Path intersection matrices and applications to networks

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Abstract

For a network G, we introduce a non-singular symmetric matrix, called a *path intersection matrix*, that will provide a new method for computing the ratio $k(G)/k(G/ab)$ where $k(G)$ is the tree-number of G and G/ab is obtained from $G \cup ab$ by contracting the new edge ab between two distinct nodes a and b. The quantities $k(G)/k(G/ab)$ appear as invariants for various networks such as effective conductance for an electrical network and an ingredient for information centrality for a social network. We will review several examples of networks where path intersection matrices can be applied.

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1. Introduction

Tree enumeration in graphs often relies on matrices. For example, the matrix-tree theorem states that every cofactor of the Laplacian matrix L of a finite graph G is the tree number $k(G)$, *i.e.*, the number of spanning trees. The tree numbers, in turn, may provide "combinatorial" interpretations of various network invariants. It is well-known that the effective conductance between two vertices a and b equals $k(G)/k(G/ab)$, where the *contraction* G/ab is obtained from G by adding a new edge ab between a and b, and contracting it. We refer the readers to [2] and [14] for derivations of this formula from the Laplacian matrix L and Kirchhoff's laws.

The purpose of this paper is to present a new matrix called a *path intersection matrix* D^{ab} for two distinct vertices a and b in a finite graph G that provides a new method for computing the ratio $k(G)/k(G/ab)$. We remark that this ratio also equals the *information* between two nodes a and b in a network proposed by Stephenson and Zelen [13]. Refer to [7] for a derivation of this fact using the combinatorial Laplacian $L + J$ where J is an all 1's matrix.

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The ingredient for constructing a path intersection matrix D^{ab} for a finite graph G is a minimal collection of paths from a to b whose pairwise differences span the cycle space of G. We will discuss motivations and details of this construction in section 4. While one has choices for paths in constructing D^{ab}, the main results of the paper will show that the determinant of D^{ab} equals $k(G/ab)$, and that the sum of all entries of its inverse matrix equals $k(G)/k(G/ab)$, which depend only on the vertices a and b.

In deriving $k(G)/k(G/ab)$, our method differs from other works in the way $k(G/ab)$ is counted. In the previous works mentioned above, $k(G/ab)$ is realized exactly as the number of spanning trees in the contraction G/ab, whereas $\det D^{ab}$ computes $k(G/ab)$ as the number of the spanning trees in $G \cup ab$ containing the edge ab. In particular, our method does not require edge-contraction, maintaining the original structure of graphs.

The paper is organized as follows. Section 2 reviews tree enumeration in a graph via a basis of its cycle space. Section 3 introduces a new method for tree enumeration in a contraction of a graph *without* edge contraction, and demonstrates its advantages through classical examples. Section 4 defines the main object of study, the path intersection matrix of a graph G, and suggests the sum of all entries in its inverse matrix as the main invariant for our purpose. Section 5 presents the main result of the paper proving a combinatorial interpretation of the main invariant as the ratio $k(G)/k(G/ab)$, which requires a careful analysis of minors of a path intersection matrix. Section 6 discusses some well-known network invariants involving this quantity.

2. Preliminaries

2.1. Spanning trees of a graph

We refer the readers to [3] for basic definitions concerning graphs. In this paper, we assume that a graph G is connected with multiple edges allowed. Also we assume that its vertex set $V(G)$ has n elements, and its edge set $E(G)$ is a multiset, having m elements.

A *spanning tree* T in a connected graph G is a spanning subgraph which is connected and has no cycle. One can show that every spanning tree has $n-1$ edges, and we call $n-1$ the *rank* and $c = m-(n-1)$ the *corank* of G. We will denote the collection of all spanning trees in G by $\mathcal{T}(G)$ and we will call the number of spanning trees in G the *tree-number* of G, denoted by $k(G)$.

The *deletion-contraction recurrence* is an important property for $k(G)$: if $e \in E(G)$ is not a loop nor an isthmus (refer to [15]),

$$k(G) = k(G \setminus e) + k(G/e)$$

where $G \setminus e$ is the subgraph of G obtained by deleting e and G/e is obtained by contracting e. For our purpose, it is important to note that *the number of spanning trees in G containing the given edge e equals the number of spanning trees in G/e.*

2.2. Cycle space and $k(G)$

Let G be a connected graph. We will assume that each edge $e = vv'$ of G is assigned an orientation, and $[e] = [vv']$ denotes the orientation of e that originates from vertex v and terminates at vertex v'. Let $C_1 = C_1(G) = \mathbb{Z}^m$ be a free abelian group generated by the oriented edges $\{[e] \mid e \in E(G)\}$, and $C_0 = C_0(G) = \mathbb{Z}^n$ generated by the set $\{[v] \mid v \in V(G)\}$ where $[v]$ denotes the unique *oriented* vertex for each $v \in V(G)$. An element $x \in C_1$ may be represented as a column vector $x = (n_e)_{e \in E(G)}$ or as a *formal* sum $x = \sum_{e \in E(G)} n_e[e]$ with $n_e \in \mathbb{Z}$ for all $e \in E(G)$, depending on the context. The elements of C_0 will be represented similarly.

The incidence matrix $\partial_1 = \partial_1(G) : C_1 \to C_0$ is defined by $\partial_1([vv']) = [v'] - [v]$ for each oriented edge $[vv'] \in C_1$. Hence, ∂_1 is an n-by-m matrix whose rows and columns are indexed by $V(G)$ and $E(G)$, respectively, and its $([v],[e])$-entry equals 1 or -1 if v is the terminal or originating vertex of e, respectively, and equals 0 if v is neither. The *cycle space* of G, denoted by $H_1(G)$, is the kernel of ∂_1. It is well-known that the rank of $H_1(G)$ as a free abelian group equals the corank c of G.

Now, we define ∂_2 for G to be an m-by-c *integer* matrix, given by

$$\partial_2 = \begin{bmatrix} z_1 & z_2 & \dots & z_c \end{bmatrix}$$

where the columns $z_1, \dots, z_c$ form a basis for the cycle space $H_1(G)$. We may regard ∂_2 as a map $C_2 \to C_1$ where $C_2 = \mathbb{Z}^c$ with a standard basis. When G is planar, for example, the z_i's may be given by the boundary cycles of the finite faces of G.

Although ∂_2 depends on the choice of a basis of $H_1(G)$, the following proposition shows an important property of ∂_2 that is independent of that choice. For $S \subset E(G)$, let $(\partial_2)_S$ denote the submatrix of ∂_2 whose rows are indexed by S. Also, let $\overline{S} = E(G) \setminus S$.

Proposition 2.1. *Let S be a subset of $E(G)$ with $|S| = c =$ corank of G. If $\overline{S}$ forms a spanning tree in G, then $|\det(\partial_2)_S| = 1$. Otherwise, it is equal to* 0.

Proof. Suppose $\partial_2 = [z_1 \cdots z_c]$ is given where z_i's form a basis for $H_1(G)$. Let S be a subset of $E(G)$ such that $\overline{S}$ forms a spanning tree in G. For each $e \in S$, there exist coefficients $a_{1,e}, a_{2,e}, \dots, a_{c,e} \in \mathbb{Z}$ such that $\sum_{i=1}^{c} a_{i,e} z_i = z_e$ where $z_e \in H_1(G)$ corresponds to the unique cycle in $\overline{S} \cup e$. Writing these equations in a matrix form and restricting it to the rows indexed by S, we obtain a matrix equation $I' = (\partial_2)_S A$ where $A = (a_{i,e})$ is the *coefficient* matrix and I' is a permutation matrix up to sign. Hence, we have $|\det(\partial_2)_S| = 1$. If $\overline{S}$ does not form a spanning tree, then $\overline{S}$ contains a cycle, *i.e.*, there is $z \in H_1(G)$ *supported* by $\overline{S}$. From this fact, one can easily deduce that $(\partial_2)_S$ is singular. □

The following theorem, which is essential for our purpose, is a well-known result from linear algebra. Refer to any standard text for a proof (for example, [6]).

Theorem 2.2 (Cauchy-Binet formula)**.** *Let A and B be s-by-t matrices with $s \geq t$ whose rows are indexed by a set R with $|R| = s$. For a t-set $I \subset R$, let A_I (resp. B_I) be a submatrix of A (resp. B) whose rows are indexed by I. Then we have*

$$\det A^t B = \sum_{I \subset R,\, |I| = t} \det A_I \det B_I.$$

Recall that the matrix-tree theorem from graph theory states that for a graph G, every cofactor of its Laplacian matrix $L(G) = \partial_1\partial_1^t$ equals $k(G)$. The following is a cycle-space version of this fact, and its proof is immediate from Proposition 2.1 and Theorem 2.2.

Theorem 2.3. *The number of spanning trees in G is*

$$k(G) = \det(\partial_2^t\partial_2).$$

Remark 2.4. The proofs of the matrix-tree theorem and Theorem 2.3 are similar in that both are given via the Cauchy-Binet Theorem. However, they differ in the ways the spanning trees of G are accounted for in the matrices ∂_1 and ∂_2. Indeed, $(\partial_1^t)_S$ with $|S| = n-1$ is a full-rank submatrix of ∂_1^t iff S forms a spanning tree whereas $(\partial_2)_S$ with $|S| = c$ is a full-rank submatrix of ∂_2 iff the *complement* of S in $E(G)$ forms a spanning tree.

Example 2.5. To illustrate Theorem 2.3, we compute $k(G)$ where G is the graph in Figure 1. Note that $z_1 = [12]+[23]+[34]-[14]$ and $z_2 = [34]-[14]+[15]-[25]+[23]$ form a basis of $H_1(G)$. Using this basis, we have

$$\partial_2 = \begin{array}{c} \\ {[12]} \\ {[14]} \\ {[15]} \\ {[23]} \\ {[25]} \\ {[34]} \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \end{array} \quad \text{and} \quad \partial_2^t\partial_2 = \begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}.$$

Therefore, Theorem 2.3 gives $k(G) = \det(\partial_2^t\partial_2) = 11$.

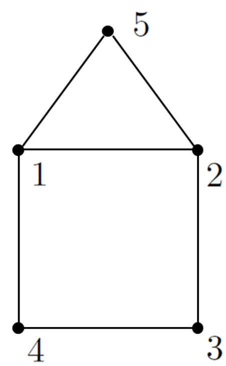

Figure 1: A graph G with $c = 2$

Example 2.6. Let P_n be a path of length $n \geq 1$, or $P_n = v_0v_1\cdots v_n$. A *fan* $\hat{P}_n$ is obtained by joining each vertex of P_n to a new vertex w by a simple edge (see Figure 2). If we define

∂_2 using cycles described in Figure 2, then $\partial_2^t \partial_2 = T_n$ where T_n is the $n \times n$ tridigonal matrix whose main diagonals are 3 and the first diagonals below and above these are -1. It follows $\det T_n = 3 \det T_{n-1} - \det T_{n-2}$ for $n \geq 2$ with $\det T_0 := 1$ and $\det T_1 = 3$, which are the same recurrence relations satisfied by the Fibonacci numbers $\{F_{2n+2}\}$ with $F_2 = 1$ and $F_4 = 3$. Hence we have $k(\hat{P}_n) = \det T_n = F_{2n+2}$.

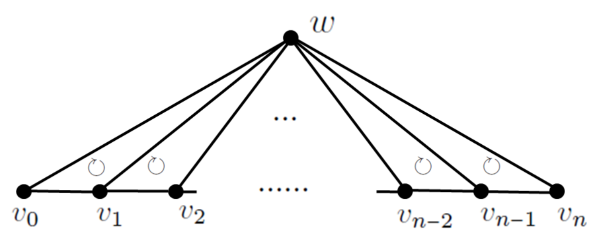

Figure 2: A fan $\hat{P}_n$

2.3. Weighted graphs

A weighted graph G is also called a *network*. For each $e \in E(G)$ in a weighted graph G, we assume that its weight w_e is postive. For a spanning tree T in G, we define its weight to be

$$w_T = \prod_{e \in E(T)} w_e \, .$$

Then the weighted tree-number $\hat{k}(G)$ of G is defined as

$$\hat{k}(G) = \sum_{T \in \mathcal{T}(G)} w_T \, .$$

3. Tree-numbers of contraction graphs

In this section, we will give a new method for computing the tree-number of a contraction graph as a determinant. Given a connected graph G and $a, b \in V(G)$ with $a \neq b$, a *path* p from a to b is an element of $C_1(G)$ such that

$$\partial_1 p = [b] - [a].$$

Hence, for our purpose, a *path* p means an element of C_1 which is a formal sum of oriented edges satisfying the above condition, and *need not* correspond to a graph-theoretic path in G. Also, a path is defined between two distinct vertices only.

Let c be the corank of G. Given a basis $\{z_1, \ldots, z_c\}$ for $H_1(G)$ and a path p from a to b, define ∂^{ab} to be an $m \times (c+1)$ matrix obtained by adding p to ∂_2 as a column, *i.e.*,

$$\partial^{ab} = \begin{bmatrix} p & z_1 & \ldots & z_c \end{bmatrix} \, .$$

Now, given $a, b \in V(G)$ with $a \neq b$, let G^{ab} denote the graph obtained from G by adding a new edge ab oriented from b to a. Hence, $\partial_1(G^{ab})$ is an n-by-$(m+1)$ matrix obtained from $\partial_1(G)$ by adding a new column that will be indexed by $[ba]$. In particular, $\partial_1(G^{ab})[ba] = [a] - [b]$, and the restriction of $\partial_1(G^{ab})$ to $C_1(G)$ equals $\partial_1(G)$.

Proposition 3.1. *The tree-number of the contraction G/ab is given by*

$$k(G/ab) = \det\left((\partial^{ab})^t \partial^{ab}\right).$$

In particular, $\det\left((\partial^{ab})^t \partial^{ab}\right)$ *depends only on the vertices a and b, and is independent of the choices of a path from a to b and a basis for $H_1(G)$.*

Proof. Assume that the corank of G is c. Then, consequently, the corank of G^{ab} is $c+1$. We claim that the columns of the following $(m+1)$-by-$(c+1)$ matrix form a basis for $H_1(G^{ab})$:

$$\begin{bmatrix} 1\ 0 \cdots 0 \\ \partial^{ab} \\ \end{bmatrix} = \begin{bmatrix} \tilde{z}_0 & \tilde{z}_1 & \tilde{z}_2 & \dots & \tilde{z}_c \end{bmatrix}$$

where the first row is indexed by the new oriented edge $[ba]$ and the rest by the oriented edges in G. As formal sums, we have $\tilde{z}_0 = [ba] + p$ and $\tilde{z}_i = z_i$ for $1 \leq i \leq c$.

For the proof of the claim, we will write ∂_1' for $\partial_1(G^{ab})$ and ∂_1 for $\partial_1(G)$. We have $\tilde{z}_1, \dots, \tilde{z}_c \in H_1(G^{ab})$ because $\partial_1' \tilde{z}_i = \partial_1 z_i = 0$ for $1 \leq i \leq c$. Also we have $\tilde{z}_0 \in H_1(G^{ab})$ because $\partial_1' \tilde{z}_0 = \partial_1'[ba] + \partial_1' p = [a] - [b] + \partial_1 p = 0$. Since $z_1, \dots, z_c$ are linearly independent, being a basis of $H_1(G)$, it follows that $\tilde{z}_0, \dots, \tilde{z}_c$ are also linearly independent. Hence, it remains to show that $\tilde{z}_i$'s span $H_1(G^{ab})$. Indeed, let $\tilde{z} \in H_1(G^{ab})$, and write $\tilde{z} = r[ba] + x$ for some integer r and some $x \in C_1(G)$. Clearly, $w := \tilde{z} - r\tilde{z}_0 \in H_1(G^{ab})$. Since $w = x - rp \in C_1(G)$, we have $0 = \partial_1' w = \partial_1 w$, which shows $w \in H_1(G)$. Hence, w is a linear combination of $z_1, \dots, z_c$. We conclude that $\tilde{z}$ is a linear combination of $\tilde{z}_0, \dots, \tilde{z}_c$ which proves the claim.

Hence, the above matrix can be taken as ∂_2 for G^{ab}. In what follows, a matrix with a subscript S means its full submatrix whose rows are indexed by a set S. Then, we have

$$\begin{aligned} \det\left((\partial^{ab})^t \partial^{ab}\right) &= \sum_{S \subset E(G),\, |S| = c+1} [\det(\partial^{ab})_S]^2 \\ &= \sum_{S \subset E(G^{ab}),\, |S| = c+1,\, ab \notin S} [\det(\partial_2)_S]^2 \\ &= k(G/ab)\,. \end{aligned}$$

The first equality uses the Cauchy-Binet formula (Theorem 2.2), and the second equality comes from the definition of ∂_2. For the last equality, note that the subsets $S \subset E(G^{ab})$ with $|S| = c+1$ and $ab \notin S$ such that $E(G^{ab}) \setminus S$ forms a spanning tree in G^{ab} correspond bijectively to the spanning trees in G^{ab}/ab. Now, the result follows from $k(G^{ab}/ab) = k(G/ab)$ and Proposition 2.1. The second statement of the proposition is clear. $\square$

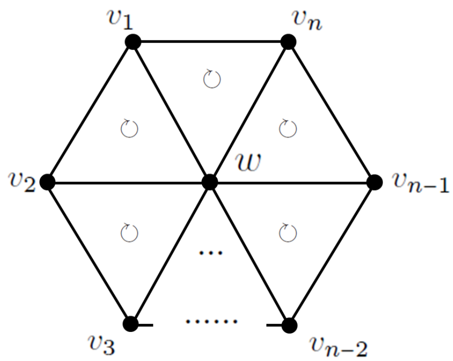

Figure 3: A wheel W_n

Example 3.2. A *wheel* W_n is a graph obtained by connecting each vertex of a cycle C_n of length $n \geq 1$ with a new vertex w by a simple edge (see Figure 3). In this example, we will compute $k(W_n/ab)$ for $a = w$ and $b = v_i$ with $1 \leq i \leq n$. Choose the oriented edge $p = [ab]$ as a path from a to b, and let $\{z_1, \ldots, z_c\}$ be the basis for $H_1(W_n)$ described in Figure 3. The matrix $(\partial^{ab})^t \partial^{ab}$ for these choices is an $(n+1)$-by-$(n+1)$ matrix

$$\begin{bmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 & 1 \\ -1 & 3 & -1 & 0 & \cdots & 0 & -1 \\ 0 & -1 & 3 & \cdots & \ddots & \vdots & 0 \\ 0 & 0 & -1 & \ddots & \ddots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 3 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 1 & -1 & 0 & \cdots & 0 & -1 & 3 \end{bmatrix}.$$

By elementary row and column operations, we have $\det(\partial^{ab})^t \partial^{ab} = \det(T_n - R_n)$ where R_n is the n-by-n matrix whose $(1,1)$-entry and (n,n)-entry are 1 and other entries are 0. By the multilinearity of determinant, we have

$$\begin{aligned} k(W_n/ab) &= \det(T_n) - 2\det(T_{n-1}) + \det(T_{n-2}) \\ &= F_{2n+2} - 2F_{2n} + F_{2n-2} = F_{2n}. \end{aligned}$$

Note that this result can be also obtained from the fact that $T_n - R_n$ is a reduced Laplacian of a fan $\hat{P}_{n-1}$. One can also compute $k(W_n/ab)$ by noting that T_n is a reduced Laplacian of the contraction W_n/ab [9]. Our method is different from these methods in that graph contractions or row and column deletions of a matrix are not required.

Example 3.3. In this example, we will rederive the formula

$$k(W_n) = L_{2n} - 2$$

where L_{2n} is the $2n$-th Lucas number (refer to [11, 10, 9, 4]). Recall that the Lucas numbers L_n are defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Expanding $\det(\partial^{ab})^t\partial^{ab}$ for W_n along the first row gives

$$k(W_n/ab) = k(W_n) - 2(k(\hat{P}_{n-1}) - k(\hat{P}_{n-2})) + 2$$

Then from Example 2.6 and 3.2, we have

$$\begin{aligned} k(W_n) &= F_{2n} + 2(F_{2n} - F_{2n-2}) - 2 \\ &= F_{2n} + 2F_{2n-1} - 2 = L_{2n} - 2. \end{aligned}$$

4. Path intersection matrix

In this section, we define a path intersection matrix for connected graphs, which is one of the main objects of study in this paper, and discuss its intriguing properties. In [13], an example of this matrix was presented to illustrate total information contained in paths, which motivated our current work.

4.1. Path intersection matrix of a graph

Let c be the corank of G. Given $a, b \in V(G)$ with $a \neq b$, a collection $P = \{p_0, \ldots, p_c\}$ of $c+1$ distinct paths each from a to b will be called *saturated* if the set of cycles $\{p_i - p_j \mid i \neq j\}$ spans $H_1(G)$. Equivalently, P is saturated if p_0 is a path from a to b and $p_i = p_0 + z_i$ for some basis $\{z_1, \ldots, z_c\}$ for $H_1(G)$. For a saturated P, let ∂^P be the m-by-$(c+1)$ matrix whose columns are p_i's:

$$\partial^P = \begin{bmatrix} p_0 & p_1 & \ldots & p_c \end{bmatrix}.$$

Definition 4.1. The *path intersection matrix* D^{ab} induced by a saturated P is the symmetric matrix of order $c+1$ given by

$$D^{ab} = (\partial^P)^t\partial^P.$$

The terminology *path intersection matrix* is motivated by the following observation: If all entries of each $p_i \in P$ are ± 1 or 0, a diagonal entry in D^{ab} is the *length* of the corresponding path, and an off-diagonal entry is the "net intersection", *i.e.*, the number of common edges with the same orientations minus that with the opposite orientations in the corresponding pair of paths.

There is an intriguing *invariant* I_{ab} associated with D^{ab} defined by

$$I_{ab} = \text{the sum of all entries in the inverse of } D^{ab}.$$

Theorem 5.1 will give a combinatorial interpretation of I_{ab}, and show that I_{ab} depends only on the vertices a and b although D^{ab} depends on the paths in its definition.

4.2. Examples

Example 4.2. (as appeared in [13]) Let G be the graph in Figure 1 with the corank $c = 2$. Let C_1 be the free abelian group generated by the oriented edges $\{[12], [14], [15], [23], [25], [34]\}$. To compute I_{13}, let $P = \{p_0, p_1, p_2\} \subset C_1$ where $p_0 = [14] - [34]$, $p_1 = [12] + [23]$, and $p_2 = [15] - [25] + [23]$. Then, it checks easily that P is saturated, and we have

$$\partial^P = \begin{array}{c} \\ [12] \\ [14] \\ [15] \\ [23] \\ [25] \\ [34] \end{array} \begin{array}{c} \begin{array}{ccc} p_0 & p_1 & p_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad D^{13} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

The inverse of D^{13} equals $\frac{1}{10}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & -2 \\ 0 & -2 & 4 \end{bmatrix}$ and $I_{13} = \frac{11}{10}$.

Example 4.3. Let H_3 be the 3-cube graph (Figure 4), and let $ab \in E(H_3)$. Suppose that a basis $\{z_1, z_2, z_3, z_4, z_5\}$ of $H_1(H_3)$ is given as in Figure 4. If we let P be the collection of paths $p_0 := [ab]$ and $p_i := p_0 + z_i$ for $i = 1, 2, \ldots, 5$, then D^{ab} is given by

$$D^{ab} = \begin{array}{c} \\ p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{array} \begin{array}{c} \begin{array}{cccccc} p_0 & p_1 & p_2 & p_3 & p_4 & p_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 3 & -1 & -1 & -1 & 0 \\ 1 & -1 & 5 & 0 & 1 & 1 \\ 1 & -1 & 0 & 5 & 0 & 2 \\ 1 & -1 & 1 & 0 & 5 & 1 \\ 2 & 0 & 1 & 2 & 1 & 7 \end{bmatrix} \end{array},$$

and one can compute $I_{ab} = 384/224$.

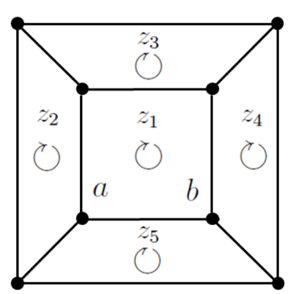

Figure 4: the 3-cube graph H_3

Example 4.4. Let $P_n^{(2)}$ be the path graph of length n each of whose edges has multiplicity 2 (Figure 5). To compute I_{0n}, let P be the collection of paths $p_0 := [e_1] + [e_3] + \cdots + [e_{2n-1}]$ and $p_i := p_0 + [e_{2i}] - [e_{2i-1}]$ for $i = 1, 2, \ldots, n$. Then D^{0n} is the $(n+1)$-by-$(n+1)$ matrix,

$$D^{0n} = \begin{bmatrix} n & n-1 & \cdots & \cdots & n-1 \\ n-1 & n & n-2 & \cdots & n-2 \\ \vdots & n-2 & n & \ddots & \vdots \\ n-1 & \vdots & \ddots & \ddots & n-2 \\ n-1 & n-2 & \cdots & n-2 & n \end{bmatrix}, \text{ and } (D^{0n})^{-1} = \begin{bmatrix} a & b & \cdots & \cdots & b \\ b & c & d & \cdots & d \\ \vdots & d & c & \ddots & \vdots \\ b & \vdots & \ddots & \ddots & d \\ b & d & \cdots & d & c \end{bmatrix},$$

where $a = (n^2 - 2n + 2)/n, b = -(n-1)/n, c = (n+1)/2n, d = 1/2n$. Then we obtain $I_{0n} = a + 2n \cdot b + n \cdot c + (n^2 - n)d = 2/n$.

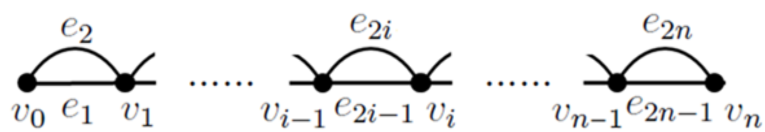

Figure 5: the path $P_n^{(2)}$ of length n each of whose edges has multiplicity 2

We will recover the computations in these three examples by using Theorem 5.1. We end this section with a lemma showing that $\det D^{ab}$ depends only on the vertices $a, b \in V(G)$.

Lemma 4.5. $\det D^{ab} = k(G/ab)$.

Proof. Note that ∂^{ab} is a matrix obtained from ∂^P by subtracting its first column from the other columns, *i.e.*, $\partial^{ab} = \partial^P A$ where A is a $(c+1)$-by-$(c+1)$ matrix such that its entries in the main diagonal are 1, its off-diagonals in the first row are -1, and the others are 0. Since $D^{ab} = (\partial^{ab} A^{-1})^t (\partial^{ab} A^{-1})$, Proposition 3.1 proves the lemma. □

5. Combinatorial interpretation of I_{ab}

5.1. The main result

Recall that I_{ab} is defined to be the sum of all entries in the inverse matrix of $D^{ab} = (\partial^P)^t \partial^P$ where P is a saturated collection of paths from a to b. The following theorem is the main result of the paper demonstrating, in particular, that I_{ab} depends only on the choices of $a, b \in V(G)$ with $a \neq b$.

Theorem 5.1. *For two distinct vertices a and b in a connected graph G,*

$$I_{ab} = \frac{k(G)}{k(G/ab)}.$$

Proof. In this proof, we will abbreviate D^{ab} as D, and denote its (i,j)-minor by D_{ij}. Let $\partial^P(i)$ denote the matrix obtained from ∂^P by removing its i-th column. Then we have $D_{ij} = \det\left((\partial^P(i))^t \partial^P(j)\right)$. Now, the formula of an inverse matrix gives

$$I_{ab} = \frac{1}{\det D} \sum_{1\le i,j\le c+1} (-1)^{i+j} \det\left((\partial^P(i))^t \partial^P(j)\right),$$

where c is the corank of G. By Lemma 4.5, it suffices to show

$$k(G) = \sum_{1\le i,j\le c+1} (-1)^{i+j} \det\left((\partial^P(i))^t \partial^P(j)\right).$$

This identity will be shown by the following equalities. As before, a matrix with a subscript S means its full submatrix whose rows are indexed by a set S.

$$\begin{aligned}
& \sum_{1\le i,j\le c+1} (-1)^{i+j} \det\left((\partial^P(i))^t \partial^P(j)\right) \\
= & \sum_{1\le i,j\le c+1} (-1)^{i+j} \sum_{S\subset E(G),\,|S|=c} \det \partial^P(i)_S \det \partial^P(j)_S \\
= & \sum_{S\subset E(G),\,|S|=c} \sum_{1\le i,j\le c+1} (-1)^i \det \partial^P(i)_S (-1)^j \det \partial^P(j)_S \\
= & \sum_{S\subset E(G),\,|S|=c} \Big[\sum_{1\le i\le c+1} (-1)^i \det \partial^P(i)_S \Big]^2 \\
= & \sum_{S\subset E(G),\,|S|=c} \left[\det \begin{bmatrix} 1 \cdots 1 \\ (\partial^P)_S \end{bmatrix} \right]^2 \\
= & \sum_{S\subset E(G^{ab}),\,|S|=c+1,\,ab\in S} \Big[\det(\partial_2)_S \Big]^2 \\
= & \; k(G).
\end{aligned}$$

The first equality uses Cauchy-Binet formula (Theorem 2.2). The second equality follows from changing the order of summation. The fifth equality follows from the definition of ∂_2 for G^{ab}. For the last equality, note that for $S \subset E(G^{ab})$ with $|S| = c+1$ and $ab \in S$, a subgraph defined by $E(G^{ab}) \setminus S$ is a spanning tree in G iff $|\det(\partial_2)_S| = 1$ by Proposition 2.1. ☐

Example 5.2. Let G be the graph in Example 4.2. One can compute $k(G) = 11$ and $k(G/13) = 10$. By the above theorem, $I_{13} = 11/10$, matching Example 4.2.

Example 5.3. Let H_3 be the 3-cube in Example 4.3. Note that $k(H_3) = 384$ (refer to [12]), and using the symmetry of H_3 we find $k(H_3/ab) = (7/12)\cdot 384 = 224$. By the above theorem, $I_{ab} = 384/224$, matching Example 4.3.

Example 5.4. Let $P_n^{(2)}$ be the path graph of length n with 2 multiple edges in Example 4.4. From $k(P_n^{(2)}) = 2^n$, $k(P_n^{(2)}/0n) = n \cdot 2^{n-1}$, we have $I_{0n} = 2/n$, matching Example 4.4.

5.2. Weighted analogue

Suppose that each $e \in E(G)$ is assigned a positive weight w_e. Let $\sqrt{W}$ be a diagonal matrix whose columns are indexed by the same manner as the rows of ∂_2 for G and whose diagonal entry corresponding to $e \in E(G)$ is $\sqrt{w_e}$. Define the second weighted boundary operator $\hat{\partial}_2$ to be $(\sqrt{W})^{-1}\partial_2$. Then the weighted analogue of Proposition 2.1 is as follows. For $S \subset E(G)$, let $(\hat{\partial}_2)_S$ be the submatrix of $\hat{\partial}_2$ whose rows are indexed by S.

Proposition 5.5. *Let $S \subset E(G)$ with $|S| =$ corank of G. If $E(G) \setminus S$ forms a spanning tree in G, then $|\det(\hat{\partial}_2)_S| = 1/\prod_{e\in S}\sqrt{w_e}$. Otherwise, it equals 0.*

Proof. The proof follows from Proposition 2.1 and the construction of $\hat{\partial}_2$. □

Now, we define the weighted analogue of ∂^P to be $\widehat{\partial^P} = (\sqrt{W})^{-1}\partial^P$. Then the corresponding weighted path intersection matrix is defined to be $\hat{D}^{ab} = (\widehat{\partial^P})^t\widehat{\partial^P} = (\partial^P)^t W^{-1}\partial^P$. For two distinct vertices $a, b \in V(G)$, let

$$I_{ab} = \text{ the sum of all entries in the inverse of } \hat{D}^{ab}.$$

The following theorem is the weighted analogue of Theorem 5.1. Denote by $\hat{k}(G)_{ab}$ the weighted tree-number of G^{ab} containing the edge ab with $w_{ab} := 1$, *i.e.*,

$$\hat{k}(G)_{ab} = \sum_{T\in\mathcal{T}(G^{ab}),\, ab\in E(T)} w_T.$$

Theorem 5.6. *For distinct vertices $a, b \in V(G)$ in a weighted graph G,*

$$I_{ab} = \frac{\hat{k}(G)}{\hat{k}(G)_{ab}}.$$

Proof. We use a similar method used in Theorem 5.1, replacing Proposition 2.1 with Proposition 5.5. Then we have

$$I_{ab} = \frac{\sum_{\substack{T\in\mathcal{T}(G^{ab}),\\ ab\notin E(T)}} \prod_{e\notin E(T)} 1/w_e}{\sum_{\substack{T\in\mathcal{T}(G^{ab}),\\ ab\in E(T)}} \prod_{e\notin E(T)} 1/w_e}.$$

Multiplying both the numerator and the denominator by $\prod_{e\in E(G^{ab})} w_e$, the result follows. □

Example 5.7. Regarding $P_n^{(2)}$ as the path graph of length n whose edges have 2 as weights, we will compute I_{0n}. Note that $\hat{D}^{0n}$ is just a 1-by-1 matrix whose entry is $n/2$. Hence, $I_{0n} = 2/n$, matching Examples 4.4 and 5.4.

6. Examples

6.1. Effective resistance

Consider an electrical network G where each edge is weighted by its conductance. Let L be the Laplacian matrix of G. Let $V = (\phi_v)^t$ and $I = (\iota_v)^t$ ($v \in V(G)$) be *voltage* and *current* vectors, respectively, satisfying $\alpha = \iota_a = -\iota_b$ and $\iota_v = 0$ for $v \neq a, b$. Solving for V in the equation $LV = I$, the *effective resistance* R_{ab} between a and b is defined to be $(\phi_b - \phi_a)/\alpha$. Using Kirchhoff's laws and Ohm's law, Thomassen [14] showed that

$$R_{ab} = \frac{\hat{k}(G/ab)}{\hat{k}(G)} = I_{ab}^{-1}.$$

For a derivation of this formula using a combinatorial Laplacian, refer to [7]. The combinatorial interpretation of I_{ab} gives a combinatorial proof of Foster's Theorem on electrical networks [5]:

Theorem 6.1. *[5, Foster's Theorem] For a weighted graph G with n vertices, the following identity holds.*

$$\sum_{ab \in E(G)} w_{ab} R_{ab} = n - 1.$$

Proof. Let $\mathcal{F} = \{(T, ab) \mid T \in \mathcal{T}(G) \text{ and } ab \in E(T)\}$. Then, we get

$$\hat{k}(G)(n-1) = \sum_{(T,ab) \in \mathcal{F}} w_T = \sum_{ab \in E(G)} w_{ab}\, \hat{k}(G/ab).$$

This together with Theorem 5.6 yields the desired identity. □

For the unweighted complete graph K_n, the effective resistance R_{ab} is constant for every edge ab by symmetry. Since $|E(K_n)| = \binom{n}{2}$, Theorem 6.1 implies $R_{ab} = (n-1)/\binom{n}{2} = 2/n$ for each pair $a, b \in V(K_n)$ with $a \neq b$.

6.2. Information centrality

Stephenson and Zelen [13] introduced the *information centrality* as follows. Let L be the Laplacian matrix of G, J the all 1's matrix, and $(L + J)^{-1} = (g_{ab})$. Based on the theory of statistical estimation, they defined the information between two nodes a and b to be

$$(g_{aa} + g_{bb} - 2g_{ab})^{-1} \qquad (1)$$

Kook [7] gave a combinatorial interpretation of the right-hand side of (1), *i.e.*,

$$(g_{aa} + g_{bb} - 2g_{ab})^{-1} = \frac{\hat{k}(G)}{\hat{k}(G/ab)} = I_{ab}.$$

The *information centrality* I_a for a node a is defined to be the harmonic mean of the information between a and other vertices [13], *i.e.*,

$$I_a = n \cdot \Big[\sum_{b \in V(G) \setminus a} \frac{1}{I_{ab}} \Big]^{-1}.$$

We end this paper with a question. Can a path intersection matrix D^{ab} be obtained from properties or laws governing electrical networks? This question was motivated by the fact that the Laplacian matrix L of a network is a direct consequence of Kirchhoff's laws.

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