Harmonic cycles for graphs

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ABSTRACT
Given a finite connected graph $G = (V(G), E(G))$ and a basis ∂ for a hyperplane in the cycle space of G, define $\lambda = \sum_Y \det(C_Y, \partial) \cdot C_Y$ summing over all connected spanning subgraphs Y of G such that $|E(Y)| = |V(G)|$ with C_Y denoting the unique cycle in Y. We will show that λ is an element of the harmonic space $\ker(\partial_1^t \partial_1 + \partial\partial^t)$ where ∂_1 is the incidence matrix of G by establishing an inner product formula $\lambda \circ z = \det(z, \partial)k(G)$ for the cycles z and the tree number $k(G)$ of G. Several examples and applications of these formulas will be given.

KEYWORDS
harmonic cycle, cycletree, winding number, tree number, funtional

1. Introduction

Let $G = (V(G), E(G))$ be a finite connected graph and ∂_1 its incidence matrix. A *second incidence matrix* for G is an integer matrix ∂_2 such that $\partial_1\partial_2 = 0$. For a pair $\mathscr{G} = (G, \partial_2)$, we define its *harmonic space* to be $\mathcal{H}(\mathscr{G}) = \ker(\partial_1^t\partial_1 + \partial_2\partial_2^t)$. An element of $\mathcal{H}(\mathscr{G})$ is called a *harmonic cycle*, which is the main object of study in this paper. (See section 3.) Refer to [4] for a concise discussion of harmonic spaces of a chain complex, and [3,8] for previous studies related to harmonic cycles.

The purpose of this paper is to present a formula for a harmonic cycle, which is given as a sum of *doubly* weighted cycles in G. To describe the weights, we will introduce *cycletrees* of G and a *winding number* for the cycles in G. (See sections 2 and 4.) A cycletree in G is a connected spanning subgraph containing a unique cycle. A cycletree can be understood as a union of a spanning tree in G and an external edge. To define the winding number, assume that the columns of ∂_2 form a basis of a hyperplane, i.e., a codimension 1 subspace, of the cycle space Z_1 of G. This condition is equivalent to $\operatorname{rk}\mathcal{H}(\mathscr{G}) = 1$ as we shall see. The winding number of $z \in Z_1$ is defined by $w(z) = \det(z, \partial_2)$. Now let λ be an element of Z_1 given by

$$\lambda = \sum_Y w(C_Y) \cdot C_Y$$

summing over all cycletrees Y in G with C_Y denoting the unique cycle in Y. (See

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section 5.) Note that in this formula each cycle C in G is doubly weighted by its winding number and the number of cycletrees it belongs to.

We will prove that λ is a harmonic cycle, i.e., an element of $\mathcal{H}(\mathscr{G})$. The proof is based another intriguing formula for λ as a functional on Z_1. Let the first chain group of G be equipped with an inner product $\circ$ where the oriented edges form an orthonormal basis. Let $k(G)$ denote the number of spanning trees in G. For $z \in Z_1$, we will show

$$\lambda \circ z = w(z)k(G)$$

from which one can deduce that λ is a harmonic cycle. (See section 6.) The functional defined by λ is called a *combinatorial harmonic functional*, or a *matrix-tree functional.*

Several examples are given in the paper to illustrate these results. Based on the rank one case, we will address harmonic spaces of arbitrary ranks. (See section 7.) As an application of the above inner product formula for λ, we introduce a *rational* winding number for paths in G. (See section 8.)

2. Preliminaries

We refer the readers to [1,2] for basic definitions concerning graphs. In this paper, we assume that a graph $G = (V(G), E(G))$ is finite and connected. Loops and multiple edges are allowed.

2.1. Spanning trees and cycle space of a graph

A subgraph of G is *spanning* if the vertex set of the subgraph equals $V(G)$. A *spanning tree* T of a connected graph G is a spanning subgraph which is connected and has no cycle. One can show that every spanning tree has $|V(G)| - 1$ edges. Let $\mathcal{T}(G)$ denote the set of all spanning trees in G. The *tree number* $k(G)$ of G is the number of spanning trees of G:

$$k(G) = |\mathcal{T}(G)|\,.$$

We assume that every edge e of G is assigned an orientation with $[e]$ denoting the oriented edge. One may regard $-[e]$ as representing the edge e with the opposite orientation. The *chain group* $C_1 = C_1(G) = \mathbb{Z}^{|E(G)|}$ is generated by the oriented edges $\{[e] \mid e \in E(G)\}$, and $C_0 = C_0(G) = \mathbb{Z}^{|V(G)|}$ by the vertex set $V(G)$. An element $x \in C_1$ may be represented either as a column vector $x = (n_e)_{e\in E(G)}$ or as a *formal* sum $x = \sum_{e\in E(G)} n_e[e]$ with $n_e \in \mathbb{Z}$ for all $e \in E(G)$. An element of C_0 will be represented similarly. The *incidence matrix* $\partial_1 = \partial_1(G) : C_1 \to C_0$ is an integer matrix defined by $\partial_1[xy] = y - x$ for an oriented edge $[xy]$. We assume that C_1 is given a *standard* inner product, denoted by $\circ$, where the oriented edges of G form an orthonormal basis.

The *cycle space* of G is $\ker \partial_1$ which we will denote by $Z_1 = Z_1(G)$. A cycle C as a subgraph of G with a given orientation corresponds to a unique element $\sum_{e\in E(C)} \epsilon_e[e] \in Z_1$ where the coefficients $\epsilon_e = \pm 1$ are determined by the orientation of C. Let $\mathcal{C}(G)$ denote the set of all cycles in G. We will assume that every $C \in \mathcal{C}(G)$ is assigned an orientation.

The rank of Z_1 for a connected G equals the *corank* $|E(G)|-|V(G)|+1$ of G (refer to [1]), and an important basis for Z_1 is given as follows. Fix a spanning tree $T \in \mathcal{T}(G)$.

For each $e \in E(G) \setminus E(T)$, there is a unique cycle in $T \cup e$ which contains e. Let z_e denote the element in Z_1 that corresponds to this cycle with a given orientation. Then the collection $\{z_e \mid e \in E(G) \setminus E(T)\}$ is a basis for Z_1 with the property that for $e, e' \in E(G) \setminus E(T)$, the coefficient of $[e]$ in $z_{e'}$ is ± 1 if $e = e'$, and 0 otherwise (refer to [1]). Hence, every $z \in Z_1$ is written uniquely as

$$z = \sum_{e \in E(G) \setminus E(T)} m_e \cdot z_e \tag{1}$$

where m_e is the product of the coefficients of $[e]$ in z and z_e.

2.2. *Cycletrees of a graph*

Let G be a finite connected graph. A *cycletree* in G is a connected spanning subgraph of G with exactly one cycle. Let $\mathcal{U}(G)$ denote the set of all cycletrees in G.

Note that $Y \in \mathcal{U}(G)$ can be expressed as a union

$$Y = T \cup e \tag{2}$$

of a spanning tree T in G and an edge $e \in E(G) - E(T)$. Hence, a connected spanning subgraph Y of G is a cycletree iff $|E(Y)| = |V(G)|$.

We will denote the unique cycle in a cycletree $Y \in \mathcal{U}(G)$ by C_Y and its corresponding element in $Z_1(G)$ by z_Y. Again, assume that an orientation of C_Y for each $Y \in \mathcal{U}(G)$ is fixed so that z_Y is well defined. As we shall see, our results are independent of these orientations. In the literature, a cycletree is also called a *cycle-rooted spanning tree* [6], or a *co-tree* [3].

Example 2.1. For the graph G on the left in Figure 1, the list of all 14 cycletrees are shown. For each cycletree, its unique cycle is given as bold red edges.

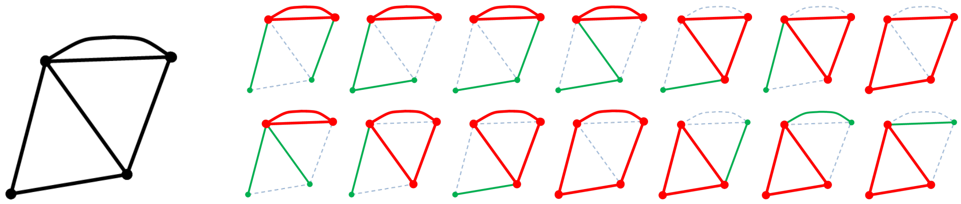

Figure 1. A graph G and its cycletrees

3. Harmonic space for a graph

We will discuss the notion of a harmonic space for a graph G. We refer the readers to [4] for harmonic spaces and combinatorial Hodge theory for a chain complex.

An integer matrix ∂_2 will be called a *second incidence matrix* for G if $\partial_1 \partial_2 = 0$. Hence the columns of ∂_2 are elements of $Z_1(G)$. Also, we regard ∂_2^t as a map on $C_1(G)$. We will use the notation $\mathscr{G} = (G, \partial_2)$ when ∂_2 is a second incidence matrix for G. The *combinatorial Laplacian* Δ for $\mathscr{G} = (G, \partial_2)$ is an operator on $C_1(G)$ defined by

$$\Delta = \partial_1^t \partial_1 + \partial_2 \partial_2^t$$

and the *harmonic space* $\mathcal{H}(\mathscr{G})$ is the kernel of Δ, i.e.,

$$\mathcal{H}(\mathscr{G}) = \{\lambda \in C_1(G) \mid \Delta\lambda = 0\}\,.$$

An element $\lambda \in \mathcal{H}(\mathscr{G})$ is called a *harmonic cycle*, the main interest of this paper.

Proposition 3.1. *Given $\mathscr{G} = (G, \partial_2)$, let $Z^1 = \ker \partial_2^t$ and $B_1 = \operatorname{im} \partial_2$. Then*

(1) $\mathcal{H}(\mathscr{G}) = Z_1 \cap Z^1$, and
(2) $\operatorname{rk} \mathcal{H}_1(\mathscr{G}) = \operatorname{rk} Z_1 - \operatorname{rk} B_1$.

Proof. (1) The backward inclusion is clear. For the forward inclusion, let $\Delta\lambda = 0$. Left-multiplying this equation by ∂_1 and using the property $\partial_1\partial_2 = 0$, we obtain $\partial_1\partial_1^t\partial_1\lambda = 0$. Since $\ker M^tM = \ker M$ for any matrix M, we see that $\lambda \in \ker \partial_1$. Similarly, left-multiplying $\Delta\lambda = 0$ by ∂_2^t reveals $\lambda \in \ker \partial_2^t$.

(2) It follows from (1) that $\mathcal{H}(\mathscr{G})$ consists of all elements in Z_1 that are orthogonal to the subspace B_1 of Z_1. Hence, $\operatorname{rk} Z_1 = \operatorname{rk} \mathcal{H}(\mathscr{G}) + \operatorname{rk} B_1$. □

Example 3.2. Given a connected cell complex X, let ∂_1 and ∂_2 be the first two boundary operators for X. (Refer to [5,7] for a definition of cell complexes.) The first harmonic space for X is defined by $\mathcal{H}_1(X) = \ker(\partial_1^t\partial_1 + \partial_2\partial_2^t)$. Now, if we let G be the 1-skeleton $X^{(1)}$ of X, then we have $\mathcal{H}_1(X) = \mathcal{H}(\mathscr{G})$ for $\mathscr{G} = (G, \partial_2)$.

The rest of this section will discuss harmonic cycles when $\operatorname{rk} \mathcal{H}_1(\mathscr{G}) = 1$. This case is a basis for understanding harmonic spaces of arbitrary rank.

Example 3.3. Each picture in Figure 2 represents a harmonic cycle λ for $\mathscr{G} = (G, \partial_2)$ where the columns of ∂_2 are the boundary cycles of the shaded faces of G and $\operatorname{rk} \mathcal{H}(\mathscr{G}) = 1$. The number attached to an oriented edge is its coefficient in λ, called the *flow*. Note that the net flow through each vertex is zero, satisfying the *cycle condition* $\lambda \in Z_1$, and the sum of the (signed) flows around each shaded square is zero, satisfying the *cocycle condition* $\lambda \in Z^1$. Note how the coefficients in a harmonic cycle reflect the symmetry (or unsymmetry) in $\mathscr{G}$.

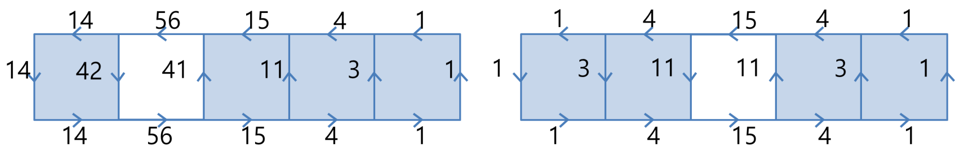

Figure 2. Examples of harmonic cycles

4. Winding number for the cycles

Given a graph G, suppose $\operatorname{rk} Z_1 = m$ with $m > 0$. A *unicyclizer* ∂ of G is a second incidence matrix for G of rank $m - 1$ with linearly independent columns. The pair $\mathscr{G} = (G, \partial)$ will be called a *unicyclization* of G. Note that in this case, we have $\operatorname{rk} \mathcal{H}(\mathscr{G}) = m - (m - 1) = 1$ by Proposition 3.1.

Let β be a basis of the cycle space $Z_1(G)$. A cycle $z \in Z_1$ will be denoted $[z]_\beta \in \mathbb{Z}^m$ when it is written with respect to β. Similarly, $[\partial]_\beta$ is the matrix obtained by writing each column vector of ∂ with respect to β. Note that the size of $[\partial]_\beta$ is $m \times (m-1)$. We are ready to present the main definition of this section.

Definition 4.1. Given a unicyclization $\mathscr{G} = (G, \partial)$ and a basis β for $Z_1(G)$, the winding number for the cycles in G is a map $w_{\mathscr{G}} : Z_1 \to \mathbb{Z}$ given by

$$w(z) = w_{\mathscr{G}}(z) = \det([z]_\beta, [\partial]_\beta)\,.$$

When β is fixed, we may simply write $w(z) = \det(z, \partial)$. If C is a cycle in G with a given orientation and $[C]$ its corresponding element in Z_1, we may write $w(C)$ instead of $w([C])$. The following proposition shows that the winding number is essentially independent of the choices of a basis for Z_1 and a basis for $\operatorname{im}\partial$.

Proposition 4.2. *Let ∂ and ∂' be unicyclizers of a graph G, and β and β' bases of Z_1. Let w denote the winding number defined by ∂ and β, and w' defined by ∂' and β'. If $\operatorname{im}\partial = \operatorname{im}\partial'$ as subgroups of Z_1, then we must have either $w = w'$ or $w = -w'$ as maps on Z_1.*

Proof. If $z \in \operatorname{im}\partial = \operatorname{im}\partial'$, then $w(z) = w'(z) = 0$. Suppose $z \notin \operatorname{im}\partial$. Clearly, the matrices $M = ([z]_{\beta'}, [\partial]_{\beta'})$ and $M' = ([z]_{\beta'}, [\partial']_{\beta'})$ have the same column spaces. Also, the set α of the columns of M forms a basis of that space, and the same is true for the set α' of the columns of M'. Therefore, we see that

$$\det([z]_\beta, [\partial]_\beta) = \det B \cdot \det([z]_{\beta'}, [\partial]_{\beta'}) = \det B \cdot \det([z]_{\beta'}, [\partial']_{\beta'}) \cdot \det A$$

where B is a change of basis matrix for β and β', and A is a change of basis matrix for α and α'. Since B and A (and their inverses) are invertible integer matrices, each has determinant ± 1. It is easy to see that A does not depend on the choice of $z \in Z_1$. Now the result is clear. □

The winding number $w_{\mathscr{G}}$ of a unicyclization $\mathscr{G} = (G, \partial)$ can be interpreted via homology. (Refer to [5,7] for homology groups.) The first homology of $\mathscr{G}$ is $H_1(\mathscr{G}) = Z_1/B_1$ where $B_1 = \operatorname{im}\partial$. One can find a basis β of Z_1 such that $[\partial]_\beta$ is an upper triangular matrix of size $m \times (m-1)$ with diagonal entries $d_1, \cdots, d_{m-1}$ satisfying $d_i > 0$ for all i. Therefore, we get $H_1(\mathscr{G}) \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_{m-1}} \oplus \mathbb{Z}$. Without loss of generality, assume that the last element in β represents a generator for the free part $\mathbb{Z}$ of $H_1(\mathscr{G})$. Hence we also assume that the last row of $[\partial]_\beta$ is indexed by that element. Let $\tau = d_1 \cdots d_{m-1}$ the size of the torsion part of $H_1(\mathscr{G})$. For $C \in \mathcal{C}(G)$,

$$|w_{\mathscr{G}}(C)| \;=\; |\det([\partial]_\beta, [C]_\beta)| = \tau c_m$$

where c_m is the last component of $[C]_\beta$. Hence, we obtain $w_{\mathscr{G}}(C)/\tau = c_m$ which counts the number of times a cycle C 'winds around' a generator for the free part of $H_1(\mathscr{G})$.

5. Standard harmonic cycle

Let us introduce the main object of study in this paper. Recall that for a connected graph G, every $Y \in \mathcal{U}(G)$ contains a unique cycle C_Y. Let z_Y denote the element in $Z_1(G)$ corresponding to C_Y. Assume a fixed basis for $Z_1(G)$.

Definition 5.1. (Standard harmonic cycle) The standard harmonic cycle for a unicyclization $\mathscr{G} = (G, \partial)$ is an element $\lambda_{\mathscr{G}}$ in $Z_1(G)$ given by

$$\lambda_{\mathscr{G}} = \sum_{Y \in \mathcal{U}(G)} w_{\mathscr{G}}(z_Y) \cdot z_Y \tag{3}$$

where $w_{\mathscr{G}}$ is the winding number for $\mathscr{G}$.

Note that $\lambda_{\mathscr{G}}$ is independent of the orientations of C_Y's. Indeed, each *weighted* cycle $w_{\mathscr{G}}(z_Y) \cdot z_Y$ in the sum is independent of the orientation of C_Y because $w_{\mathscr{G}}(z_Y) \cdot z_Y = \det(z_Y, \partial) \cdot z_Y = \det(-z_Y, \partial) \cdot (-z_Y) = w_{\mathscr{G}}(-z_Y) \cdot (-z_Y)$. We also note that if $\mathscr{G} = (G, \partial)$ and $\mathscr{G}' = (G, \partial')$ are unicyclizations of G with $\operatorname{im} \partial = \operatorname{im} \partial'$, then we have $\lambda_{\mathscr{G}} = \pm \lambda_{\mathscr{G}'}$ by Proposition 4.2. Later, we will show in Theorem 6.3 that $\lambda_{\mathscr{G}}$ is indeed a nonzero harmonic cycle for $\mathscr{G}$.

Example 5.2. The picture on the left in Figure 3 describes the standard harmonic cycle λ for $\mathscr{G} = (G, \partial)$ where the columns of ∂ are the boundary cycles of the two shaded faces of G. The pictures on the right show all cycletrees $Y \in \mathcal{U}(G)$ with nonzero $w(C_Y)$, which are all equal to 1 in this example, with their unique cycles in red. The sum of these cycles equals λ.

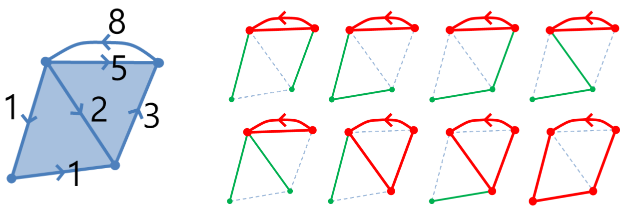

Figure 3. An example of standard harmonic cycle

The following proposition gives a description of $\lambda_{\mathscr{G}}$ via tree numbers. Let $\mathcal{C}(G)$ be the set of all cycles in G. For $C \in \mathcal{C}(G)$, let G/C be the graph obtained by contracting C to a point. Let $k(G/C)$ denote the number of spanning trees in G/C. (Refer to [9] for graph contractions.)

Proposition 5.3. *The standard harmonic cycle for a unicyclization $\mathscr{G} = (G, \partial)$ is*

$$\lambda_{\mathscr{G}} = \sum_{C \in \mathcal{C}(G)} k(G/C) w_{\mathscr{G}}(C) \cdot [C] \,. \tag{4}$$

Proof. Given $C \in \mathcal{C}(G)$, let $\mathcal{U}(G)_C = \{Y \in \mathcal{U}(G) \mid C_Y = C\}$. The map $Y \mapsto Y/C$ defines a bijection from $\mathcal{U}(G)_C$ to $\mathcal{T}(G/C)$. The proposition follows from $\mathcal{U}(G) = \amalg_{C \in \mathcal{C}(G)} \mathcal{U}(G)_C$ and the definition of $\lambda_{\mathscr{G}}$. Details will be omitted. □

6. Main results

In this section, we present the main results of the paper that will justify our terminology standard *harmonic cycle* for a graph. To begin, we will derive an intriguing formula that relates the standard harmonic cycle and the tree number of a graph.

Theorem 6.1. *(Inner product formula) Let $\lambda_{\mathscr{G}}$ be the standard harmonic cycle for a unicyclization $\mathscr{G} = (G, \partial)$. For $z \in Z_1(G)$, we have*

$$\lambda_{\mathscr{G}} \circ z = w_{\mathscr{G}}(z)k(G) \tag{5}$$

where $\circ$ denotes a standard inner product for $C_1(G)$.

Proof. By (1) and the linearity of $w = w_{\mathscr{G}}$, it is enough to prove the theorem when $z \in Z_1(G)$ corresponds to a cycle $C \in \mathcal{C}(G)$. Again, assume that every $C \in \mathcal{C}(G)$ is assigned an orientation, and let $\epsilon(C, e)$ denote the coefficient of $[e]$ in $[C]$. If C' is also a cycle in G, then note that

$$[C] \circ [C'] = \sum_{e} \epsilon(C, e)\epsilon(C', e) \tag{6}$$

where the sum may be taken over $E(C)$ or $E(C')$ (or, $E(G)$).

Given a spanning tree $T \in \mathcal{T}(G)$, let $E(\overline{T}) = E(G) \setminus E(T)$. For $e \in E(\overline{T})$, let C_e denote the unique cycle in $T \cup e$. Now, given a cycle $C \in \mathcal{C}(G)$, we have

$$\begin{aligned} w(C)k(G) &= \sum_{T \in \mathcal{T}(G)} w(C) \\ &= \sum_{T \in \mathcal{T}(G)} \sum_{e \in E(\overline{T})} \epsilon(C, e)\epsilon(C_e, e)w(C_e) \\ &= \sum_{Y \in \mathcal{U}(G)} \sum_{e \in E(C_Y)} \epsilon(C, e)\epsilon(C_Y, e)w(C_Y) \\ &= \sum_{Y \in \mathcal{U}(G)} ([C] \circ [C_Y])w(C_Y) \end{aligned}$$

where the equalities are justified as follows. The first equality is clear. The second equality follows from (1) and the linearity of w the winding number. The third equality holds because the set $\{(T, e) \mid T \in \mathcal{T}(G),\ e \in E(\overline{T})\}$ corresponds bijectively to the set $\{(Y, e) \mid Y \in \mathcal{U}(G),\ e \in E(C_Y)\}$ via the map $(T,\ e) \mapsto (T \cup e,\ e)$, and because C_e equals C_Y for $Y = T \cup e$. The fourth equality follows from (6). Since the last expression in the above equations equals $\lambda_{\mathscr{G}} \circ [C]$, the proof is complete. □

The following corollary provides a new method for computing the tree number $k(G)$ of G. We will present an example illustrating this formula in the last section.

Corollary 6.2. *Let $w = w_{\mathscr{G}}$, and let C_0 be a cycle in G. Then we have*

$$w(C_0)k(G) = \sum_{C \in \mathcal{C}(G)} ([C_0] \circ [C])w(C)k(G/C)\,. \tag{7}$$

Proof. The equation follows immediately from Theorem 6.1 and Proposition 5.3. □

Theorem 6.3. *Let $\mathscr{G} = (G, \partial)$ be a unicyclization of a graph G. Then*

$$\lambda_{\mathscr{G}} = \sum_{Y \in \mathcal{U}(G)} w_{\mathscr{G}}(z_Y) \cdot z_Y \tag{8}$$

is a nonzero harmonic cycle for $\mathscr{G}$.

Proof. Let $\lambda = \lambda_{\mathscr{G}}$ and $w = w_{\mathscr{G}}$. For any column z of ∂, we have $w(z) = \det(z, \partial) = 0$. Hence $\lambda \circ z = w(z)k(G) = 0$ by Theorem 6.1, which implies $\lambda \in Z^1 = \ker \partial^t$. Therefore we have $\lambda \in Z_1 \cap Z^1$, and λ is a harmonic cycle for $\mathscr{G}$ by Proposition 3.1. Since $\operatorname{rk} \partial = \operatorname{rk} Z_1 - 1$, there is $x \in Z_1$ such that $w(x) = \det(x, \partial) \neq 0$. Hence, $\lambda \circ x = w(x)k(G) \neq 0$, which shows $\lambda \neq 0$. □

7. Harmonic spaces of arbitrary ranks

Based on our results for unicyclizations of a graph G, we will describe the harmonic space of $\mathscr{G} = (G, \partial_2)$ for an arbitrary ∂_2.

Proposition 7.1. *Any nonzero cycle $z \in Z_1(G)$ is a standard harmonic cycle of a unicyclization of G up to a scalar (possibly rational) multiplication.*

Proof. Let $\operatorname{rk} Z_1 = m$ $(m > 0)$. Let $z^{\perp} = \{c \in Z_1 \mid z \circ c = 0\}$. Since $z \neq 0$, we have $\operatorname{rk} z^{\perp} = m - 1$. Let ∂ be an integer matrix whose columns form a basis for $z^{\perp}$. Then $\mathscr{G} = (G, \partial)$ is a unicyclization of G. For $c \in \lambda^{\perp}$, we have $w_{\mathscr{G}}(c) = \det(c, \partial) = 0$, and consequently, $\lambda_{\mathscr{G}} \circ c = w_{\mathscr{G}}(c)k(G) = 0$ by Theorem 6.1. Hence, $\lambda_{\mathscr{G}}$ is orthogonal to $z^{\perp}$, and the result follows. □

Given $\mathscr{G} = (G, \partial_2)$, suppose $\operatorname{rk} \partial_2 = m - r$ where $m = \operatorname{rk} Z_1(G)$ and $r > 0$. Then, we have $\operatorname{rk} \mathcal{H}(\mathscr{G}) = r$. Let $\{z_i \mid 1 \leq i \leq m\}$ be a collection of linearly independent vectors in Z_1 such that $\{z_{r+1}, \ldots, z_m\}$ forms a basis for $\operatorname{im} \partial_2$. For $1 \leq k \leq r$, let D_k be the matrix having all z_i's as columns except z_k. In particular, we have $\operatorname{rk} D_k = m - 1$. Then, for each $1 \leq k \leq r$, the pair (G, D_k) is a unicyclization of G. Let w_k and λ_k be the corresponding winding number and standard harmonic cycle, respectively. Hence, we have $w_k(z) = \det(z, D_k)$ for $z \in Z_1(G)$, and $\lambda_k = \sum_{Y \in \mathcal{U}(G)} w_k(C_Y) \cdot [C_Y]$. Note that $w_k(z_j)$ is nonzero iff $k = j$.

Theorem 7.2. *Given $\mathscr{G} = (G, \partial_2)$, the collection $\{\lambda_k \mid 1 \leq k \leq \operatorname{rk} \mathcal{H}(\mathscr{G})\}$ of standard harmonic cycles is linearly independent in the harmonic space $\mathcal{H}(\mathscr{G})$.*

Proof. Suppose $\sum_{1 \leq k \leq r} n_k \lambda_k = 0$. We claim that $n_j = 0$ for each $1 \leq j \leq r$. Applying the inner product to this equation with z_j for a fixed j shows

$$\begin{aligned} 0 &= \sum_{1 \leq k \leq r} n_k (\lambda_k \circ z_j) \\ &= \sum_{1 \leq k \leq r} n_j w_k(z_j) k(G) \\ &= n_j w_j(z_j) k(G) \end{aligned}$$

where the second equality follows from Theorem 6.1, and the last equality from the fact $w_k(z_j)$ is nonzero iff $k = j$. We conclude that $n_j = 0$ for each $1 \leq j \leq r$, and the proof is complete. □

8. Examples and applications

Example 8.1. Each picture in Figure 4 shows the standard harmonic cycle λ of a unicyclization of a graph G where the coefficients of the oriented edges form Fibonacci numbers $f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$. The well-known equalities $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$ and $f_2 + f_4 + \cdots + f_{2n-2} = f_{2n-1} - f_1$ correspond to the cycle or cocycle conditions of a harmonic cycle. Details will be left to the readers.

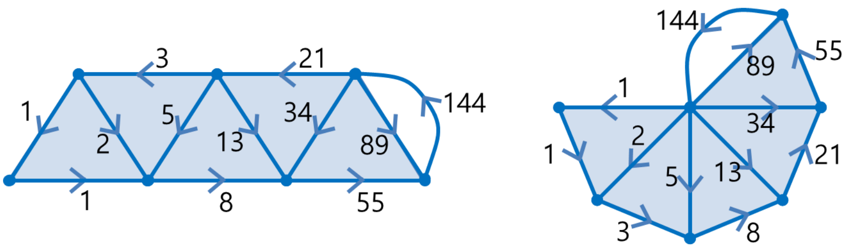

Figure 4. Fibonacci numbers in standard harmonic cycles

Example 8.2. Let us count the number of spanning trees of the graph G on the left in Figure 5 by applying (7). Assume all cycles are oriented counterclockwise. Let β be the basis of Z_1 given by the boundary cycles of four finite faces of G. Let ∂ be given by the boundary cycles of the three shaded faces, and let C_0 be the green cycle winding around the 'hole', the unshaded face. One can check that $w(C_0) = \det I_4 = 1$ where I_4 is the identity matrix of order 4.

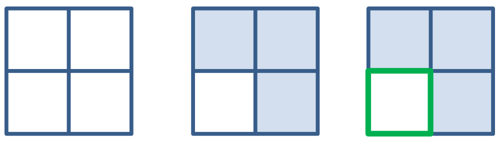

Figure 5. A graph G, a unicyclizer ∂, and a cycle C_0

By (7), we need to consider only those $C \in \mathcal{C}(G)$ with $w(C) \neq 0$, and they are the cycles which wind around the hole. Figure 6 shows the full list of those cycles in red.

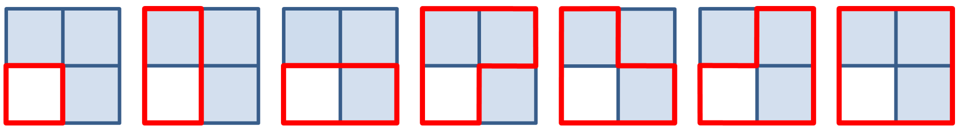

Figure 6. Cycles in G with nontrivial winding numbers

It is easy to show that $w(C) = 1$ for each red cycle C, and we have the equality

$$\begin{aligned} 192 = k(G) &= \sum_{C:\ \textit{red cycles}} ([C_0] \circ [C]) k(G/C) \\ &= 4 \cdot 30 + 3 \cdot 8 + 3 \cdot 8 + 3 \cdot 2 + 2 \cdot 2 + 3 \cdot 2 + 2 \cdot 4 \end{aligned}$$

where the terms in the last sum correspond to the red cycles in the same order. Each of the tree numbers that appear in the right hand side of this equation may be computed also by applying (7).

Example 8.3. The picture on the left in Figure 7 shows the standard harmonic cycle λ of a Möbius strip represented as a unicyclization of a graph G. The pictures on the right illustrate a computation of λ by Proposition 5.3. Note that there is a cycle C with $w(C) = 2$ in this computation.

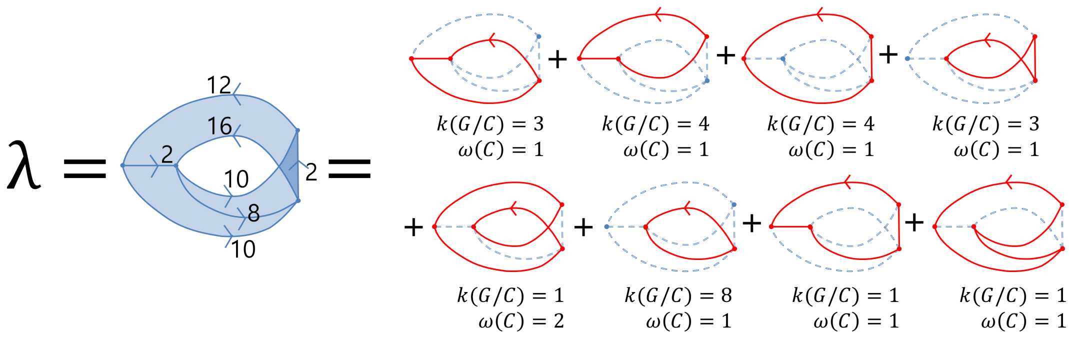

Figure 7. Standard harmonic cycle of a Möbius strip

Winding number for paths

Using the inner product formula for the standard harmonic cycle, the winding number $w_{\mathscr{G}}$ for a unicyclization $\mathscr{G} = (G, \partial)$ can be extended to a map on $C_1(G)$. For $P \in C_1(G)$, we define its winding number to be

$$w_{\mathscr{G}}(P) = \frac{\lambda_{\mathscr{G}} \circ P}{k(G)} \in \mathbb{Q}\,.$$

This definition is intended to assess the degree to which an arbitrary path 'winds around' a homology generator in a cell complex. The following example illustrates that the winding number, being a rational number, is a finer invariant than homology.

Example 8.4. Let $P \in C_1(G)$ be a path where G is the 1-skeleton of the Möbius strip in the previous example. In each figure below, P is marked red, and we have $w(P) = (\lambda \circ P)/k(G)$. Note that $k(G) = 24$ in this example.

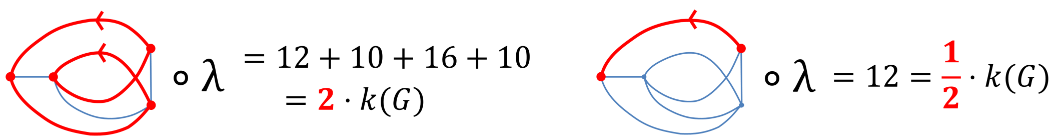

Figure 8. Winding numbers of paths

Note that the path P in the last figure has a nonzero rational winding number whereas it would belong to the same trivial class as a point in homology.

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