A FORMULA FOR SIMPLICIAL TREE NUMBERS OF MATROID COMPLEXES

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ABSTRACT. We give a formula for the simplicial tree numbers of the independent set complex of a finite matroid M as a product of eigenvalues of the total combinatorial Laplacians on this complex. Two matroid invariants emerge naturally in describing the multiplicities of these eigenvalues in the formula: one is the unsigned reduced Euler characteristic of the independent set complex and the other is the β-invariant of a matroid. We will demonstrate various applications of this formula including a "matroid theoretic" derivation of Kalai's simplicial tree numbers of a standard simplex.

1. INTRODUCTION

In this paper, we will prove a formula for the simplicial tree numbers of the independent set complex $IN(M)$ of a finite matroid M. Simplicial trees for simplicial complexes have been studied as a generalization of spanning trees for graphs [1,4,7]. In these studies, combinatorial Laplacians for simplicial complexes play a role analogous to that of graph Laplacians for graphs. We will review simplicial tree numbers and combinatorial Laplacians in section 2. Since eigenvalues of combinatorial Laplacians on matroid complexes are known [10], one may ask whether the simplicial tree numbers for these complexes can be computed [4]. We give an answer to this question in this paper.

The formula is given as a product of eigenvalues of the combinatorial Laplacians on $IN(M)$. We refer the readers to [10] for the integrality of these eigenvalues. Two matroid invariants will appear naturally in describing the multiplicities of the eigenvalues in this product. One is the unsigned reduced Euler charcteristic of $IN(M)$, and the other is Crapo's β-invariant of M. For the sake of simplicity, we will refer to the former as the α-invariant of M, and denote the two invariants by $\alpha(M)$ and $\beta(M)$, respectively. We will review these invariants in section 2, also.

Our main result (Theorem 5) can be stated as follows. Let M be a matroid of rank $d+1$ on a finite ground set E, and $L(M)$ its lattice of flats. For $V \in L(M)$, let M/V be the contraction of V from M. For an integer $\lambda \in [0, |E|]$, we define $L(M)_\lambda = \{ V \in L(M) : |E \setminus V| = \lambda \}$, and define the *convolution* $\alpha \circ_\lambda \beta$ of α-invariant and β-invariant for M with respect to the given λ to be

$$\alpha \circ_\lambda \beta = \sum_{V \in L(M)_\lambda} \alpha(V)\beta(M/V)\,.$$

Then, the simplicial tree number k_d of $IN(M)$ in the top dimension d is

$$k_d = \prod_{\lambda \in (0,|E|]} \lambda^{\alpha \circ_\lambda \beta}$$

where the product is over all positive integers $\lambda \in (0, |E|]$ (with $L(M)_\lambda \neq \emptyset$).

This formula can be used to give simplicial tree numbers k_i for $IN(M)$ in other dimensions i. For $-1 \le i \le d$, we observe that k_i is the top dimensional simplicial tree number of $IN(T^{i+1}(M))$, where $T^{i+1}(M)$ is the *truncation* obtained from M by ignoring all independent sets of rank $> i+1$. Therefore, the main result can be applied to the matroid $T^{i+1}(M)$ to compute k_i . We shall demonstrate these computations via examples later.

This paper is organized as follows. In section 2, we will review all prelimainary definitions and results concerning simplicial tree numbers, combinatorial Laplacians, and $\alpha(M)$ and $\beta(M)$ of a finite matroid M. In section 3, we will present a proof of the main result, Theorem 5. Section 4 will discuss applications of Theorem 5, including a new derivation of Kalai's classical result on standard simplex [7]. We refer the readers to [12] or [15] for terminologies and definitions from matroid theory that are used in this paper.

2. Preliminaries

2.1. **Simplicial tree numbers.** In this paper, *simplicial tree numbers* mean high-dimensional tree numbers of a simplicial complex (refer to [4, 5, 8]). A non-empty simplicial complex Γ is said to be $\mathbb{Z}$*-acyclic in positive codimensions* (shortly, $\mathbb{Z}$-*APC*) if the reduced integral homology $\tilde{H}_i(\Gamma) = 0$ for $i < \dim \Gamma$ (refer to [4]). Note that the independent set complex $IN(M)$ of a finite (non-empty) matroid M is $\mathbb{Z}$-*APC* because it is shellable (refer to [2]).

Let Γ be a $\mathbb{Z}$-APC complex of dimension d. For $i \in [0, d]$, let Γ_i denote the set of all i-dim simplices in Γ. The i-skeleton of Γ is $\Gamma^{(i)} = \Gamma_{-1} \cup \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_i$, where we define $\Gamma_{-1} = \{\emptyset\}$. For a non-empty subset $S \subset \Gamma_i$, define $\Gamma_S := S \cup \Gamma^{(i-1)}$ as an i-dimensional subcomplex of Γ. For $i \in [-1, d]$, a non-empty subset $B \subset \Gamma_i$ is an *i-dimensional simplicial tree* (or simply, *i-tree*) if

1. $\tilde{H}_i(\Gamma_B) = 0$,
2. $|\tilde{H}_{i-1}(\Gamma_B)|$ is finite, and
3. $\tilde{H}_j(\Gamma_B) = 0$ for $j \le i-2$.

Note that condition (3) is a consequence of the fact $\Gamma_B^{(i-1)} = \Gamma^{(i-1)}$. We will denote the set of all i-trees in Γ by $\mathcal{B}_i = \mathcal{B}_i(\Gamma)$ with $\mathcal{B}_{-1} = \{\{\emptyset\}\}$.

Define the i-th simplicial tree number (or simply, i-th *tree number*) of Γ to be

$$k_i = k_i(\Gamma) = \sum_{B \in \mathcal{B}_i} |\tilde{H}_{i-1}(\Gamma_B)|^2.$$

We have $k_{-1} = 1$ by definition, and $k_0 = |\Gamma_0|$. If Γ is a connected graph, then k_1 is the number of spanning trees in Γ because $|\tilde{H}_0(\Gamma_B)| = 1$ for $B \in \mathcal{B}_1$. However, $|\tilde{H}_{i-1}(\Gamma_B)|$ may not equal 1 for $B \in \mathcal{B}_i$ when $i > 1$. Refer to [7] for an example.

2.2. **Combinatorial Laplacians and tree numbers.** Let $\{C_i, \partial_i\}$ be an augmented chain complex of a finite $\mathbb{Z}$-APC complex Γ of dimension d with the augmentation $\partial_0 : C_0 \to \mathbb{Z}$ given by $\partial_0(v) = 1$ for every vertex v in Γ_0. We will represent ∂_i as a $|\Gamma_{i-1}| \times |\Gamma_i|$ integer matrix. For $i \in [-1, d]$, the i-th combinatorial Laplacian $\Delta_i : C_i \to C_i$ is defined by

$$\Delta_i = \partial_i^t \partial_i + \partial_{i+1} \partial_{i+1}^t ,$$

where ∂_{-1} and ∂_{d+1} are defined to be zero maps.

An important property of the combinatorial Laplacians is that Δ_i is non-singular iff $\mathrm{rk}\tilde{H}_i(\Gamma) = 0$ (refer to [6, Proposition 2.1]). In particular, for a finite $\mathbb{Z}$-APC complex Γ of dimension d, we have $\det \Delta_i > 0$ for $i \in [-1, d-1]$. Furthermore, the following proposition shows a connection between simplicial tree numbers and the determinant of combinatorial Laplacians. (The following proposition also appeared in [4, 5].)

Proposition 1. *[8, Prop 7] Let Γ be a $\mathbb{Z}$-APC complex of dimension d, and let Δ_i be its combinatorial Laplacians for $i \in [-1, d]$. Then*

1. $\det \Delta_{-1} = k_0$,
2. $\det \Delta_i = k_{i-1} k_i^2 k_{i+1}$ *for $i \in [0, d-1]$; and*
3. $\det \Delta_d = k_{d-1}$ *if Γ is acyclic, and 0 otherwise.* □

For example, if Γ is a finite graph with n vertices, and $k(\Gamma)$ is the number of spanning trees in Γ, then the Temperley's tree number formula $\det \Delta_0 = n^2 k(\Gamma)$ follows from this proposition (refer to [13]).

Note that if Γ is a $\mathbb{Z}$-APC complex of dimension d, Proposition 1 together with the properties $\det \Delta_i > 0$ for $i \in [-1, d-1]$ implies that all simplicial tree numbers of Γ are non-zero. Furthermore, the simplicial tree numbers of Γ can be expressed in terms of eigenvalues of the total combinatorial Laplacian as follows. Note that the following corollary appeared in [5, Eq.(11)].

Corollary 2. *Let Γ be a $\mathbb{Z}$-APC complex of dimension d, and Λ the set of all distinct eigenvalues of its total combinatorial Laplacian $\oplus_{i=-1}^{d} \Delta_i$. For $\lambda \in \Lambda$ and $j \in [-1, d]$, let $m_{\lambda,j}$ be the multiplicity of λ as an eigenvalue of Δ_j. For $i \in [0, d]$, the i-th tree number k_i of Γ is*

$$k_i = \prod_{\lambda \in \Lambda \setminus \{0\}} \lambda^{a_{\lambda,i}}$$

where $a_{\lambda,i} = \sum_{j=-1}^{i} (-1)^{i-j-1}(i-j) m_{\lambda,j}$.

Proof. For each $i \in [0, d]$, Proposition 1 implies $k_i = \prod_{j=-1}^{i-1} \det \Delta_j^{(-1)^{i-j-1}(i-j)}$. The formula for $a_{\lambda,i}$ follows from this and the identities $\det \Delta_j = \prod_{\lambda \in \Lambda \setminus \{0\}} \lambda^{m_{\lambda,j}}$ for $j < d$. Note that the i-th term in the formula for $a_{\lambda,i}$ is zero. □

2.3. **α-invariant and β-invariant of a matroid M.** We define the α-invariant $\alpha(M)$ of a finite matroid M to be the unsigned reduced Euler characteristic of the independent set complex $IN(M)$, i.e.,

$$\alpha(M) = |\tilde{\chi}(IN(M))| .$$

This terminology, α-invaraint, was introduced first in [9]. We refer the readers to [2] for a comprehensive discussion of this invariant.

For a finite matroid M, let $L(M)$ be its lattice of flats, and $\mu(V, W)$ the Möbius function on $L(M) \times L(M)$. Also, let r denote the rank function on M. The β-invariant of M is defined as

$$\beta(M) = (-1)^{r(M)} \sum_{V \in L(M)} \mu(\hat{0}, V) r(V). \tag{1}$$

This is not the original definition introduced first in [3], but this is an equivalent definition used in [16]. For our purpose, it is useful to take (1) as the definition of the β-invariant. The β-invariant is also equal to the unsigned reduced Euler characteristic of the *reduced broken circuit complex* [2].

3. The main result

In this section, M denotes a non-empty matroid of rank $r(M) = d+1$ on a finite ground set E, and $IN(M)$ its independent set complex. Hence the dimension of $IN(M)$ is d. The following result contains necessary information for our purpose concerning the eigenvalues of the combinatorial Laplacians on $IN(M)$.

Proposition 3. *[10, Corollary 18] For $\lambda \in \mathbb{R}$ and $i \in [-1, d]$, let $m_{\lambda,i}$ be the multiplicity of λ as an eigenvalue of the combinatorial Laplacian Δ_i on $IN(M)$. Then we have*

$$\sum_{\lambda\in\mathbb{R},\, i\in[-1,d]} m_{\lambda,i} t^i q^\lambda = \sum_{V,W\in L(M)} \alpha(V)\mu(W/V) t^{r(W)-1} q^{|E\setminus V|}$$

where we define $\mu(W/V) = |\mu(V,W)|$ if $V \subset W$, and 0 otherwise.

Proof. Corollary 18 in [10] originally states that

$$\sum_{\lambda\in\mathbb{R},\, i\in[-1,d]} \dim_{\mathbb{R}}(\Delta_i^M)_\lambda\, t^i q^\lambda = t^{-1} q^{|E|} \mathrm{Spec}_M(t, q^{-1})$$

where $(\Delta_i^M)_\lambda$ is the real λ-eigenspace of the combinatorial Laplacian Δ_i^M on $IN(M)$, and $\mathrm{Spec}_M(t,q)$, called the *spectrum polynomial* of M, is given by (2.2) in [10]:

$$\mathrm{Spec}_M(t,q) = \sum_{W\in L(M)} t^{r(W)} \sum_{V\in L(M),\, V\subset W} |\tilde{\chi}(IN(V))| \cdot |\mu(V,W)| q^{|V|}\,.$$

The proposition follows from these two equations and the observations that Δ_i^M is the same integer matrix as Δ_i on $IN(M)$ with a fixed linear ordering on E, and that $\dim_{\mathbb{R}}(\Delta_i^M)_\lambda = m_{\lambda,i}$. □

From this proposition, it is clear that every eigenvalue of the combinatorial Laplacians on $IN(M)$ is an integer of the form $|E \setminus V|$ for some $V \in L(M)$. For an integer $\lambda \in [0, |E|]$, define $L(M)_\lambda = \{\, V \in L(M) : |E \setminus V| = \lambda \,\}$. Note that this set may be empty. Also, let $L(M)^j$ denote the collection of all flats of rank j in M. The following lemma is immediate from Proposition 3.

Lemma 4. *For integers $\lambda \in [0, |E|]$ and $i \in [-1, d]$, the multiplicity $m_{\lambda,i}$ of λ as an eigenvalue of Δ_i on $IN(M)$ is*

$$m_{\lambda,i} = \sum_{V\in L(M)_\lambda} \sum_{W\in L(M)^{i+1}} \alpha(V)\mu(W/V)\,. \quad \square$$

From this lemma, we see that λ is an eigenvalue of the total combinatorial Laplacian $\oplus_{i=-1}^{d}\Delta_i$ if and only if $\lambda = |E \setminus V|$ for some $V \in L(M)$ with $\alpha(V) \neq 0$. We will let Λ_M denote the set of all distinct eigenvalues of $\oplus_{i=-1}^{d}\Delta_i$ on $IN(M)$.

Now, we proceed to the main result of this paper. Recall that, for $\lambda \in \Lambda_M$, we define the *convolution* $\alpha \circ_\lambda \beta$ of α-invariant and β-invariant for M with respect to the given λ to be

$$\alpha \circ_\lambda \beta = \sum_{V\in L(M)_\lambda} \alpha(V)\beta(M/V)\,.$$

Theorem 5. *The simplicial tree number k_d of $IN(M)$ in the top dimension d is*

$$k_d = \prod_{\lambda\in\Lambda_M\setminus\{0\}} \lambda^{\alpha\circ_\lambda\beta}\,.$$

Proof. Since $IN(M)$ is a $\mathbb{Z}$-APC complex, we can apply Corollary 2 to obtain for each $i \in [0, d]$

$$k_i = \prod_{\lambda \in \Lambda_M \setminus \{0\}} \lambda^{a_{\lambda,i}}$$

where $a_{\lambda,i} = \sum_{j=-1}^{i} (-1)^{i-j-1}(i-j)m_{\lambda,j}$. Now, we reformulate $a_{\lambda,i}$ as follows:

$$\begin{aligned} a_{\lambda,i} &= \sum_{j=0}^{i+1} (-1)^{i-j}(i+1-j)\Big(\sum_{V \in L(M)_\lambda} \sum_{W \in L(M)^j} \alpha(V)\mu(W/V) \Big) \\ &= \sum_{V \in L(M)_\lambda} \alpha(V) \sum_{j=0}^{i+1} \sum_{W \in L(M)^j} (-1)^{i-r(V)}(i+1-j)\mu(V, W) \end{aligned}$$

where the first equality follows from Lemma 4, and the second equality from a change of summation order and $\mu(W/V) = |\mu(V, W)| = (-1)^{r(W)-r(V)}\mu(V, W)$.

Note that since $\lambda \in \Lambda_M \setminus \{0\}$, we have $V \neq \hat{1}$ for any $V \in L(M)_\lambda$. Letting $i = d$ in the above equation for $a_{\lambda,i}$, we get

$$\begin{aligned} a_{\lambda,d} &= \sum_{V \in L(M)_\lambda} \alpha(V)(-1)^{d+1-r(V)} \sum_{W \in L(M)} (r(W) - d - 1)\mu(V, W) \\ &= \sum_{V \in L(M)_\lambda} \alpha(V)(-1)^{r(M)-r(V)} \sum_{W \in L(M)} (r(W) - r(V))\mu(V, W) \\ &= \sum_{V \in L(M)_\lambda} \alpha(V)(-1)^{r(M/V)} \sum_{W/V \in L(M/V)} r(W/V)\mu(\hat{0}, W/V) \\ &= \sum_{V \in L(M)_\lambda} \alpha(V)\beta(M/V) \\ &= \alpha \circ_\lambda \beta \end{aligned}$$

where the second equality uses the fact that $\sum_{W \in L(M)} c \cdot \mu(V, W) = 0$ for any constant c and any $V \in L(M)$ with $V \neq \hat{1}$, and the third equality uses the fact that the interval $[V, \hat{1}]$ in $L(M)$ is isomorphic to $L(M/V)$. The fourth equality follows from the definition of $\beta(M)$, equation (1). ☐

4. Examples

4.1. **Standard simplexes.** In this example, we recover Kalai's formula [7, Theorem 1] for the simplicial tree numbers of a standard simplex Σ with n vertices. We regard Σ as the independent set complex of the uniform matroid $U_{n,n}$ on the ground set $[n]$. For $0 \leq i \leq n-1$, the i-skeleton $\Sigma^{(i)}$ is $IN(U_{i+1,n})$, and k_i for Σ is the top-dimensional tree number for $IN(U_{i+1,n})$.

Theorem 6. *[7, Theorem 1] Let Σ be a standard simplex with n vertices. For $0 \leq i \leq n-1$, its i-th tree number k_i equals*

$$k_i = n^{\binom{n-2}{i}}.$$

Proof. The only flat V in $L(U_{i+1,n}) \setminus \hat{1}$ with $\alpha(V) \neq 0$ is the empty flat $\emptyset$ with $\alpha(\emptyset) = 1$. Hence the only non-zero eigenvalue for the total combinatorial Laplacian on $IN(U_{i+1,n})$ is $\lambda = \mathrm{card}([n] \setminus \emptyset) = n$. By Theorem 5, we have $k_i = n^{\alpha \circ_n \beta}$ where

$$\alpha \circ_n \beta = \alpha(\emptyset)\beta(U_{i+1,n}/\emptyset) = \beta(U_{i+1,n}) = \binom{n-2}{i}.$$

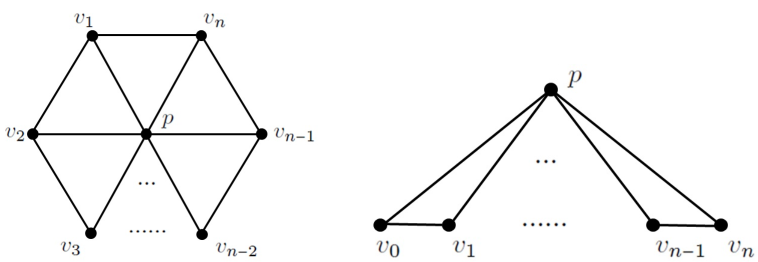
The last equality is well-known for uniform matroids [3, Proposition 7]. $\square$

4.2. **Cycle matroid complexes.** In this subsection, we assume that a finite graph G is equipped with a cycle matroid structure $M(G)$ on the ground set $E(G)$. In particular, an induced subgraph of G is regarded as a flat in $M(G)$. Also, we will write $\alpha(G) = \alpha(M(G))$ and $\beta(G) = \beta(M(G))$.

Recall that if $\tilde{G}$ is obtained from G by replacing all multiple edges by simple edges, then $\beta(G) = \beta(\tilde{G})$ (refer to [3]). Also, for a tree G, we have $\beta(G) = 1$ if G has one edge, and 0 otherwise (refer to the equation 1).

In what follows, we will compute the top-dimensional tree numbers for $IN(M(G))$ when G is a wheel or a fan. A *wheel* W_{n+1} is obtained by joining each vertex of a cycle C_n of length $n \geq 1$ to a new vertex p by a simple edge (see Fig. 1(a)). The number of edges in W_{n+1} is $2n$. Moreover, it is known that $\alpha(W_{n+1}) = 2^n - 2$ and $\beta(W_{n+1}) = n$ (refer to [3, 9]).

Let P_n be a path of length $n \geq 1$, or $P_n = v_0 v_1 \cdots v_n$. A *fan* $\hat{P}_n$ is obtained by joining each vertex of P_n to a new vertex p by a simple edge (see Fig. 1(b)). Then the number of edges in $\hat{P}_n$ is $2n + 1$. Moreover, $\alpha(\hat{P}_n) = 2^{n-1}$ and $\beta(\hat{P}_n) = 1$.



(a) a wheel W_{n+1} (b) a fan $\hat{P}_n$

Theorem 7. *The top-dimensional tree number k_{n-1} for $IN(M(W_{n+1}))$ equals*

$$(2n)^n \cdot n \cdot \prod_{k=1}^{n-2} (2n - (2k+1))^{n \cdot 2^{k-1}}.$$

Proof. In this proof, we let $G = W_{n+1}$ and let $E = E(W_{n+1})$. In order to apply Theorem 5, we will identify all induced subgraphs H of G with $\alpha(H) \neq 0$ and $\beta(G/H) \neq 0$. There are three cases to consider.

The first is the empty graph $\emptyset$, i.e., a spanning subgraph of G with no edges. In this case, we have $\alpha(\emptyset) = 1$ and $\beta(G) = n$, and the corresponding eigenvalue is $\lambda = |E| = 2n$, which explains the first factor $(2n)^n$ in the above formula.

The second is H induced by the vertex set $\{v_1, v_2, \ldots, v_n\}$. Hence $H = C_n$ and G/H is the complete graph K_2 with multiple edges. Therefore, we have $\alpha(H) = 1$ and $\beta(G/H) = \beta(K_2) = 1$. The corresponding eigenvalue is $\lambda = |E - E(C_n)| = n$, which explains the second factor n in the above formula.

The third case is $H_{i,k}$ induced by $\{p, v_i, v_{i+1}, \ldots, v_{i+k}\}$ for $i \in [1, n]$ and $k \in [1, n-2]$, where we identify v_{j+n} with v_j for all j. Note that each $H_{i,k}$ is a fan $\hat{P}_k$ and that $G/H_{i,k}$ is also a fan $\hat{P}_{n-k}$ with multiple edges. Therefore, we have $\alpha(H_{i,k}) = 2^{k-1}$ and $\beta(G/H_{i,k}) = 1$, and the corresponding eigenvalue is $\lambda = 2n - (2k+1)$, from which we obtain the last factor $\prod_{k=1}^{n-2}(2n-(2k+1))^{n \cdot 2^{k-1}}$.

One can check that any other induced subgraph H either contains an isthmus, which results in $\alpha(H) = 0$, or produces G/H which is a *tree* with multiple edges and more than 2 vertices, which results in $\beta(G/H) = 0$. $\square$

Theorem 8. *The top-dimensional tree number k_n for $IN(M(\hat{P}_n))$ equals*

$$(2n+1) \prod_{k=1}^{n-1} (2n-2k)^{2 \cdot 2^{k-1}} \prod_{k=2}^{n-2} (2k-1)^{(n-k-1) \cdot 2^{n-k-2}}.$$

Proof. Similar to the proof of the previous theorem. Details will be omitted. $\square$

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