rk} \partial_B = |B|$. Since $\Gamma_B^{(i-1)} = \Gamma^{(i-1)}$ and $\tilde{H}_{i-1}(\Gamma_B)$ is finite, we must have $\operatorname{rk} \partial_B = n_{i-1}$ the rank of $\operatorname{Ker} \partial_{i-1}$. However, $\tilde{H}_{i-1}(\Gamma) = 0$ implies $r_i = n_{i-1}$, and we have $|B| = \operatorname{rk} \partial_B = r_i$. The inclusion of the right-hand side of (3.1) in $\mathcal{B}_i$ is proved similarly. The second statement follows from $n_{i-1} = |\Gamma_{i-1}| - r_{i-1}$. □

Remarks 1. In matroid theoretic terms, $\mathcal{B}_i$ is the set of all bases of a matroid whose ground set is Γ_i and the independent sets are the subsets $I \subset \Gamma_i$ such that $\operatorname{Ker} \partial_I = 0$ or $I = \emptyset$. (Refer to [9] for the definition of a matroid.)
2. If Γ is also acyclic, then there is exactly one d-tree, namely $B = \Gamma_d$. Since $\operatorname{Ker} \partial_d = \tilde{H}_d(\Gamma) = 0$, the only base of the matroid just mentioned is Γ_d. In this case, it also follows that $k_d = 1$ because $\tilde{H}_{d-1}(\Gamma_B) = \tilde{H}_{d-1}(\Gamma) = 0$.
3. We also remark that if X is a cell complex satisfying (γ2) but $r_i = 0$ for some i, then X has no i-tree. Indeed, for any non-empty subset $S \subset \Gamma_i$, if any, we would have $\tilde{H}_i(\Gamma_S) = \mathbb{Z}^{|S|} \neq 0$.

The following theorem will play an essential role in section 3. Given non-empty subsets $S \subset \Gamma_{i-1}$ and $T \subset \Gamma_i$, let $\partial_{S,T}$ be the $|S| \times |T|$ submatrix of ∂_i whose rows and columns are indexed by S and T, respectively. Denote $\bar{S} = \Gamma_{i-1} \setminus S$.

Theorem 5. *Let Γ be a γ-complex of dimension d. Let $r_i = \operatorname{rk} \partial_i$ for $0 \leq i \leq d$. Then the set of all $r_i \times r_i$ non-singular submatrices of ∂_i is given by*

$$\mathcal{B}(\partial_i) := \{\, \partial_{\bar{A},B} \mid A \in \mathcal{B}_{i-1} \text{ and } B \in \mathcal{B}_i \,\}.$$

Moreover, we have $|\det \partial_{\bar{A},B}| = w(A)w(B)$ for $\partial_{\bar{A},B} \in \mathcal{B}(\partial_i)$.

Proof. Let $S \subset \Gamma_{i-1}$ with $|S| = r_{i-1}$ and let $T \subset \Gamma_i$ with $|T| = r_i$. Then $\partial_{\bar{S},T}$ is a square submatrix of ∂_i of order r_i by Prop. 4. First, we will show that $\partial_{\bar{S},T}$ is singular if $S \notin \mathcal{B}_{i-1}$ or $T \notin \mathcal{B}_i$. Regard $\partial_{\bar{S},T}$ as the top boundary operator for the relative complex (Γ_T, Γ_S). Note that $\tilde{H}_i(\Gamma_T) = \operatorname{Ker} \partial_T$, $\tilde{H}_i(\Gamma_T, \Gamma_S) = \operatorname{Ker} \partial_{\bar{S},T}$, and $\tilde{H}_{i-1}(\Gamma_S) = \operatorname{Ker} \partial_S$. Since $\tilde{H}_i(\Gamma_S) = 0$, we obtain the following exact sequence from the long exact homology sequence of the pair (Γ_T, Γ_S):

$$0 \to \operatorname{Ker} \partial_T \to \operatorname{Ker} \partial_{\bar{S},T} \to \operatorname{Ker} \partial_S \to \tilde{H}_{i-1}(\Gamma_T).$$

If $T \notin \mathcal{B}_i$, then $\operatorname{Ker} \partial_T \neq 0$ by Remark 1 above. Hence, we have $\operatorname{Ker} \partial_{\bar{S},T} \neq 0$. Similarly, if $S \notin \mathcal{B}_{i-1}$, then $\operatorname{rk}(\operatorname{Ker} \partial_S) \neq 0$. If $T \notin \mathcal{B}_i$, we are done. If $T \in \mathcal{B}_i$, then $\operatorname{Ker} \partial_T = 0$ and $\tilde{H}_{i-1}(\Gamma_T)$ is finite. Therefore, it is clear that $\operatorname{Ker} \partial_{\bar{S},T} \neq 0$.

Now we proceed to prove the second statement, which will also complete the proof of the first statement. Consider the following portion of the long exact homology sequence of the pair (Γ_B, Γ_A) with $A \in \mathcal{B}_{i-1}$ and $B \in \mathcal{B}_i$:

$$\tilde{H}_{i-1}(\Gamma_A) \to \tilde{H}_{i-1}(\Gamma_B) \to \tilde{H}_{i-1}(\Gamma_B, \Gamma_A) \to \tilde{H}_{i-2}(\Gamma_A) \to \tilde{H}_{i-2}(\Gamma_B)\,.$$

Since $\tilde{H}_{i-1}(\Gamma_A) = \tilde{H}_{i-2}(\Gamma_B) = 0$, it follows that

$$|\tilde{H}_{i-1}(\Gamma_B, \Gamma_A)| = |\tilde{H}_{i-2}(\Gamma_A)| \cdot |\tilde{H}_{i-1}(\Gamma_B)| = w(A)w(B)\,.$$

Note that $C_j(\Gamma_B, \Gamma_A) = \mathbb{Z}^{r_i}$ if $j = i-1$, and 0 if $j < i-1$. Therefore, we have $|\tilde{H}_{i-1}(\Gamma_B, \Gamma_A)| = |\mathbb{Z}^{r_i} / \operatorname{Im} \partial_{\bar{A},B}| = |\det \partial_{\bar{A},B}|$. □

4. Main Results

The main results will consist of a generalization of Temperley's tree-number formula for γ-complexes and its logarithmic version for acylic γ-complexes.

Proposition 6. *Let Γ be a γ-complex of dimension d, and let Δ_i be its combinatorial Laplacians for $-1 \leq i \leq d$. Then*
(1) $\det \Delta_{-1} = k_0$*,*
(2) $\det \Delta_i = k_{i-1} k_i^2 k_{i+1}$ *for* $0 \leq i \leq d-1$*, and*
(3) $\det \Delta_d = k_{d-1}$ *if* Γ *is acyclic, and 0 otherwise.*

Proof.(1) In section 2, we noted that $\Delta_{-1} = L_{-1} : \mathbb{Z} \to \mathbb{Z}$ is a multiplication by $|\Gamma_0|$. In section 3, we also saw that $k_0 = |\Gamma_0|$. Hence $\det \Delta_{-1} = k_0$.
(2) Note that we have $\operatorname{rk} \partial_i \partial_i^t = \operatorname{rk} \partial_i > 0$ for $0 \leq i \leq d$. Therefore, $\partial_i \partial_i^t$ has non-zero eigenvalues. Let π_i denote the product of all non-zero eigenvalues of $\partial_i \partial_i^t$. By Lemma 2 and Theorem 5, we have

$$\pi_i = \sum_{\substack{A \in \mathcal{B}_{i-1} \\ B \in \mathcal{B}_i}} (\det \partial_{\bar{A},B})^2 = \sum_{\substack{A \in \mathcal{B}_{i-1} \\ B \in \mathcal{B}_i}} w(A)^2 w(B)^2 = k_{i-1} k_i\,.$$

Now recall that $\partial_i^t \partial_i$ and $\partial_i \partial_i^t$ have the same multiset of non-zero eigenvalues. Therefore, for $0 \leq i \leq d-1$, Lemma 1 and Lemma 3 imply

$$\det \Delta_i = \det(\partial_i^t \partial_i + \partial_{i+1} \partial_{i+1}^t) = \pi_i \pi_{i+1} = k_{i-1} k_i^2 k_{i+1}\,.$$

(3) If Γ is acyclic, then $k_d = 1$ because Γ_d is the only d-tree in Γ. Therefore

$$\det \Delta_d = \det(\partial_d^t \partial_d) = \pi_d = k_{d-1} k_d = k_{d-1}\,.$$

If Γ is not acyclic, then $\operatorname{rk} \tilde{H}_d(\Gamma) > 0$ and $\det \Delta_d = 0$ by (2.3). □

Recall that the Matrix-Tree theorem states that for a finite graph G, every cofactor of its Laplacian matrix equals the number of spanning trees in G ([7]). The following corollary is an analogue of this theorem by Temperley [10].

Corollary 7. *(Temperley) Given a finite graph G with n vertices, let L be its Laplacian matrix, $k(G)$ the number of spanning trees in G, and J the $n \times n$ all 1's matrix. Then,*

$$\det(L + J) = n^2 k(G)\,. \tag{4.1}$$

Proof. Regarding G as a γ-complex, we have $L = \partial_1\partial_1^t$, $J = \partial_0^t\partial_0$, and $L + J = \Delta_0$. Since $k_{-1} = 1$, $k_0 = n$, and $k_1 = k(G)$, the result follows from Proposition 6 (2). □

The following theorem, which is the main result of the paper, is a logarithmic version of Proposition 6 for acyclic complexes.

Theorem 8. *Let Γ be an acyclic complex of dimension d such that $\partial_i \neq 0$ for $0 \le i \le d$. Let $D(x) = \sum_{i=-1}^{d} \omega_i x^{i+1}$ and $K(x) = \sum_{i=0}^{d-1} \kappa_i x^i$, where $\omega_i = \log\det\Delta_i$ and $\kappa_i = \log k_i$. Then we have*

$$D(x) = (1 + x)^2 K(x)\,.$$

Proof. Since Γ is a γ-complex, we have $\mathcal{B}_i \neq \emptyset$ and $k_i \ge 1$ for $0 \le i \le d$. Hence $K(x)$ is well defined. By Proposition 6, we see that $\det\Delta_i \ge 1$ for $-1 \le i \le d$, and $D(x)$ is well defined. The rest of the proof is checking the following details. Proposition 6 (1) implies $\omega_{-1} = \kappa_0$. Proposition 6 (2) implies $\omega_i = \kappa_{i-1} + 2\kappa_i + \kappa_{i+1}$ for $0 \le i \le d-1$. In particular, $k_{-1} = 1$ implies $\omega_0 = 2\kappa_0 + \kappa_1$, which also follows from (4.1). Also, $k_d = 1$ because Γ is acyclic, and we have $\omega_{d-1} = \kappa_{d-2} + 2\kappa_{d-1}$. Finally, Proposition 6 (3) implies $\omega_d = \kappa_{d-1}$. The result follows. □

5. Examples

5.1. **Tree-number of a planar graph.** Given a connected finite planar graph G, let $\bar{G}$ be the 2-dimensional complex whose 1-skeleton is G and 2-dimensional cells are the finite faces of G. Note that $\bar{G}$ is an acyclic complex. By (3) in Proposition 6, the number of spanning trees of G is the determinant of Δ_2 for $\bar{G}$. As an example, we prove the following theorem. Recall that the Fibonacci sequence $\{F_n\}$ is defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$.

Theorem 9. *Let G be a triangulation of an $(n + 2)$-gon P $(n \ge 1)$ without any internal triangle, i.e., a triangle with no edge from P. Then the number of spanning trees in G equals F_{2n+1}.*

Proof. Let $f_1, \dots, f_n$ be the faces of G. Since G has no internal triangle, each f_i contains one or two edges of P. Since each edge of P belongs to a unique face, there are exactly two faces, say f_1 and f_n, each of which contains two edges of P. If G^* is the dual graph of G and v_∞ the vertex corresponding to the infinite face of G, then $G^* - v_\infty$ is a path whose two end points correspond to f_1 and f_n. By arranging the faces of G in the same order as the vertices appear in this path, we may assume that f_i shares one edge each with f_{i-1} and f_{i+1} for $1 < i < n$. The remaining edge of f_i is an edge of P.

Assume that each face of G is oriented counterclockwise. Then each column of ∂_2 for $\bar{G}$ has exactly three non-zero entries, which are ± 1's, and the inner product of two columns is -1 if they are adjacent, and 0 otherwise. Therefore, $\Delta_2 = \partial_2^t\partial_2$ is an $n \times n$ tridiagonal matrix with 3's on the main diagonal and -1's just below and above it. If δ_n denotes the determinant of this tridigonal matrix, we have the recurrence relations $\delta_n = 3\delta_{n-1} - \delta_{n-2}$ for $n \ge 2$ with $\delta_1 = 3$ and $\delta_0 := 1$. These are the same recurrence relations satisified by the subsequence $\{F_{2n+1}\}$ of Fibonacci numbers. Hence, we have $\det\Delta_2 = \delta_n = F_{2n+1}$. □

In particular, the number of spanning trees of the *fan* with $n+2$ vertices is F_{2n+1} by this theorem.

5.2. **Standard simplexes.** Let Σ be the standard simplex on n vertices (hence $\dim \Sigma = n-1$). Then, Σ is acyclic and $|\Sigma_i| = \binom{n}{i+1}$ for $-1 \le i \le n-1$. If $[\sigma]$ denotes an oriented simplex for $\sigma \in \Sigma_i$, one can check that $\Delta_i[\sigma] = n[\sigma]$, which follows directly from the definition of the boundary operators ∂_i and ∂_{i+1} (and their transpose). Therefore, we have $\Delta_i = nI$, where I is the identity matrix of order $\binom{n}{i+1}$, and $\det \Delta_i = n^{\binom{n}{i+1}}$. Letting $\omega_i = \log_n \det \Delta_i = \binom{n}{i+1}$, we see that

$$D(x) = \sum_{i=-1}^{n-1} \omega_i x^{i+1} = \sum_{i=-1}^{n-1} \binom{n}{i+1} x^{i+1} = (1+x)^n \,.$$

By Theorem 8, we obtain

$$K(x) = \sum_{i=0}^{n-2} \kappa_i x^i = (1+x)^{n-2} \,,$$

where $\kappa_i = \log_n k_i = \binom{n-2}{i}$. Hence, we have $k_i = n^{\binom{n-2}{i}}$ for $0 \le i \le n-2$. This result was originally obtained by Kalai [6].

5.3. **Cubical complexes.** The n-cube Q_n ($n \ge 1$) is an n-dimensional cell complex that is the n-fold product $I \times \cdots \times I$, where I is the unit interval regarded as a cell complex with two 0-cells and one 1-cell. Hence Q_n is a cell complex of dimension n, and is the convex hull of the 2^n points in $\mathbb{R}^n$ whose coordinates are all 0 or 1. One can see that Q_n is acyclic by induction on n together with the fact that Q_{n-1} is a deformation retract of Q_n for $n \ge 2$.

In [3], Duval, Klivans, and Martin showed that the tree-numbers for Q_n are

$$k_i = \prod_{j=2}^{n} (2j)^{\binom{j-2}{i-1}\binom{n}{j}} \qquad (1 \le i \le n-1) \tag{5.1}$$

based on the spectra (the multisets of eigenvalues) of $\partial_* \partial_*^t$, which are, in turn, obtained from those of Δ_*. In what follows, we will derive (5.1) directly from the spectra $\mathit{Spec}(\Delta_*)$ of Δ_* via Theorem 8. We will start with the following generating function for the eigenvalues of Δ_* for Q_n ([3, Theorem 3.4]):

$$\sum_{i=0}^{\dim Q_n} \sum_{\lambda \in \mathit{Spec}(D_i)} t^i r^\lambda = (1 + r^2 + tr^2)^n = \sum_{k=0}^{n} t^k \binom{n}{k} r^{2k} (1+r^2)^{n-k} \,, \tag{5.2}$$

where $D_i = \Delta_i$ for $i \ge 1$ and $D_0 = \partial_1 \partial_1^t$. From (5.2), one can deduce that $\det \Delta_i = \prod_{j=1}^n (2j)^{\binom{n}{j}\binom{j}{i}}$ for $1 \le i \le n$, and that $\pi_{D_0} = k_0 k_1 = \prod_{j=1}^n (2j)^{\binom{n}{j}}$. By Corollary 7, we also obtain $\det \Delta_0 = 2^n \prod_{j=1}^n (2j)^{\binom{n}{j}}$. Now, let $\omega_i = \log_2 \det \Delta_i$, and let $\alpha_j = \binom{n}{j} \log_2(2j)$. Then,

$$\omega_{-1} = n, \quad \omega_0 = n + \sum_{j=1}^{n} \alpha_j \,, \quad \text{and } \omega_i = \sum_{j=1}^{n} \binom{j}{i} \alpha_j \text{ for } 1 \le i \le n \,,$$

and we have

$$\begin{aligned} D(x) &= \sum_{i=-1}^{n} \omega_i x^{i+1} \\ &= n + \Big(n + \sum_{j=1}^{n} \alpha_j\Big)x + \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} \binom{j}{i}\alpha_j\Big)x^{i+1} \\ &= n(1+x) + \sum_{i=0}^{n}\Big(\sum_{j=1}^{n} \binom{j}{i}\alpha_j\Big)x^{i+1} \\ &= n(1+x) + x\sum_{j=1}^{n} \alpha_j(1+x)^j \quad \text{(by interchanging the sums)} \\ &= n(1+x)^2 + x\sum_{j=2}^{n} \alpha_j(1+x)^j\,. \quad \text{(because } \alpha_1 = n\text{)} \end{aligned}$$

By Theorem 8, we obtain

$$K(x) = \sum_{i=0}^{n-1} \kappa_i x^i = n + x\sum_{j=2}^{n} \alpha_j(1+x)^{j-2}\,,$$

where $\kappa_i = \log_2 k_i$. By identifying the coefficients of x^i for $1 \le i \le n-1$, we obtain $\kappa_i = \sum_{j=2}^{n} \binom{j-2}{i-1}\alpha_j$, and $k_i = \prod_{j=2}^{n}(2j)^{\binom{j-2}{i-1}\binom{n}{j}}$ for $1 \le i \le n-1$.

6. Acknowledgments

The authors would like to thank Prof. M. Wachs for her encouragement to generalize Temperley's tree-number formula.

References

[1] R. Adin, Counting colorful multi-dimensional trees, Combinatorica. **12**, no. 3 (1992) 247-260.
[2] N. Biggs, Algebraic Graph Theory (2nd ed.), Cambridge University Press, Cambridge, 1993.
[3] A. Duval, C. Klivans, and J. Martin, Cellular spanning trees and Laplcians of cubical complexes, Advances in Applied Mathematics. **46** (2011) 247-274.
[4] J. Friedman, *Computing Betti numbers via combinatorial Laplacians*, in Proc. 28th Annual ACM Symposium on the Theory of Computing, ACM: New York, 1996, 386–391.
[5] F. R. Gantmacher, The Theory of Matrices, vol I, Chelsea, New York, 1960.
[6] G. Kalai, Enumeration of **Q**-acyclic simplicial complexes, Israel J. Math. **45** (1983), 337-351.
[7] G. Kirkhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme gefürht wird, Ann. Phys. Chem. **72** (1847) 497-508.
[8] J.R. Munkres, Elements of Algebraic Topology, Addison-Wesley, Reading, MA, 1984.
[9] J. Oxley, Matroid Theory, Oxford University Press, Oxford, England, 1992.
[10] H.N.V. Temperley, On the mutual cancellation of cluster integrals in Mayer's fugacity series, Proc. Phys. Soc. **83** (1964) 3-16.

E-mail address: `hyukkim@snu.ac.kr`

Department of Mathematical Sciences, Seoul National University, Seoul, Korea

E-mail address: `andrewk@math.uri.edu`

Department of Mathematics, University of Rhode Island, Kingston, RI 02881