Weighted Tree Numbers of Matroid Complexes

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Abstract. We give a new formula for the weighted high-dimensional tree numbers of matroid complexes. This formula is derived from our result that the spectra of the weighted combinatorial Laplacians of matroid complexes are polynomials in the weights. In the formula, Crapo's β-invariant appears as the key factor relating weighted combinatorial Laplacians and weighted tree numbers for matroid complexes.

Résumé. Nous présentons une nouvelle formule pour les nombres d'arbres pondérés de grande dimension des matroïdes complexes. Cette formule est dérivée du résultat que le spectre des Laplaciens combinatoires pondérés des matroïdes complexes sont des polynômes à plusieurs variables. Dans la formule, le β-invariant de Crapo apparaît comme étant le facteur clé reliant les Laplaciens combinatoires pondérés et les nombres d'arbres pondérés des matroïdes complexes.

Keywords: matroid complex, weighted combinatorial Laplacians, weighted tree numbers

1 Introduction

The purpose of this paper is to give a new formula of the weighted tree numbers of matroid complexes. As a high-dimensional analogue of Cayley-Prüfer theorem [29], Kalai [19] found the formula for the weighted tree numbers of standard simplexes. Continuing his study, Adin [1] presented a formula for the tree numbers of complete colorful complexes and posed the problem of finding their weighted tree numbers. Duval, Klivans, and Martin [11] obtained a formula of the weighted tree numbers of shifted complexes, developing simplicial matrix-tree theorem. We derive a formula of the weighted tree numbers of the independent set complex of matroids (Theorem 9). We also confirm the conjecture about the formula for the weighted tree numbers of hypercubes [12, Conjecture 4.3] by applying a similar method studied in this paper. (This result will be given in a forthcoming paper.)

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2 Combinatorial Laplacians and high-dimensional tree numbers

To begin, we will review a close relationship between combinatorial Laplacians and high-dimensional tree numbers of cell complexes. This relation was studied by Adin [1] for the first time and has been studied in [8, 11–13, 20, 26–28, 30, 34]. For a graph (a 1-dimensional complex), Temperley's formula [36] shows this relationship.

Theorem 1 *[36, Temperley's formula] Let G be a finite loopless graph with n vertices with its Laplacian matrix $L(G)$, and J the all $1'$s matrix. If we denote the number of spanning trees of G by $k(G)$, then*

$$\det(L(G) + J) = n^2 \cdot k(G).$$

Here, $L(G) + J$ is the 0-th combinatorial Laplacian of G. □

Temperley's formula has been generalized to high-dimensional complexes [11, 12, 20]. In this paper, we will focus on the following type of complexes. Refering to the terminologies in [11], a finite simplicial complex will be called $\mathbb{Z}$-APC (*$\mathbb{Z}$-acyclic in positive codimension*) if its reduced homology over $\mathbb{Z}$ is trivial except possibly in the top dimension. The independent set complex $IN(M)$ of a matroid M, which is the main object of study in this paper, is $\mathbb{Z}$-APC because it is shellable [3]. We may refer to $IN(M)$ as a *matroid complex*, also.

Let $\{C_i, \partial_i\}$ be an augmented chain complex of a finite $\mathbb{Z}$-APC complex Γ of dimension d with the augmentation $\partial_0 : C_0 \to \mathbb{Z}$ given by $\partial_0(v) = 1$ for every vertex v in Γ. Recall that, for $i \in [-1, d]$, the i-th combinatorial Laplacian $\Delta_i : C_i \to C_i$ is defined by

$$\Delta_i = \partial_i^t \partial_i + \partial_{i+1} \partial_{i+1}^t.$$

Let Γ_i be the set of all i-simplices, and $\Gamma^{(i)}$ the i-skeleton of Γ. For a non-empty subset $S \subset \Gamma_i$, define $\Gamma_S = S \cup \Gamma^{(i-1)}$ as an i-dimensional subcomplex of Γ. For $i \in [-1, d]$, a non-empty subset $B \subset \Gamma_i$ is an *i-dimensional tree* (or, simply, i-tree) if

(1) $H_i(\Gamma_B) = 0$,

(2) $|H_{i-1}(\Gamma_B)|$ is finite, and

(3) $H_j(\Gamma_B) = 0$ for $j \le i - 2$.

Note that condition (3) is a consequence of the fact $\Gamma_B^{(i-1)} = \Gamma^{(i-1)}$. We will denote the set of all i-trees in Γ by $\mathcal{B}_i = \mathcal{B}_i(\Gamma)$ with $\mathcal{B}_{-1} = \{\varnothing\}$. Define the i-th tree number of Γ to be

$$k_i = k_i(\Gamma) = \sum_{B \in \mathcal{B}_i} |H_{i-1}(\Gamma_B)|^2 .$$

The following is a generalization of Temperley's formula showing a relationship between Δ_i and unweighted high dimensional tree numbers k_i.

Theorem 2 *[20, Proposition 7] Let k_i be the i-th tree number of a $\mathbb{Z}$-APC complex Γ .*

(1) $\det \Delta_{-1} = k_0$

(2) $\det \Delta_i = k_{i-1} k_i^2 k_{i+1}$ *for* $i \in [0, d-1]$

(3) $\det \Delta_d = k_{d-1}$ *if Γ is acyclic, and 0 otherwise.* □

3 Weighted combinatorial Laplacians and weighted tree numbers

As a refined enumerator of tree numbers, we discuss weighted tree numbers. For example, Cayley-Prüfer theorem [29] gives an enumeration of the spanning trees of complete graphs according to their vertex degrees, and Kalai's formula [19, Theorem 1, 3'] gives an enumeration of high-dimensional tree numbers of standard simplexes according to their vertex degrees. Other examples of weighted Laplacians and weighted tree numbers can be found in [11, 12, 30]. In [30], the weights of different dimensions were considered simultaneously, and we will develop similar ideas for matroid complexes in this paper.

Let Γ be $\mathbb{Z}$-APC. For each vertex $v \in \Gamma_0$, let x_v be an indeterminate and define the weight of v to be $X_v = x_v^2$. For each face $\sigma \in \Gamma_i$, define its weight of a face σ to be

$$X_\sigma = \prod_{v \in \sigma} X_v = (x_\sigma)^2.$$

Denote by $\mathbb{F}$ the field containing $\mathbb{R}$ and all the indeterminates x_v. Let $\hat{C}_i$ be the $\mathbb{F}$-vector space of i-chains in Γ. The *weighted boundary operator* $\hat{\partial}_i : \hat{C}_i \to \hat{C}_{i-1}$ is defined as follows. For each oriented i-face $[\sigma] = [v_0, v_1, \ldots, v_i]$,

$$\hat{\partial}_i[\sigma] = \sum_{j=0}^{i} (-1)^j x_{v_j} [\sigma - v_j].$$

This $\hat{\partial}_i$ can be also written as

$$\hat{\partial}_i = W_{i-1}^{-1} \partial_i W_i$$

where W_i is the diagonal matrix whose diagonal entry corresponding to each i-face $\sigma \in \Gamma_i$ is x_σ. Define the i-th *weighted combinatorial Laplacian* $\hat{\Delta}_i : \hat{C}_i \to \hat{C}_i$ to be the combinatorial Laplacian of the weighted boundary operator $\hat{\partial}_i$, *i.e.*,

$$\hat{\Delta}_i = \hat{\partial}_i^t \hat{\partial}_i + \hat{\partial}_{i+1} \hat{\partial}_{i+1}^t.$$

To illustrate the weighted boundary operater $\hat{\partial}_i$, we present an example.

Example 1 Let $\mathcal{K}$ be an (abstract) simplicial complex on a vertex set $\{1, 2, 3, 4, 5\}$ whose facets are $\{124, 125, 134, 135, 145, 234, 235, 245\}$ (see Fig. 1). Suppose that the rows and columns of $\hat{\partial}_2 : \hat{C}_2 \to \hat{C}_1$ are ordered lexicographically. Then $\hat{\partial}_2$ is given by

$$\begin{array}{c|cccccccc|}
 & [124] & [125] & [134] & [135] & [145] & [234] & [235] & [245] \\
\hline
[12] & x_4 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
[13] & 0 & 0 & x_4 & x_5 & 0 & 0 & 0 & 0 \\
[14] & -x_2 & 0 & -x_3 & 0 & x_5 & 0 & 0 & 0 \\
[15] & 0 & -x_2 & 0 & -x_3 & -x_4 & 0 & 0 & 0 \\
[23] & 0 & 0 & 0 & 0 & 0 & x_4 & x_5 & 0 \\
[24] & x_1 & 0 & 0 & 0 & 0 & -x_3 & 0 & x_5 \\
[25] & 0 & x_1 & 0 & 0 & 0 & 0 & -x_3 & -x_4 \\
[34] & 0 & 0 & x_1 & 0 & 0 & x_2 & 0 & 0 \\
[35] & 0 & 0 & 0 & x_1 & 0 & 0 & x_2 & 0 \\
[45] & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_2
\end{array}.$$

□

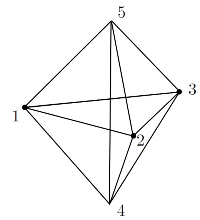

Fig. 1: a realization of $\mathcal{K}$

We introduce the definition of weighted high-dimensional tree numbers originated from Kalai [19].

Definition 3 For $i \in [0, d]$, define the i-th *weighted tree number* of Γ to be

$$\hat{k}_i = \hat{k}_i(\Gamma) = \sum_{B \in \mathcal{B}_i} |H_{i-1}(\Gamma_B)|^2 X_B. \tag{3.1}$$

where $X_B = \prod_{\sigma \in B} X_\sigma$ is the weight of $B \in \mathcal{B}_i$. Define $\hat{k}_{-1} = 1$.

For each $B \in \mathcal{B}_i$, define the *degree* of a vertex $v \in \Gamma_0$ in B to be the number of facets in B containing v, denoted by $\deg_B v$. When Γ is a graph, this degree is the same as the degree in graph theory. Then equation (3.1) becomes

$$\sum_{B \in \mathcal{B}_i} |H_{i-1}(\Gamma_B)|^2 \prod_{v \in \Gamma_0} X_v{}^{\deg_B v},$$

which explains why weighted tree numbers are often called degree-weighted tree numbers. Note that, if $X_v = 1$ for all $v \in \Gamma_0$, then we recover the i-th (unweighted) tree number k_i.

An example of degree-weighted tree numbers is Cayley-Prüfer [29] theorem for complete graphs as follows.

$$\sum_{T \in \mathcal{B}_1(K_n)} \prod_{i=1}^{n} X_i^{\deg_T(i)} = X_1 X_2 \cdots X_n (X_1 + X_2 + \cdots + X_n)^{n-2}.$$

A high-dimensional analogue of this theorem is Kalai's formula [19, Theorem 3']

$$\sum_{B \in \mathcal{B}_i(\Sigma)} |H_{i-1}(\Gamma_B)| \prod_{i=1}^{n} X_i^{\deg_B(i)} = (X_1 X_2 \cdots X_n)^{\binom{n-2}{i-1}} (X_1 + X_2 + \cdots + X_n)^{\binom{n-2}{i}} \tag{3.2}$$

where Σ is the standard simplex on n vertices.

The following is the weighted version of Theorem 2.

Theorem 4 *The following holds for a* $\mathbb{Z}$*-APC complex* Γ*:*

(1) $\det \hat{\Delta}_{-1} = \hat{k}_0$

(2) $\det \hat{\Delta}_i = (\prod_{\sigma \in \Gamma_{i-1}} X_\sigma)^{-1} (\prod_{\sigma \in \Gamma_i} X_\sigma)^{-1} \hat{k}_{i-1} \hat{k}_i^2 \hat{k}_{i+1}$ *for* $i \in [0, d-1]$

(3) $\det \hat{\Delta}_d = (\prod_{\sigma \in \Gamma_{d-1}} X_\sigma)^{-1} \hat{k}_{d-1} (\prod_{\sigma \in \Gamma_d} X_\sigma)$ *if* Γ *is acyclic, and* 0 *otherwise.* □

From now on, let Γ be an acyclic complex of dimension $d+1$. The reason for considering dimension $d+1$ is that we will apply an acyclization to a $\mathbb{Z}$-APC complex of dimension d.

By using Theorem 2, a relation was found between the generating function of the logarithmic determinants of combinatorial Laplacians and that of the logarithmic determinants of tree numbers, which makes it efficient to compute the tree numbers [20, Theorem 8]. The following theorem is the weighted version of this relation. We introduce *formal logarithm* having the following property: $\log XY = \log X + \log Y$ for nonzero $X, Y \in \mathbb{F}$.

Theorem 5 *Let $\hat{D}(x)$, $\hat{K}(x)$, and $F(x)$ be given as follows.*

(1) $\hat{D}(x) = \sum_{i=-1}^{d+1} \hat{\omega}_i x^{i+1}$ *where* $\hat{\omega}_i = \log \det \hat{\Delta}_i$

(2) $\hat{K}(x) = \sum_{i=0}^{d} \hat{\kappa}_i x^i$ *where* $\hat{\kappa}_i = \log \hat{k}_i$

(3) $F(x) = \sum_{v \in \Gamma_0} \log X_v \Big(\big(\sum_{i=0}^{d} f_{v,i} x^{i+1}\big) - f_{v,d+1} x^{d+1} \Big)$ *where $f_{v,i}$ is the number of i-faces in X containing v.*

Then we have

$$\begin{aligned} \hat{D}(x) &= (1+x)^2 \hat{K}(x) - (1+x) F(x), \text{ or} \\ \hat{K}(x) &= (1+x)^{-2} \hat{D}(x) + (1+x)^{-1} F(x). \end{aligned}$$

□

From this theorem, one can recover Kalai's formula (equation (3.2)).

In addition, we express weighted tree numbers in terms of all the monomials corresponding to all the vertices and eigenvalues of weighted combinatorial Laplacians. The following theorem is the weighted version of [12, equation (11)] obtained from Theorem 5.

Theorem 6 *Let $\hat{\Lambda}$ be the set of all distinct eigenvalues of the total weighted Laplacian $\bigoplus_{i=-1}^{d+1} \hat{\Delta}_i$ and let $m_{\hat{\lambda},i}$ be the multiplicity of $\hat{\lambda}$ in $\hat{\Delta}_i$, i.e., $\det \hat{\Delta}_i = \prod_{\hat{\lambda} \in \hat{\Lambda}} \hat{\lambda}^{m_{\hat{\lambda},i}}$. The d-th weighted tree number $\hat{k}_d$ of Γ is*

$$\prod_{\text{vertices } v \text{ of } \Gamma} (X_v)^{|\tilde{\chi}((lk\ v)^{(d-2)})|} \prod_{\hat{\lambda} \in \hat{\Lambda}} \hat{\lambda}^{a_{\hat{\lambda},d}}$$

where $a_{\hat{\lambda},d} = \sum_{j=-1}^{d-1} (-1)^{d-j-1} (d-j) m_{\hat{\lambda},j}$ *and the* link *of a vertex v is given by*

$$lk\ v = \{\sigma \in \Gamma \mid \sigma \cap \{v\} = \varnothing, \sigma \cup \{v\} \in \Gamma\}.$$

□

4 Weighted combinatorial Laplacians of matroid complexes

An interesting question for combinatorial Laplacians is which complexes have integral spectra. There are some known complexes with this property: chessboard [18], matching [10], matroid [25], shifted [14], and shifted cubical complexes [12]. Then a natural question for weighted combinatorial Laplacains is which complexes have spectra that are polynomials. Duval, Klivans, and Martin showed that the spectra of the weighted combinatorial Laplacians of shifted complexes are polynomials and used their result to give the weighted tree numbers of shifted complexes [11]. We show that matroid complexes have polynomial spectra and will use these to find the weighted tree numbers of matroid complexes.

First, we review the spectra of the unweighted combinatorial Laplacians of matroid complexes [25]. Let M be a loopless matroid, r its rank function, $L(M)$ its lattice of flats, and $\mu(V,W)$ the Möbius function on $L(M)\times L(M)$. Define the α-invariant $\alpha(M)$ of M to be the unsigned reduced *Euler characteristic* of its matroid complex $IN(M)$. For convenience, we will denote $\mu(W/V) = |\mu(V,W)|$ and $d = r(M)-1$.

Theorem 7 *[25, Corollary 18] Let Λ be the set of all distinct eigenvalues of the total Laplacian $\bigoplus_{i=-1}^{d}\Delta_i$ of a matroid complex $IN(M)$. Then*

$$\Lambda = \{|E \smallsetminus V| : V \in L(M) \text{ and } \alpha(V) \neq 0\}$$

and, for each $\lambda \in \Lambda$, its multiplicity $m_{\lambda,i}$ in Δ_i is given by

$$\sum_{V:|E\smallsetminus V|=\lambda}\ \sum_{W:r(W)=i+1} \alpha(V)\mu(W/V).$$

□

We present the weighted version of the above theorem. Let $\mathbb{F}$ be the field containing $\mathbb{R}$ and all the indeterminates x_e for each element e in the ground set E of M. For each $e \in E$, define the weight of e to be $X_e = x_e^2$. For each non-empty set $S \subset E$, define $\|S\| = \sum_{e\in S} X_e$ and $\|\varnothing\| = 0$.

Theorem 8 *Let $\hat{\Lambda}$ be the set of all distinct eigenvalues of the total weighted Laplacian $\bigoplus_{i=-1}^{d}\hat{\Delta}_i$ of a matroid complex $IN(M)$ where $\hat{\Delta}_i$ is the i-th weighted combinatorial Laplacian of M. Then*

$$\hat{\Lambda} = \{\|E \smallsetminus V\| : V \in L(M) \text{ and } \alpha(V) \neq 0\}$$

and, for each $\hat{\lambda} = \|E \smallsetminus V\| \in \hat{\Lambda}$, its multiplicity $m_{\hat{\lambda},i}$ in $\hat{\Delta}_i$ is given by

$$\sum_{W:r(W)=i+1} \alpha(V)\mu(W/V).$$

In particular, the spectra of $\bigoplus_{i=-1}^{d}\hat{\Delta}_i$ are polynomials in X_e's. □

5 Weighted tree numbers of matroid complexes

We show that the weighted tree numbers of matroid complexes have a nice factorization according to the degrees of their vertices. Our method is different from that which was used to find the formula for the weighted tree numbers of a shifted complex [11]. While the reduced Laplacian in the top dimension was used in [11], we use all of the combinatorial Laplacians except the top-dimension.

To begin, we review two important invariants of a matroid M which will appear in the formula. One is $\alpha(M)$ which equals the unsigned reduced Euler characteristic $|\tilde{\chi}(IN(M))|$ of $IN(M)$. Note that $\alpha(M)$ has other interpretations as follows:

$$\alpha(M) = |\mu_{L(M^*)}(\hat{0},\hat{1})| = \text{rk } \tilde{H}_{r(M)-1}(IN(M)) = T_M(0,1),$$

where M^* is the dual matroid of M, and $T_M(x,y)$ the Tutte polynomial of M.

The other is Crapo's $\beta(M)$ which is defined as follows [6]:

$$\beta(M) = (-1)^{r(M)} \sum_{A \subset E(M)} (-1)^{|A|} r(A)\,.$$

For our purpose, it will be useful to take the following equivalent definition of $\beta(M)$ (used in [38, Chapter 7.3]).

$$\beta(M) = (-1)^{r(M)} \sum_{V \in L(M)} \mu(\hat{0}, V) r(V).$$

It is also known that $\beta(M)$ also equals the unsigned reduced Euler characteristic of the *reduced broken circuit complex* [3]. The following is the main theorem of this paper.

Theorem 9 *The d-th weighted tree number $\hat{k}_d(M) = \hat{k}_d(IN(M))$ of a matroid complex $IN(M)$ is*

$$\prod_{e \in E} X_e^{(|\mathcal{B}(M/e)| - \alpha(M/e))} \prod_{\text{flats } V \text{ of } M} (\sum_{e \notin V} X_e)^{\alpha(V)\beta(M/V)}.$$

where $|\mathcal{B}(M)|$ denotes the number of bases of a matroid M. Here,

$$|\mathcal{B}(M/e)| - \alpha(M/e) = |\tilde{\chi}(IN(M/e)^{(d-2)})|.$$

(This equality comes from the shellability of matroid complexes.) □

This theorem is proved using Theorem 6 and Theorem 8.

By setting $X_e = 1$ for all $e \in E$, we can recover (unweighted) tree numbers of matroid complexes [24, Theorem 2]. To simplify their formulas, we introduce a convolution of α-invariant and β-invariant.

Definition 10 For $\lambda \in \Lambda = \{|E \setminus V| : V \in L(M) \text{ and } \alpha(V) \neq 0\}$, define a convolution of α-invariant and β-invariant with respect to λ as

$$\alpha \circ_\lambda \beta = \sum_{V \in L(M): |E \setminus V| = \lambda} \alpha(V)\beta(M/V).$$

Theorem 11 *[24, Theorem 2] The d-th tree number $k_d(M) = k_d(IN(M))$ of a matroid complex $IN(M)$ is*

$$\prod_{\lambda \in \Lambda} \lambda^{\alpha \circ_\lambda \beta}.$$

□

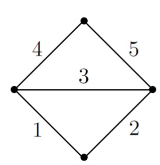

Fig. 2: a graph $G = K_4 - e$

Example 2 Let $M = M(G)$ be the cycle matroid of G where G is a graph $K_4 - e$ (see Fig.2). Then the cycle matroid complex $IN(M)$ of M is the simplicial complex $\mathcal{K}$ in Example 1 (see Fig.1). We apply our theorem to compute the tree numbers of the matroid complex $IN(M)$.

First, for each vertex e in $IN(M)$, the contracted matroid complex $IN(M/e)$ consists of 4 vertices and so $|\hat{\chi}((IN(M/e))^{(0)})| = 3$.

Second, for each flat V in M, let us compute $\alpha(V)$ and $\beta(M/V)$.

- If $V = \varnothing$, then $\alpha(V) = 1$ and $\beta(M/V) = \beta(M) = 1$.
- If V has only one element, then $\alpha(V) = 0$.
- If V is $\{1, 2, 3\}$ or $\{3, 4, 5\}$, then $\alpha(V) = 1$ and $\beta(M/V) = 1$.
- If $V = M$, then $\beta(M/V) = \beta(\varnothing) = 0$.

Therefore,

$$\hat{k}_2(M) = X_1^3 X_2^3 X_3^3 X_4^3 X_5^3 (X_1 + X_2 + X_3 + X_4 + X_5)(X_1 + X_2)(X_4 + X_5)$$

and we obtain $k_2(M) = 2^2 \cdot 5$. □

6 Applications: Complete colorful complexes

We give the weighted version of Adin's formula for tree numbers of complete colorful complexes, answering the question posed in [1, Section 6 (b)]. Define a *complete colorful complex* as follows. For each $t \in [r]$, let $E_t = \{e_{1,t}, e_{2,t}, \ldots, e_{n_t,t}\}$ be a set of *color* t. Let $E = \cup_{t=1}^r E_t$ be a vertex set. Define *complete colorful complex* $K = K(n_1, \ldots, n_r)$ as a simplicial complex on a vertex set E whose faces are subsets of E each containing at most one element from each E_t, *i.e.*,

$$K = \{F \subset E \mid |F \cap E_t| \leq 1 \text{ for } t = 1, \ldots, r\}.$$

Note that K is isomorphic to the matroid complex of $\bigoplus_{t=1}^r U_{1,n_t}$ where U_{1,n_t} is a rank 1 uniform matroid on n_t elements. For simplicity, denote the dimension of K by $d = r - 1$. In addition, for each $i \in [1, d]$, the i-th skeleton $K^{(i)}$ is a matroid complex.

For each $t \in [r]$, denote weights of $e_{1,t}, e_{2,t}, \ldots, e_{n_t,t}$ by $X_{1,t}, X_{2,t}, \ldots, X_{n_t,t}$, respectively. For each $S \subset [r]$, define $\pi_S = \prod_{s \in S} (n_s - 1)$.

Theorem 12 *For $i \in [1, d]$, we have*

$$\hat{k}_i(K) = \prod_{t=1}^{r} (X_{1,t} \cdots X_{n_t,t})^{\sum_{j=0}^{i-1} (-1)^{i-1-j} e_j(n_1, \ldots, \hat{n}_t, \ldots, n_r)} \prod_{|S| \leq i} (\sum_{s \notin S} X_{1,s} + \cdots + X_{n_s,s})^{\pi_S \binom{r-2-|S|}{i-|S|}}$$

where $e_j(Y_1, \ldots, Y_n)$ is the j-th elementary symmetric polynomial.
In particular,

$$\hat{k}_d(K) = \prod_{t=1}^{r} \Big((X_{1,t} \cdots X_{n_t,t})^{(\prod_{s \neq t} (n_s) - \prod_{s \neq t} (n_s - 1))} (X_{1,t} + \cdots + X_{n_t,t})^{\prod_{s \neq t} (n_s - 1)} \Big).$$

□

The weighted top-dimensional tree number of a complete colorful complex was computed by Aalipour and Duval.

We recover Adin's formula for the unweighted tree numbers of complete colorful complexes from the above weighted version, by setting $X_{1,t} = \cdots = X_{n_t,t} = 1$ for all $t \in [r]$. (The top-dimensional tree number of a complete colorful complex was suggested by Bolker [5].)

Corollary 13 *[1, Theorem 1.5] For $i \in [1, d]$, we have*

$$k_i(K) = \prod_{|S| \le i} (\sum_{s \notin S} n_s)^{\pi_S \binom{r-2-|S|}{i-|S|}}.$$

In particular,

$$k_d(K) = \prod_{t=1}^{r} n_t^{(\prod_{s \neq t} (n_s - 1))}.$$

□

Note that the 1-dimensional skeleton of a complete colorful complex is a complete multipartite graph. By using the above theorem, we obtain the weighted spanning tree numbers of complete multipartite graphs (For that of a complete bipartite graph, see [35, Exercise 5.30]).

Let $K_{n_1,\ldots,n_r}$ be a complete multipartite graph with an r-partition $(V_1, \ldots, V_r)$. For each $t \in [r]$, let $V_t = \{v_{1,t}, \ldots, v_{n_t,t}\}$, and denote the weights of $v_{1,t}, \ldots, v_{n_t,t}$ by $X_{1,t}, \ldots, X_{n_t,t}$, respectively. For a complete bipartite graph $K_{m,n}$ with a bipartition (A, B) where $A = \{u_1, \ldots, u_m\}$ and $B = \{v_1, \ldots, v_n\}$, let $X_1, \ldots, X_m$ (*resp.* $Y_1, \ldots, Y_n$) be the weights of $u_1, \ldots, u_m$ (*resp.* $v_1, \ldots, v_n$).

Corollary 14 *The weighted spanning tree number of $K_{n_1,\ldots,n_r}$ is given by*

$$\hat{k}(K_{n_1,\ldots,n_r}) = (\prod_{t=1}^{r} X_{1,t} \cdots X_{n_t,t}) \Big(\sum_{t=1}^{r} (X_{1,t} + \cdots + X_{n_t,t}) \Big)^{r-2} \prod_{t=1}^{r} \Big(\sum_{s \neq t} (X_{1,s} + \cdots + X_{n_s,s})^{n_s - 1} \Big)$$

In particular, the weighted spanning tree number of $K_{m,n}$ is given by

$$\hat{k}(K_{m,n}) = (X_1 \cdots X_m)(Y_1 \cdots Y_n)(X_1 + \cdots + X_m)^{n-1}(Y_1 + \cdots + Y_n)^{m-1}.$$

□

When each color set has only one element, we recover Kalai's formula for the weighted tree numbers of standard simplexes.

Corollary 15 *[19, Theorem 1, 3'] Let Σ be the standard simplex on n vertices. For each vertex $v_j \in (\Sigma)_0$, let X_j be its weight. Then the i-th weighted tree number is given by*

$$\hat{k}_i(\Sigma) = (X_1 X_2 \cdots X_n)^{\binom{n-2}{i-1}} (X_1 + X_2 + \cdots + X_n)^{\binom{n-2}{i}}.$$

In particular, its i-th tree number is given by

$$k_i(\Sigma) = n^{\binom{n-2}{i}}.$$

□

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References

[1] R. Adin, *Counting colorful multi-dimensional trees*, Combinatorica 12(3) (1992), 247-260.

[2] N. Biggs. *Algebraic Graph Theory* (2nd ed.), Cambridge University Press, Cambridge, 1993.

[3] A. Björner, The homology and shellability of matroids and geometric lattices, in *Matroid Applications* (ed. N. White) of *Encyclopedia of Mathematics and Its Applications vol. 40*, Cambridge Univ. Press, Cambridge, 1992, 226-283.

[4] A. Björner and G. Kalai, On f-vectors and homology, in: Combinatorial Mathematics, *Proc. of 3rd International Conference* (New York, 1985), N.Y. Acad. Sci., New York, (1989), 63-80.

[5] E. D. Bolker, Simplicial geometry and transportation polytopes, *Trans. Amer. Math. Soc.* 218 (1976) 121-142.

[6] H. Crapo, A higher invariant for matroids, *J. Comb. Theory* 2 (1967) 406-417.

[7] H. Crapo, The Tutte polynomial, *Aequationes Math.* 3 (1969) 211-229.

[8] M. Catanzaro, V. Chernyak, and J. Klein, Kirchhoff's theorems in higher dimensions and Reidemeister torsion, arXiv:1206.6783.

[9] G. Denham, The Combinatorial Laplacian of the Tutte complex, *J. Algerba* 242 (2001) 160-175.

[10] X. Dong and M. Wachs, Combinatorial Laplacian of the matching complex, *Electron. J. Combin.* 9 (2002), #R17, 11 pp.

[11] A. Duval, C. Klivans, and J. Martin, Simplicial matrix-tree theorems, *Trans. Amer. Math. Soc.* 361 (2009) 6073-6114.

[12] A. Duval, C. Klivans, and J. Martin, Cellular spanning trees and Laplacians of cubical complexes, *Adv. Appl. Math. 46* (2011) 247-274.

[13] A. Duval, C. Klivans, and J. Martin, Cuts and flows of cell complexes, preprint arXiv:1206.6157v1 (2012).

[14] A. Duval and V. Reiner, Shifted simplicial complexes are Laplacian integral, *Trans. Amer. Math. Soc.* 354 (2002) 4313-4344.

[15] G. Etienne and M. Las Vergnas, External and internal elements of a matroid basis, *Disc. Math.* 179 (1998) 111-119.

[16] J. Friedman, Computing Betti numbers via combinatorial Laplacians, in *Proc. 28th Annual ACM Symposium on the Theory of Computing*, ACM: New York, (1996) 386-391.

[17] J. Friedman, Computing Betti numbers via combinatorial Laplacians, *Algorithmica* 21 (1998) 331-346.

[18] J. Friedman and P. Hanlon, On the Betti numbers of chessboard complexes, *J. Algebraic Combin.* 8 (1998) 193-203.

[19] G. Kalai, Enumeration of $\mathbb{Q}$-acyclic simplicial complexes, *Israel J. Math.* 45 (1983) 337-351.

[20] H. Kim and W. Kook, Logarithmic tree-numbers for acyclic complexes, *Electron. J. Combin.* 21 (2014) #P1.50, 12 pp.

[21] W. Kook, The homology of the cycle matroid of a coned graph, *European J. Combin.* 28 (2007) 734-741.

[22] W. Kook, Edge-rooted forest and the α-invariant of cone graphs, *Disc. Appl. Math.* 155 (2007) 1071-1075.

[23] W. Kook. Combinatorial Green's function of a graph and applications to networks, *Adv. Appl. Math.* 46 (2011) 417-423.

[24] W. Kook and K. Lee, A formula for simplicial tree numbers of matroid complexes, preprint (2014).

[25] W. Kook, V. Reiner, and D. Stanton, Combinatorial Laplacians of matroid complexes, *J. Amer. Math. Soc.* 13 (2000) 129-148.

[26] R. Lyons, Determinantal probability measures, *Publ. Math. Inst. Hautes Eťudes Sci.* 98 (2003) 167-212.

[27] R. Lyons, Random complexes and l^2-Betti numbers, *J. Topol. Anal.* 1 (2009) 153-175.

[28] M. Maxwell, Enumerating bases of self-dual matroids, *J. Combin. Theory Ser. A* 116 (2009) 351-378.

[29] J.W. Moon, *Counting labelled trees*, Canadian Mathematical Monographs 1, Canad. Math. Cong., 1970.

[30] J. Martin, M. Maxwell, V. Reiner, and S. Wilson, Pseudodeterminants and perfect square spanning tree counts, arXiv:1311.6686 (2013).

[31] J. Martin and V. Reiner, Factorization of some weighted spanning tree enumerators, *J. Combin. Theory Ser. A* 104 (2003) 287-300.

[32] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Reading, MA, 1984.

[33] J. Oxley, *Matroid Theory*, Oxford University Press, Oxford, 1992.

[34] A. Petersson, *Enumeration of spanning trees in simplicial complexes*, Masters thesis, Uppsala Univ., 2008.

[35] R. Stanley, *Enumerative Combinatorics*, vol. II. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999.

[36] H.N.V. Temperley, On the mutual cancellation of cluster integrals in Mayer's fugacity series, *Proc. Phys. Soc.*, 83:3-16 (1964).

[37] D. J. A. Welsh, *Matorid Theory*, Dover Books on Mathematics, 2010.

[38] N. White, *Combinatorial Geometries*, *Encyclopedia of Mathematics and Its Applications vol. 29*, Cambridge Univ. Press, Cambridge, 1987.