Möbius coinvariants and bipartite edge-rooted forests

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Abstract

The Möbius coinvariant $\mu^{\perp}(G)$ of a graph G is defined to be the Möbius invariant of the dual of the cycle matroid of G. This invariant is known to equal the rank of the reduced homology of the cycle matroid complex of G. For a complete graph K_{m+1}, W. Kook gave an interpretation of $\mu^{\perp}(K_{m+1})$ as the number of edge-rooted forests in K_m. In this paper, we obtain a new combinatorial interpretation of $\mu^{\perp}(K_{m+1,n+1})$ as the number of B-edge-rooted forests in $K_{m,n}$, which is a bipartite analogue of the previous result.

Based on these interpretations, we will give new bijective proofs of the formulas for $\mu^{\perp}(K_{m+1})$ and $\mu^{\perp}(K_{m+1,n+1})$ given by I. Novik, A. Postnikov, and B. Sturmfels in terms of the Hermite polynomials. In addition, we will construct a homology basis for the cycle matroid complex of $K_{m+1,n+1}$ indexed by the B-edge-rooted forests. Also we will discuss the Möbius coinvariant of *bi-coned* graphs which generalize complete bipartite graphs.

Keywords:
B-edge-rooted forest, Möbius coinvariant, matroid complex, homology, group action, bi-coned graph
2010 MSC: Primary 05E45, Secondary 05A15, 05B35, 05C31, 05E18, 05E40

1. Introduction

The Möbius invariant $\mu(M)$ of a matroid M is defined to be $|\mu_{L(M)}(\hat{0}, \hat{1})|$ where $\mu_{L(M)}$ is the Möbius function on the lattice of flats $L(M)$ of M. We define the Möbius coinvariant of M to be $\mu^{\perp}(M) := \mu(M^*)$ where M^* is the dual matroid of M. It is well-known that $\mu^{\perp}(M)$ equals an evaluation $T_M(0,1)$ of the Tutte polynomial $T_M(x,y)$ of M, which also equals the rank of the reduced homology of the independent set complex $IN(M)$ of M [1, 3]. For a graph G, we define its Möbius coinvariant to be $\mu^{\perp}(G) := \mu(M(G)^*)$ where $M(G)$ is the cycle matroid of G. In this paper, we will give a new interpretation of the Möbius coinvariant for a class of graphs generalizing complete bipartite graphs.

Some of the previous results that motivated the current work are as follows. In the context of hyperplane arrangement and commutative algebra, D. Bayer, S. Popescu, and B. Sturmfels posed a problem of computing $\mu^{\perp}(G)$ [4]. In response to this problem, I. Novik, A. Postnikov,

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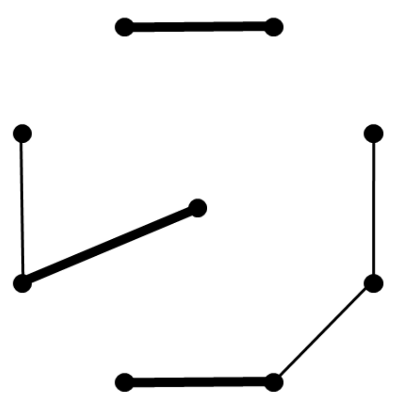
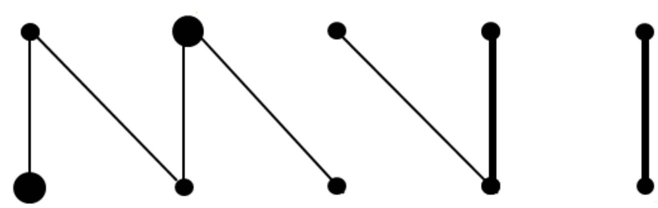
and B. Sturmfels [20] gave formulas for $\mu^{\perp}(K_{m+1})$ and $\mu^{\perp}(K_{m+1,n+1})$ for complete graphs K_{m+1} and complete bipartite graphs $K_{m+1,n+1}$, studying cographic ideals and hyperplane arrangements. They expressed these formulas in terms of Hermite polynomials, a generating function for partial matchings. In a study of group action on the homology of matroid complexes, W. Kook ([12] or [17, Theorem 20]) gave a simple combinatorial interpretation for $\mu^{\perp}(K_{m+1})$, and more generally for the Möbius coinvariant of a *coned graph* [13], via *edge-rooted forests* (see Figure 1(a)). However, a direct correspondence relating the edge-rooted forests and the formula for $\mu^{\perp}(K_{m+1})$ has not appeared before. We will establish this correspondence in this paper.

More importantly, we will give a new combinatorial interpretation for the Möbius coinvariant of a complete bipartite graph $K_{m+1,n+1}$, which can be outlined as follows. A tree in $K_{m,n}$ is *bi-rooted* if it has two root vertices, one in each bipartite set of $K_{m,n}$. A tree in $K_{m,n}$ is edge-rooted if one edge is marked as an *edge-root*. Define a *B-edge-rooted forest* in $K_{m,n}$ to be a spanning forest with exactly one component bi-rooted and the remaining components edge-rooted (see Figure 1(b)). By identifying all spanning trees with zero internal activity, we will show that $\mu^{\perp}(K_{m+1,n+1})$ equals the number of all B-edge-rooted forests in $K_{m,n}$ (see Theorem 3.4), which is an analogue of the result for a complete graph.



(a) an edge-rooted forest



(b) a B-edge-rooted forest

Figure 1: An edge-rooted forest and a B-edge-rooted forest

As a consequence of our combinatorial interpretations for the invariants $\mu^{\perp}(K_{m+1})$ and $\mu^{\perp}(K_{m+1,n+1})$, we will give new bijective proofs of their formulas given by I. Novik, A. Postnikov, and B. Sturmfels [20]. In Theorem 4.3, we will show that $\mu^{\perp}(K_{m+1})$ equals

$$\sum_{k\geq 1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} (2k \cdot m^{m-1-2k})(2k-1)!!$$

where the k-th term in this sum is the number of edge-rooted forests in K_m with k components. In Theorem 4.8, we will show that $\mu^{\perp}(K_{m+1,n+1})$ equals

$$\sum_{k=0}^{\min(m-1,n-1)} \binom{m}{k}\binom{n}{k} k!(m-k)(n-k)n^{m-1-k}m^{n-1-k}$$

where the k-th term in the sum is the number of B-edge-rooted forests in $K_{m,n}$ with $k+1$ components (*i.e.*, the number of B-edge-rooted forests with one bi-rooted tree and k edge-rooted trees). As a tool for proving this fact, we develop a bipartite analogue of Lemma 4.2 about the number of the vertex-rooted forests in a complete graph with given roots (see Theorem 4.6).

B-edge-rooted forests are not only a combinatorial interpretation, but also a "code" for constructing a homology basis for the cycle matroid complex of $K_{m+1,n+1}$. W. Kook [12] showed how edge-rooted forests in K_m could be used to construct a homology basis for the independent set complex $I(K_{m+1}) := IN(M(K_{m+1}))$ (see Example 3 in [17]). This basis also reveals that the action of the symmetric group S_m (as a subgroup of S_{m+1}) on this homology group is isomorphic up to sign to that on the set of all edge-rooted forests in K_m.

As a bipartite analogue of these results, we will construct a basis for the top reduced homology group of $I(K_{m+1,n+1})$ from the B-edge-rooted forests in $K_{m,n}$ (Theorem 5.3). Furthermore, we will show that this construction is equivariant up to sign under the action of $S_m \times S_n$ as a subgroup of $S_{m+1} \times S_{n+1}$ (Theorem 5.5). This result requires identifying special facets that depend on the bi-roots, a subtle difference from the case of complete graphs.

The paper is organized as follows. Section 2 is a brief review of several interpretations of Möbius coinvariant. Section 3 presents the main definition of the paper, B-edge-rooted forests, and shows a new interpretation for the Möbius coinvariant of a complete bipartite graph, as an analogue of that of a complete graph. Section 4 provides bijective proofs of the formulas for the Möbius coinvariants of complete graphs and complete bipartite graphs given by I. Novik, A. Postnikov, and B. Sturmfels. Section 5 constructs a new homology basis for $I(K_{m+1,n+1})$ based on our combinatorial interpretation, and analyses the symmetric group action on the homology. Section 6 defines *bi-coned* graphs which generalize complete bipartite graphs and discuss their Möbius coinvariants.

2. Background: Interpretations of Möbius coinvariant

In this section, we will review relevant interpretations of the Möbius coinvariant for graphs. We refer the readers to [22, 26] for basic definitions and notations concerning matroids. A *matroid* M on a finite *ground set* E is a pair $(E, \mathcal{I})$ with $\mathcal{I} \subset 2^E$ satisfying

(i) $\emptyset \in \mathcal{I}$,

(ii) if $I \in \mathcal{I}$ and $I' \subset I$, then $I' \in \mathcal{I}$, and

(iii) if $I, I' \in \mathcal{I}$ and $|I| > |I'|$, then there is $e \in I - I'$ such that $I' \cup e \in \mathcal{I}$.

The collection $\mathcal{I}$ is called the *independent sets*. A maximal independent set is called a *basis* of M. For a matroid M, its dual matroid $M^* = (E, \mathcal{I}^*)$ is defined by $\mathcal{I}^* = \{E - I \mid I \in \mathcal{I}\}$.

The axioms (i) and (ii) imply that $\mathcal{I}$ forms an *abstract simplicial complex*. This complex is called the *independent set complex*, or the *matroid complex* of M, denoted by $IN(M)$. A matroid complex is shellable [1], hence homotopy equivalent to a wedge of spheres of the same dimension so that its reduced homology groups vanish except for the top dimension.

An important example for our purpose is the *cycle matroid* of a graph. For a graph $G = (V(G), E(G))$, its cycle matroid is $M(G) = (E, \mathcal{I})$ where $E = E(G)$ and a subset $I \subset E$ is an element of $\mathcal{I}$ if I is acyclic. A basis of $M(G)$ is a spanning tree of G for a connected graph G. Its dual matroid $M(G)^*$ is called a *cographic* matroid. For simplicity, we will write $I(G)$ for the matroid complex $IN(M(G))$. Now, we will introduce the definition of Möbius coinvariant of a graph along with a review of related matroidal invariants.

2.1. The Möbius coinvariant of a graph

The *Möbius invariant* of a matroid M is defined to be $\mu(M) := |\mu_{L(M)}(\hat{0}, \hat{1})|$ where $L(M)$ is the lattice of flats of M and $\mu_{L(M)}(\cdot, \cdot)$ the *Möbius function* on $L(M)$. The *Möbius coinvariant* of M is defined to be $\mu^{\perp}(M) := \mu(M^*)$. Now, we define the *Möbius coinvariant* of a graph G, denoted by $\mu^{\perp}(G)$, to be

$$\mu^{\perp}(G) = \mu(M(G)^*)\,.$$

2.2. An evaluation of the Tutte polynomial of a graph

For a matroid M, its Tutte polynomial $T_M(x, y)$ is defined by (refer to [3])

$$T_M(x, y) = \sum t_{i,j} x^i y^j$$

where $t_{i,j}$ is the number of bases of M with internal activity i and external activity j. We will review basis activity in Section 3. From the well-known identities $\mu(M) = T_M(1, 0)$ and $T_M(x, y) = T_{M^*}(y, x)$, we have

$$\mu^{\perp}(G) = T_G(0, 1)$$

where $T_G(x, y) = T_{M(G)}(x, y)$. For a connected G, $\mu^{\perp}(G)$ is the number of the spanning trees in G with zero internal activity.

2.3. The rank of the reduced homology of $I(G)$

Suppose that G is a connected graph with n vertices. The facets (maximal faces) of $I(G)$ correspond to the spanning trees of G, and the dimension of $I(G)$ is $n - 2$. The rank of the top reduced homology group $\tilde{H}_{n-2}(I(G))$ is well-known to equal $T_G(0, 1)$ (refer to [1]). Hence we have

$$\mu^{\perp}(G) = \operatorname{rk} \tilde{H}_{n-2}(I(G))\,.$$

Since a matroid complex is shellable [1], it follows that $\mu^{\perp}(G)$ equals the unsigned reduced Euler characteristic $|\tilde{\chi}(I(G))|$ of $I(G)$. This invariant was also called the α-invariant of G in [11, 13] where a symmetric group action on $\tilde{H}_{n-2}(I(K_n))$ was studied.

3. Combinatorial interpretations for $\mu^{\perp}(K_{m+1})$ and $\mu^{\perp}(K_{m+1,n+1})$

The definition of internal activity for a spanning tree in a graph G is as follows. Suppose that the set of edges $E(G)$ is linearly ordered by ω. Let T be a spanning tree in G. Deleting an edge e in T creates a forest with two components, say T_1 and T_2. The *fundamental bond* of e with respect to T is the set $E_G(T_1, T_2)$ of all edges in G having one vertex in T_1 and the other in T_2. Then $e \in T$ is said to be *internally active* if e is ω-smallest in its fundamental bond with respect to T. Otherwise e is *internally passive*. The *internal activity* of T is the number of internally active edges. This section will assume the interpretation of $\mu^{\perp}(G)$ as the number of spanning trees in G with zero internal activity.

3.1. The Möbius coinvariant of a complete graph

We will review a combinatorial interpretation for the Möbius coinvariant of a complete graph (refer to [12] or [17, Theorem 20]). A tree with at least one edge is said to be *edge-rooted* if exactly one edge is marked as an edge-root. An *edge-rooted forest* in a graph is a spanning forest with every component edge-rooted. Alternatively, an edge-rooted forest in K_m is a pair $(F, \mathbf{e})$, where F is a spanning forest in K_m and $\mathbf{e}$ is a subset of $E(F)$ containing exactly one edge from each component of F (see Figure 1(a)).

Theorem 3.1. *[12] For $m \geq 1$, the Möbius coinvariant $\mu^{\perp}(K_{m+1})$ of a complete graph K_{m+1} equals the cardinality of the set of all edge-rooted forests in K_m.*

3.2. The Möbius coinvariant of a complete bipartite graph

In this paper, we assume that the vertex bipartition of the complete bipartite graph $K_{m+1,n+1}$ is given by $([m] \cup 0, [\overline{n}] \cup \overline{0})$ where $[m] = \{1, \cdots, m\}$ and $[\overline{n}] = \{\overline{1}, \cdots, \overline{n}\}$. Denote by $K_{m,n}$ the subgraph induced by $([m], [\overline{n}])$. For our purpose, we will order the edges of $K_{m+1,n+1}$ as follows. First, order its vertices by

$$\overline{0} < 0 < 1 < 2 < \cdots < m < \overline{1} < \cdots < \overline{n}.$$

Then, we will use the resulting lexicographic ordering on $E(K_{m+1,n+1})$. In this ordering, every edge in $K_{m,n}$ is larger than any edge incident to 0 or $\overline{0}$.

To describe the Möbius coinvariant of $K_{m+1,n+1}$, we introduce a new concept which did not appear in the case of a complete graph. Define a *bi-rooted tree* to be a tree in $K_{m,n}$ which has one root-vertex in each bipartite set. A vertex of a spanning tree T in $K_{m+1,n+1}$ is called a *connecting root* if it is adjacent to 0 or $\overline{0}$ in the spanning tree T. If T is a spanning tree in $K_{m+1,n+1}$ and C a component in the spanning forest $K_{m,n} \cap T$ of $K_{m,n}$, the component C is also called *bi-rooted* if it has a connecting root in each bipartite set of $K_{m,n}$. Note that there is no bi-rooted component in $K_{m,n} \cap T$ if $0\overline{0} \in E(T)$, and exactly one bi-rooted component if $0\overline{0} \notin E(T)$. In the following lemma, C_x denotes the component of $K_{m,n} \cap T$ containing a vertex x.

Lemma 3.2. *An edge e in a spanning tree T in $K_{m+1,n+1}$ is internally active (with respect to the order defined above) if and only if e is in one of the following three cases:*

(i) e *is* $0\overline{0}$.

(ii) e *is* $\overline{0}v$ *with* $v \in [m]$ *where* C_v *is not bi-rooted and* v *is the smallest vertex in* C_v.

(iii) e *is* $0\overline{w}$ *with* $\overline{w} \in [\overline{n}]$ *where* $C_{\overline{w}}$ *consists of the vertex* $\overline{w}$ *only.*

Proof. It is easy to check the sufficiency. The necessity can be proved by the following observations whose proofs are simple and will be omitted. If e ($\neq 0\overline{0}$) is the smallest in its basic bond, then it is of the form $\overline{0}v$ or $0\overline{w}$ for some $v \in [m]$ or $\overline{w} \in [\overline{n}]$. Also, if $\overline{0}v$ or $0\overline{w}$ is incident to a bi-rooted component in $K_{m,n} \cap T$, it is not internally active because its basic bond contains $0\overline{0}$. Therefore, e must be incident to a "single-rooted" component, and (ii) and (iii) cover these cases. $\square$

The following is the main definition of this section which is a bipartite analogue of an edge-rooted forest. As we shall see, it will provide an indexing set for the spanning trees with internal activity zero in $K_{m+1,n+1}$.

Definition 3.3. A *B-edge-rooted forest* in a complete bipartite graph $K_{m,n}$ $(m, n \geq 1)$ is a spanning forest in $K_{m,n}$ composed of two kinds of components such that

- exactly one component is *bi-rooted*, *i*.e., has one vertex-root in each bipartite set, and
- each remaining component is edge-rooted, *i*.e., has one edge marked as an edge-root.

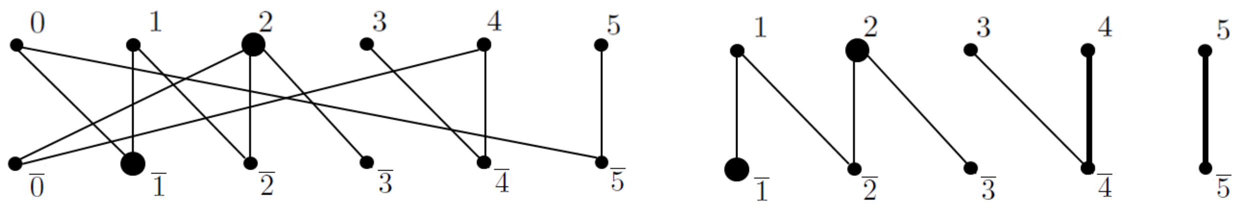
Equivalently, we define the set $\mathcal{F}_e^B(K_{m,n})$ of all B-edge-rooted forests in $K_{m,n}$ to be the set of all triples $(F, \mathbf{b}, \mathbf{e})$ where F is a spanning forest in $K_{m,n}$ with each component having at least one edge, $\mathbf{b} = \{v_1, \overline{v_2}\}$ with $v_1 \in [m], \overline{v_2} \in [\overline{n}]$ is a bi-root for the component $C_{\mathbf{b}}$ of F, and $\mathbf{e}$ is a subset of $E(F) - E(C_{\mathbf{b}})$ containing exactly one edge from each component of $F - C_{\mathbf{b}}$. We allow the possibility $F = C_{\mathbf{b}}$, and in that case, $\mathbf{e}$ is empty.

Based on this definition, we now give a combinatorial interpretation for $\mu^{\perp}(K_{m+1,n+1})$ that is independent of ordering of $E(K_{m+1,n+1})$.

Theorem 3.4. *The Möbius coinvariant* $\mu^{\perp}(K_{m+1,n+1})$ *of* $K_{m+1,n+1}$ *equals the cardinality of the set of all B-edge-rooted forests in* $K_{m,n}$ *:*

$$\mu^{\perp}(K_{m+1,n+1}) = |\mathcal{F}_e^B(K_{m,n})| \,.$$

Proof. It suffices to construct a bijection between $\mathcal{F}_e^B(K_{m,n})$ and the set $\mathcal{T}^0(K_{m+1,n+1})$ of all internal activity zero spanning trees in $K_{m+1,n+1}$. Take $T \in \mathcal{T}^0(K_{m+1,n+1})$. The graph obtained by deleting two vertices $0, \overline{0}$ from T is a spanning forest F in $K_{m,n}$. Since we have $0\overline{0} \notin E(T)$, exactly one component in F is bi-rooted, say, by $\mathbf{b}$. Let $C_1, C_2, \cdots, C_d$ be other components in F with a connecting root c_i and the smallest vertex $s_i \in V(C_i)$ for each $i \in [d]$. Since $c_i \neq s_i$ by Lemma 3.2, there is a unique path in C_i from s_i to c_i. Let e_i be the last edge in the path and define $\mathbf{e} = \{e_1, e_2, \cdots, e_d\}$. The correspondence $T \mapsto (F, \mathbf{b}, \mathbf{e})$ is reversible by Lemma 3.2, which completes the proof. (See Figure 2.) $\square$



(a) a tree $T \in \mathcal{T}^0$

(b) B-edge-rooted forest

Figure 2: A tree $T \in \mathcal{T}^0$ and its corresponding B-edge-rooted forest

4. Formulas for $\mu^\perp(K_{m+1})$ and $\mu^\perp(K_{m+1,n+1})$

In this section, we will give bijective proofs of the formulas in [20] for $\mu^\perp(K_{m+1})$ and $\mu^\perp(K_{m+1,n+1})$. The problem of computing these numbers was posed by D. Bayer, S. Popescu and B. Sturmfels [4] studying the hyperplane arrangements and commutative algebra arising from graphs. I. Novik, A. Postnikov, and B. Sturmfels [20] continued these studies in a similar context, and gave formulas for $\mu^\perp(K_{m+1})$ and $\mu^\perp(K_{m+1,n+1})$ in terms of *Hermite polynomials*, obtained via recurrence relations. We will recover the formulas via the combinatorial interpretations discussed in the previous section. Our method also gives more information about each term in the formulas.

4.1. The Möbius coinvariant of a complete graph

We first review the *Hermite polynomial* $H_n(x)$. For a partial matching π in K_n, let $a(\pi)$ be the number of vertices in K_n of degree 0. The Hermite polynomial $H_n(x)$ is defined to be a generating function for partial matchings in K_n, *i.e.*, $H_n(x) = \sum_\pi x^{a(\pi)}$ where the sum is over all partial matchings π in K_n. Then the following recurrence holds for $n \geq 0$:

$$H_{n+1}(x) = xH_n(x) + nH_{n-1}(x).$$

where $H_{-1}(x) = 0$ and $H_0(x) = 1$. In addition, the Hermite polynomial $H_n(x)$ has the following explicit form for $n \geq 1$:

$$H_n(x) = x^n + \sum_{k\geq 1}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!x^{n-2k}. \tag{1}$$

Theorem 4.1. *[20, Theorem 5.8] The Möbius coinvariant $\mu^\perp(K_{m+1})$ of a complete graph K_{m+1} with $m \geq 1$ equals $(m-1)H_{m-2}(m)$. Hence, we obtain*

$$\mu^\perp(K_{m+1}) = \sum_{k\geq 1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} (2k-1)!!(2k \cdot m^{m-1-2k}) \tag{2}$$

by evaluating $H_{m-2}(m)$ using (1).

Several initial values of $\mu^{\perp}(K_{m+1})$ are given below:

m	2	3	4	5	6	7	8	9
$\mu^{\perp}(K_{m+1})$	1	6	51	560	7575	122052	2285353	48803904

Now, we recall an important lemma concerning the number of the vertex-rooted forests in a complete graph as a generalization of Cayley's formula. See [21] for a proof.

Lemma 4.2. *[21] Let S be a subset of vertices in K_n with $|S| = s$. Then the number of the vertex-rooted forests in K_n with s components each of which contains a distinct element (vertex) in S as a vertex-root is $s \cdot n^{n-s-1}$.*

The following theorem is one of the main results of this section providing a combinatorial interpretation of Theorem 4.1 via the edge-rooted forests in a complete graph.

Theorem 4.3. *The k-th term in the expression for*

$$\mu^{\perp}(K_{m+1}) = \sum_{k\geq 1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k} (2k \cdot m^{m-1-2k})(2k-1)!!$$

equals the number of edge-rooted forests in K_m with exactly k components.

Proof. By Theorem 3.1, $\mu^{\perp}(K_{m+1})$ equals the number of edge-rooted forests in K_m. Note that the number of components of an edge-rooted forest in K_m is at most $\lfloor \frac{m}{2} \rfloor$. Hence, it suffices to show that the number of the edge-rooted forests in K_m with exactly k components is

$$\binom{m}{2k} (2k \cdot m^{m-1-2k})(2k-1)!!\,.$$

Fix $k \geq 1$. If R is a vertex-rooted forest on K_m with $2k$ components and $\mathbf{e}$ is a perfect matching of the $2k$ vertex-roots of R, then $R \cup \mathbf{e}$ is an edge-rooted forest with k components when $\mathbf{e}$ is regarded as edge-roots. The map $(R, \mathbf{e}) \mapsto R \cup \mathbf{e}$ is clearly a bijection between the set of all pairs $(R, \mathbf{e})$ and the set of all edge-rooted forests with k components. By Lemma 4.2 and the fact that $(2k-1)!!$ is the number of perfect matchings on $2k$ vertices, the cardinality of the former set is seen to be the desired formula. $\square$

Since the number of the edge-rooted spanning trees in K_n is $(n-1)n^{n-2}$, an exponential generating function for $\mu^{\perp}(K_{m+1})$ is given by ([13])

$$\sum_{m\geq 0} \mu^{\perp}(K_{m+1}) \frac{x^m}{m!} = \exp\left(\overline{T}(x)\right)$$

where $\overline{T}(x) = \sum_{n\geq 2} (n-1)n^{n-2} \frac{x^n}{n!}$. This formula can be obtained from [8, Theorem 7] also.

4.2. The Möbius coinvariant of a complete bipartite graph

We review a bipartite analogue $H_{m,n}(x,y)$ of Hermite polynomials. For a partial matching π in $K_{m,n}$, let $a(\pi)$ (*resp.* $b(\pi)$) be the number of vertices in $[m]$ (*resp.* $[n]$) of degree 0. Then $H_{m,n}(x,y)$ is defined to be a generating function for partial matchings in $K_{m,n}$, *i.e.*, $H_{m,n}(x,y) = \sum_{\pi} x^{a(\pi)}y^{b(\pi)}$ where the sum is over all partial matchings π in $K_{m,n}$, with $H_{m,0}(x,y) = x^m$, $H_{0,n}(x,y) = y^n$, and $H_{m,-1}(x,y) = H_{-1,m}(x,y) = 0$. Then $H_{m,n}(x,y)$ has the following explicit form

$$H_{m,n}(x,y) = \sum_{k\geq 0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k! x^{m-k} y^{n-k}. \tag{3}$$

Theorem 4.4. *[20, Theorem 5.14] The Möbius coinvariant $\mu^{\perp}(K_{m+1,n+1})$ of a complete bipartite graph $K_{m+1,n+1}$ with $m,n \geq 0$ equals $mn \cdot H_{m-1,n-1}(n,m)$. Hence, we obtain*

$$\mu^{\perp}(K_{m+1,n+1}) = \sum_{k=0}^{\min(m-1,n-1)} \binom{m}{k}\binom{n}{k} k!(m-k)(n-k)n^{m-1-k}m^{n-1-k} \tag{4}$$

by evaluating $H_{m-1,n-1}(n,m)$ using (3)

Several initial values of $\mu^{\perp}(K_{m+1,m+1})$ are given below:

m	1	2	3	4	5	6
$\mu^{\perp}(K_{m+1,m+1})$	1	20	1071	107104	17201225	4053135456

Now, we need the following lemma that gives a bijective proof for the number of the spanning trees in $K_{m,n}$ using, for example, *function graphs*. For a function $f : A \to B$ where A is a non-empty subset of a finite set B, the *graph* G_f of f is a directed graph on the vertex set B with directed edges (a,b) iff $f(a) = b$.

Lemma 4.5. *Let $K_{m,n}$ be the complete bipartite graph with a bipartition $[m] \cup [\overline{n}]$. Then there is a bijection from the set of all spanning trees of $K_{m,n}$ to the set of all functions $f : \{2, \cdots, m, \overline{2}, \cdots, \overline{n}\} \to \{1, \cdots, m, \overline{1}, \cdots, \overline{n}\}$ such that*

(i) if $i \in \{2, \cdots, m\}$, then $f(i) \in [\overline{n}]$, and

(ii) if $i \in \{\overline{2}, \cdots, \overline{n}\}$, then $f(i) \in [m]$.

Proof. We will sketch a bijection from [7, Theorem 2.1]. Given a spanning tree T in $K_{m,n}$, direct all edges towards the vertex 1. Then, there is a unique directed path from $\overline{1}$ to 1 on T. Recall that $2 < \cdots < m < \overline{2} < \cdots < \overline{n}$. Let $r_0 = \overline{1}$ and define r_i $(i > 0)$ to be the maximum vertex on the path strictly between r_{i-1} and 1, recursively. Let l_i be the vertex immediately after r_{i-1} on the path. Now, delete the edge (r_{i-1}, l_i) and add a new edge (r_i, l_i). Then the resulting directed graph is the graph of the desired function f. One can check that the correspondence $T \mapsto f$ is bijective. □

We present the following theorem as a bipartite analogue of Lemma 4.2, which will be used to give a combinatorial interpretation for $\mu^{\perp}(K_{m+1,n+1})$.

Theorem 4.6. *The number of B-edge-rooted forests in $K_{m,n}$ consisting of $k+1$ components with a given bi-root and a given collection of k edge-roots is $n^{m-1-k}m^{n-1-k}$.*

Proof. Without loss of generality, we will assume that the given bi-root is $\{1,\overline{1}\}$, and the given edge-roots are $2\overline{2}, 3\overline{3}, \ldots, (k+1)\overline{(k+1)}$. We will construct a bijection between the set of the B-edge-rooted forests in $K_{m,n}$ with these given roots and the set of the functions $f : \{k+2, \cdots, m, \overline{k+2}, \cdots, \overline{n}\} \to \{1, \cdots, m, \overline{1}, \cdots \overline{n}\}$ such that

(i) if $i \in \{k+2, \cdots, m\}$, then $f(i) \in [\overline{n}]$, and

(ii) if $i \in \{\overline{k+2}, \cdots, \overline{n}\}$, then $f(i) \in [m]$.

Take a B-edge-rooted forest F in $K_{m,n}$ with these roots. Deleting the edge-roots from F yields $2k$ vertex-rooted trees (with no change in the original bi-rooted component). Direct all edges in each of these $2k$ trees towards the vertex-root to create directed trees, and apply Lemma 4.5 to the bi-rooted component of F. These operations produce a directed graph which is the graph of a desired function f satisfying (i) and (ii).

We will show that the correspondence given by $F \mapsto f$ is a bijection by describing its inverse. Indeed, suppose f satisfies (i) and (ii). For a vertex $x \in [m] \cup [\overline{n}]$, let C_x denote the component of the graph G_f of f containing x. For $X = \{2, \cdots, k+1\} \cup \{\overline{2}, \cdots, \overline{k+1}\}$, it is easy to check that $C_x \neq C_y$ if $x, y \in X$ and $x \neq y$, and that C_x is acyclic for every $x \in X$. Add the given edge-roots to $\cup_{x \in X} C_x$ to create k edge-rooted trees. Apply Lemma 4.5 to the restriction of f to $\{k+2, \cdots, m, \overline{k+2}, \cdots, \overline{n}\} \setminus \cup_{x \in X} V(C_x)$ to create a tree containing the vertices $\{1, \overline{1}\}$, and designate them as the bi-root. This bi-rooted tree together with the k edge-rooted trees is a B-edge-rooted forest F with the given roots. One can check that the map $f \mapsto F$ is the desired inverse. $\square$

Example 4.7. This example illustrates the proof of the above theorem. Consider a B-edge-rooted forest in Figure 3 with one bi-rooted component and one edge-rooted component. First, delete the edge-root $2\overline{2}$, and we get $f(\overline{3}) = 2, f(\overline{4}) = 2, f(3) = \overline{2}$. Next, applying Lemma 4.5 to the bi-rooted component (with the bi-roots $\{1, \overline{1}\}$), we see that $f(4) = \overline{5}, f(\overline{5}) = 5, f(5) = \overline{6}, f(\overline{6}) = 4, f(6) = \overline{6}, f(\overline{7}) = 6$. These assignments give the desired function. Figure 4 is the graph of this f.

The following theorem is another main result of this section providing a combinatorial interpretation of Theorem 4.4 via B-edge-rooted forests in a complete bipartite graph.

Theorem 4.8. *The k-th term in the expression for*

$$\mu^{\perp}(K_{m+1,n+1}) = \sum_{k=0}^{\min(m-1,n-1)} \binom{m}{k}\binom{n}{k} k!(m-k)(n-k)n^{m-1-k}m^{n-1-k}$$

equals the number of the B-edge-rooted forests in $K_{m,n}$ with $k+1$ components (i.e., the number of the B-edge-rooted forests in $K_{m,n}$ with one bi-rooted tree and k edge-rooted trees).

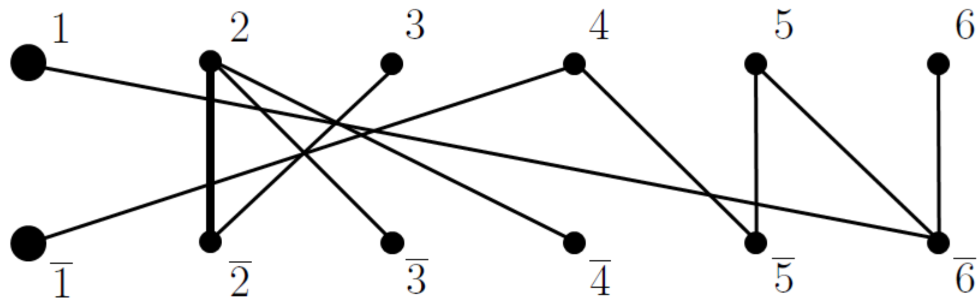


Figure 3: a B-edge-rooted forest

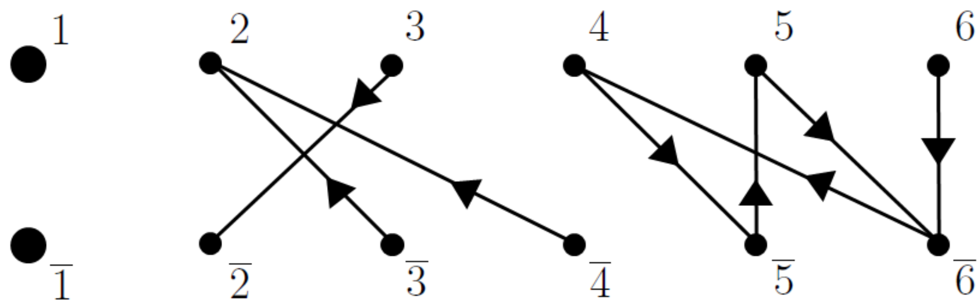


Figure 4: the function graph G_f

Proof. By Theorem 3.4, $\mu^{\perp}(K_{m+1,n+1})$ equals the number of the B-edge-rooted forests in $K_{m,n}$. Since the number of components in a B-edge-rooted forest on $K_{m,n}$ is at most $\min(m-1, n-1)+1$, it suffices to show that the number of the B-edge-rooted forests in $K_{m,n}$ with $k+1$ components equals the k-th term in the above formula

$$\binom{m}{k}\binom{n}{k}k!(m-k)(n-k)n^{m-1-k}m^{n-1-k}.$$

There are $\binom{m}{k}\binom{n}{k}k!$ ways to choose k-matchings in $K_{m,n}$ which we use as edge-roots for k edge-rooted components. There are $(m-k)(n-k)$ pairs of vertices each consisting of one vertex from each bipartite set, disjoint from a given k-matching. Each of these pairs is used as a bi-root for a bi-rooted component. Thus, the proof follows from Theorem 4.6. $\square$

We can obtain an exponential generating function for $\mu^{\perp}(K_{m+1,n+1})$ from its combinatorial interpretation. Since the number of all bi-rooted spanning trees in $K_{s,t}$ is $(st)s^{t-1}t^{s-1} = s^t t^s$, an exponential generating function for this sequence is

$$\overline{B}(x,y) = \sum_{s,t\geq 1} s^t t^s \frac{x^s y^t}{s!t!}.$$

Since the number of all edge-rooted spanning trees in $K_{s,t}$ is $(s+t-1)s^{t-1}t^{s-1}$, an exponential generating function for this sequence is

$$\overline{T}(x,y) = \sum_{s,t\geq 1} (s+t-1)s^{t-1}t^{s-1}\frac{x^s y^t}{s!t!}.$$

Therefore, an exponential generating function for $\mu^{\perp}(K_{m+1,n+1})$ is

$$\sum_{m,n\geq 0} \mu^{\perp}(K_{m+1,n+1})\frac{x^m y^n}{m!n!} = \overline{B}(x,y)\exp\left(\overline{T}(x,y)\right).$$

5. Homology of $I(K_{m+1,n+1})$

In this section, we present a homology basis for $I(K_{m+1,n+1})$. This basis is indexed by the set of all B-edge-rooted forests in $K_{m,n}$ introduced in section 3.2. In section 5.2, this set is shown to be isomorphic as a poset to the indexing set for the homology basis of a matroid complex given by A. Björner [1, Theorem 7.8.4].

5.1. Preliminaries

The *join* $K * K'$ of simplicial complexes K and K' with mutually disjoint vertex sets is defined by

$$K * K' = \{\tau \cup \tau' : \tau \in K \text{ and } \tau' \in K'\}.$$

Refer to [18] or [19] for details concerning join. If K and K' are homeomorphic to spheres of dimensions i and j, then $K*K'$ is homeomorphic to a sphere of dimension $i+j+1$. If $K*K'$ is a subcomplex of a complex A, and if σ is an isomorphism on A, then $\sigma(K*K') = \sigma(K)*\sigma(K')$.

For a simplicial complex K, its *exterior face ring* $\Lambda(K)$ is isomorphic to the simplicial chain complex of K as a $\mathbb{Z}$-module. As for the boundary operator ∂ of $\Lambda(K)$, one can check that if τ and τ' are disjoint with $\tau \cup \tau' \in K$, then we have

$$\partial(\tau \wedge \tau') = \partial\tau \wedge \tau' + (-1)^{|\sigma|}\tau \wedge \partial\tau'\,. \tag{5}$$

Note that, if σ is an automorphism of K, then σ induces a ring automorphism of $\Lambda(K)$ and an automorphism of the reduced homology groups of K. In describing a homology basis of K, we may work with its exterior face ring $\Lambda(K)$ rather than simplicial chain complex.

For a simplicial complex S that is homeomorphic to d-dimensional sphere, the cycle z in $\tilde{H}_d(S)$, which is unique up to sign, is called a *fundamental cycle* if every oriented facet of S appears in z with coefficient 1 or -1. When S is a join of spheres, its fundamental cycle can be described as follows. Given a finite set E with $|E| \geq 2$, let $D(E)$ denote the set of all proper subsets of E so that as a simplicial complex $D(E)$ is homeomorphic to $S^{|E|-2}$. For each $i = 1, \cdots, t$, let $E_1, E_2, \ldots, E_t$ be disjoint finite sets. Then $S_1 := D(E_1), S_2 := D(E_2), \ldots, S_t := D(E_t)$ are disjoint spheres. Let z_{S_i} denote a fundamental cycle of S_i for each $i = 1, \cdots, t$. For the join $S := S_1 * S_2 * \cdots * S_t$, let

$$z_S = \bigwedge_{1\leq i\leq t} z_{S_i}. \tag{6}$$

By applying Eq. (5) repeatedly, we see that z_S is a cycle of S. Moreover, it follows from the construction of S that z_S is a fundamental cycle of S.

5.2. A partial order on $\mathcal{F}_e^B$

Let $(F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B$, and assume $\mathbf{e} = \{e_1, \ldots, e_d\}$. For each $e_i \in \mathbf{e}$, let $\nu(e_i)$ be the vertex of the edge e_i that is farther from the smallest vertex in the component containing e_i, and let $\mathbf{v} = \{\nu(e_1), \ldots, \nu(e_d)\}$. Define $\mathcal{F}_v^B$ to be the collection of all $(F, \mathbf{b}, \mathbf{v})$ thus obtained. As we shall see, $\mathcal{F}_e^B$ and $\mathcal{F}_v^B$ can be made isomorphic *as posets* via the isomorphism $\mathcal{V} : \mathcal{F}_e^B \to \mathcal{F}_v^B$ given by $\mathcal{V}(F, \mathbf{b}, \mathbf{e}) = (F, \mathbf{b}, \mathbf{v})$. Moreover, we may identify $\mathcal{F}_v^B$ and $\mathcal{T}^0$ by Theorem 3.4. In particular, an element of $\mathcal{F}_v^B$ may be regarded as representing a spanning tree of $K_{m+1,n+1}$, or as a facet of $I(K_{m+1,n+1})$. We give partial orders Ω_e^B and Ω_v^B on $\mathcal{F}_e^B$ and $\mathcal{F}_v^B$, respectively, as follows. First, define Ω_e^B by $(F', \mathbf{b}', \mathbf{x}') \leq (F, \mathbf{b}, \mathbf{x})$ iff

(i) $\mathbf{b} = \mathbf{b}'$ and F' is a proper subgraph of F, or

(ii) $(F, \mathbf{b}) = (F', \mathbf{b}')$ and for each $x_i \in \mathbf{x}$ and $x_i' \in \mathbf{x}'$, x_i' lies on the path from x_i to the smallest vertex of the component containing x_i and x_i'.

The partial order Ω_v^B can be defined similarly. The following lemma is immediate from the definition of $\mathcal{V}$.

Lemma 5.1. *Suppose $(F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B$. Then $\mathcal{V}(F, \mathbf{b}, \mathbf{e})$ is the Ω_v^B-largest element in the set of all $(F, \mathbf{b}, \mathbf{v}) \in \mathcal{F}_v^B$ such that every vertex in $\mathbf{v}$ is a vertex of an edge in $\mathbf{e}$.* $\square$

5.3. The homology group of $I(K_{m+1,n+1})$

For each $(F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B(K_{m,n})$, we construct a subcomplex $S_{F,\mathbf{b},\mathbf{e}}$ of $I(K_{m+1,n+1})$ as follows. Again, for a finite set E, let $D(E)$ be the set of all proper subsets of E so that $D(E)$ is homeomorphic to the sphere $S^{|E|-2}$.

First, consider the bi-root $\mathbf{b} = \{b_1, \overline{b_2}\}$ where $b_1 \in [m]$ and $\overline{b_2} \in [\overline{n}]$. Let P be the unique path in F from b_1 to $\overline{b_2}$, and let Q be the cycle $P \to 0 \to \overline{0} \to b_1$. We define $S_{\mathbf{b}} := D(E(Q))$. Hence, $S_{\mathbf{b}}$ is a subcomplex of $I(K_{m+1,n+1})$ homeomorphic to the sphere S^{2l+1} where $2l - 1$ is the length of P.

Second, for each $e = v_1\overline{v_2} \in \mathbf{e}$ $(v_1 \in [m], \overline{v_2} \in [\overline{n}])$, define $S_e := D(\{e, 0\overline{v_2}, v_1\overline{0}\})$. Then S_e is a subcomplex of $I(K_{m+1,n+1})$ homeomorphic to the 1-dimensional sphere S^1.

Lastly, take an edge $e = u_1\overline{u_2}$ $(u_1 \in [m], \overline{u_2} \in [\overline{n}])$ in $E(F) \setminus (\mathbf{e} \cup P)$. If u_1 lies farther than $\overline{u_2}$ from a root (vertex or edge) in the component containing e, define $S_e := D(\{e, u_1\overline{0}\})$. If $\overline{u_2}$ lies farther than u_1, we similarly define $S_e := D(\{e, \overline{u_2}0\})$. Then, S_e is a subcomplex of $I(K_{m+1,n+1})$ homeomorphic to the 0-dimensional sphere S^0.

Finally, define $S_{F,\mathbf{b},\mathbf{e}}$ to be the join of $S_{\mathbf{b}}$, all of S_e for $e \in \mathbf{e}$, and all of S_e for $e \in E(F) \setminus (\mathbf{e} \cup P)$.

Proposition 5.2. *For every $(F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B(K_{m,n})$, the sphere $S_{F,\mathbf{b},\mathbf{e}}$ is a full-dimensional subcomplex of $I(K_{m+1,n+1})$.*

Proof. It suffices to show that for any given $(F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B$, every facet of $S_{F,\mathbf{b},\mathbf{e}}$ is a spanning tree in $K_{m+1,n+1}$. First, every facet of $S_{\mathbf{b}}$ is Q minus an edge, hence is a path containing all verticies of Q which, in particular, contains both 0 and $\overline{0}$.

Next, assume $\mathbf{e} = \{e_1, \ldots, e_d\}$. Since each e_i has no common vertex with Q, it is easy to see that every facet of $S_{\mathbf{b}} * S_{e_i}$ is a tree on the vertex set $V(Q) \cup V(e_i)$. Further, since no two edges in $\mathbf{e} = \{e_1, \ldots, e_d\}$ have a common vertex, it follows that every facet of $S_{\mathbf{b}} * S_{e_1} * \cdots * S_{e_d}$ is a tree on the vertex set $V(Q) \cup V(e_1) \cup \cdots \cup V(e_d)$.

Finally, let $e \in E(F) \setminus (\mathbf{e} \cup P)$ be an edge incident to $\mathbf{e} \cup P$. Note that one end of e is a "new" vertex x that is not in $V(Q) \cup V(e_1) \cup \cdots \cup V(e_d)$. Hence it follows that each facet of $S_{\mathbf{b}} * S_{e_1} * \cdots * S_{e_d} * S_e$ is a tree on $V(Q) \cup V(e_1) \cup \cdots \cup V(e_d) \cup \{x\}$. One can make similar observations for any edge in $E(F) \setminus (\mathbf{e} \cup P \cup \{e\})$ that is incident to $\mathbf{e} \cup P \cup \{e\}$. Repeating this process, we see that every facet of $S_{F,\mathbf{b},\mathbf{e}}$ is a spanning tree in $K_{m+1,n+1}$. □

For $(F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B(K_{m,n})$, let $z_{F,\mathbf{b},\mathbf{e}}$ denote a fundamental cycle of $S_{F,\mathbf{b},\mathbf{e}}$ constructed as in Eq. (6). We will proceed to show that the set of these $z_{F,\mathbf{b},\mathbf{e}}$ forms a basis for the top reduced homology group of $I(K_{m+1,n+1})$.

To this end, we recall a known basis [1, Theorem 7.7.2 and 7.8.4] for this homology group. It was shown that there is a basis $Z_v := \{z_{F,\mathbf{b},\mathbf{v}} \mid (F, \mathbf{b}, \mathbf{v}) \in \mathcal{F}_v^B\}$ of $\tilde{H}_{m+n}(I(K_{m+1,n+1}))$ such that the coefficient of the oriented facet for $(F, \mathbf{b}, \mathbf{v})$ in $z_{F,\mathbf{b},\mathbf{v}}$ is 1, and that of any other element of $\mathcal{F}_v^B$ is zero. Thus, when we express $z_{F,\mathbf{b},\mathbf{e}} \in Z_e$ as a linear combination of the elements in Z_v, the coefficient of $z_{F',\mathbf{b}',\mathbf{v}}$ is ± 1 if $(F', \mathbf{b}', \mathbf{v})$ gives a facet of $S_{F,\mathbf{b},\mathbf{e}}$.

By the proof of the above proposition, if $(F', \mathbf{b}', \mathbf{v})$ is a facet of $S_{F,\mathbf{b},\mathbf{e}}$, then one can check $(F', \mathbf{b}') = (F, \mathbf{b})$. Moreover, the elements in $\mathbf{v}$ are vertices of the edges in $\mathbf{e}$. From this observation and Lemma 5.1, it follows that if $(F, \mathbf{b}, \mathbf{v}) \in \mathcal{F}_v^B$ is a facet of $S_{F,\mathbf{b},\mathbf{e}}$, then

$$(F, \mathbf{b}, \mathbf{v}) \leq \mathcal{V}(F, \mathbf{b}, \mathbf{e}) \text{ in } \mathcal{F}_v^B. \tag{7}$$

Theorem 5.3. *The set of the cycles $Z_e := \{z_{F,\mathbf{b},\mathbf{e}} : (F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B(K_{m,n})\}$ forms a basis for $\tilde{H}_{m+n}(I(K_{m+1,n+1}))$.*

Proof. Let M be a matrix representing each $z_{F,\mathbf{b},\mathbf{e}} \in Z_e$ as a linear combination of elements in Z_v. To show that M is a transition matrix from Z_e to Z_v, it suffices to show that M is an upper triangular matrix whose diagonal entries are ± 1.

Suppose that the columns of M are indexed by the elements of $\mathcal{F}_e^B$ in a list that preserves the partial order Ω_e^B, and the rows of M indexed by the image of this list under $\mathcal{V}$. Note that the rows of M are indexed by the elements of $\mathcal{F}_v^B$ in a list that preserves Ω_v^B. The diagonal entries of M are the coefficients of $\mathcal{V}(F, \mathbf{b}, \mathbf{e})$ in $z_{F,\mathbf{b},\mathbf{e}}$ for $(F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B$, and they are clearly ± 1. From Eq. (7) and the definition of Ω_v^B, it follows that the only non-zero entries in the column indexed by $(F, \mathbf{b}, \mathbf{e})$ are on or above the row indexed by $\mathcal{V}(F, \mathbf{b}, \mathbf{e})$, as desired. □

5.4. A group action on the homology group of $I(K_{m+1,n+1})$

We will describe the action of $S_m \times S_n$ on $\tilde{H}_{m+n}(I(K_{m+1,n+1}))$ induced by that on $K_{m,n}$. Every $\sigma \in S_m \times S_n$ induces a permutation on the set of all subgraphs of $K_{m,n}$. For any subset $\mathbf{e} = \{e_1, \ldots, e_d\}$ of $E(K_{m,n})$ and $\mathbf{b} = \{b_1, \overline{b_2}\} \in V(K_{m,n})$, define $\sigma(\mathbf{e}) := \{\sigma(e_1), \ldots, \sigma(e_d)\}$ and $\sigma(\mathbf{b}) := \{\sigma(b_1), \sigma(\overline{b_2})\}$. Then we have the action of $S_m \times S_n$ on $\mathcal{F}_e^B$ defined by $\sigma(F, \mathbf{b}, \mathbf{e}) := (\sigma(F), \sigma(\mathbf{b}), \sigma(\mathbf{e}))$. We also note that since $\sigma \in S_m \times S_n$ induces an automorphism of $I(K_{m+1,n+1})$, it also induces a (ring) automorphism of $\Lambda(I(K_{m+1,n+1}))$.

For $(F, \mathbf{b}, \mathbf{e}) \in \mathcal{F}_e^B(K_{m,n})$, we write (with the sign to be determined),

$$z_{F,\mathbf{b},\mathbf{e}} = \pm z_{\mathbf{b}} \wedge \big(\bigwedge_{e \in \mathbf{e}} z_e\big) \wedge \Big(\bigwedge_{e \in E(F) \setminus (\mathbf{e} \cup P)} z_e\Big)$$

where $z_{\mathbf{b}} := z_{S_{\mathbf{b}}}$, $z_e := z_{S_e}$ for $e \in \mathbf{e}$, and $z_e := z_{S_e}$ for $e \in E(F) \setminus (\mathbf{e} \cup P)$.

Now, we determine the sign of $z_{F,\mathbf{b},\mathbf{e}}$ as follows. Let P be the unique path in F with $\mathbf{b}$ as end points written as $b_1 = p_1 \to \cdots \to p_{2l-1} \to p_{2l} = \overline{b_2}$ so that $p_{\text{odd}} \in [m]$ and $p_{\text{even}} \in [\overline{n}]$. Note that an edge set

$$T_P := (E(Q) \setminus 0\overline{b_2}) \cup \{\overline{0}u \mid u \in [m], u \notin V(P)\} \cup \{0\overline{v} \mid \overline{v} \in [\overline{n}], \overline{v} \notin V(P)\}$$

is a facet of $S_{F,\mathbf{b},\mathbf{e}}$. Note that $\sigma(T_P) = T_{\sigma(P)}$ for $\sigma \in S_m \times S_n$. The corresponding element of T_P in the chain group is

$$t_P := 0\overline{0} \wedge \big(1v_P(1) \wedge 2v_P(2) \wedge \cdots \wedge mv_P(m)\big) \wedge \big(\overline{1}u_P(\overline{1}) \wedge \overline{2}u_P(\overline{2}) \wedge \cdots \wedge \overline{n}u_P(\overline{n})\big)$$

where $v_P(i) = \overline{0}$ if i is b_1 or i is not a vertex of P with $v_P(i) = p_{\text{even}}$ otherwise, and $u_P(i) = \overline{0}$ if $\overline{j}$ is not a vertex of P with $u_P(\overline{j}) = p_{\text{odd}}$ otherwise. Now, we choose the sign of $z_{F,\mathbf{b},\mathbf{e}}$ so that the coefficent of t_P is positive.

Example 5.4. Consider a B-edge-rooted forest $(F, \mathbf{b}, \mathbf{e})$ in Figure 2(b). The path P equals $2 = p_1 \to p_2 \to p_3 \to p_4 = \overline{1}$, and $T_P = \{0\overline{0}, \overline{0}2, 2\overline{2}, \overline{2}1, 1\overline{1}\} \cup \{\overline{0}3, \overline{0}4, \overline{0}5\} \cup \{0\overline{3}, 0\overline{4}, 0\overline{5}\}$. Note that $0\overline{1}$ is missing (see Figure 5). Then $v_P(1) = \overline{2}, v_P(2) = v_P(3) = v_P(4) = v_P(5) = \overline{0}$, and $u_P(\overline{1}) = 1, u_P(\overline{2}) = 2, u_P(\overline{3}) = u_P(\overline{4}) = u_P(\overline{5}) = 0$. Hence,

$$t_P = 0\overline{0} \wedge \big(1\overline{2} \wedge 2\overline{0} \wedge 3\overline{0} \wedge 4\overline{0} \wedge 5\overline{0}\big) \wedge \big(\overline{1}1 \wedge \overline{2}2 \wedge \overline{3}0 \wedge \overline{4}0 \wedge \overline{5}0\big).$$

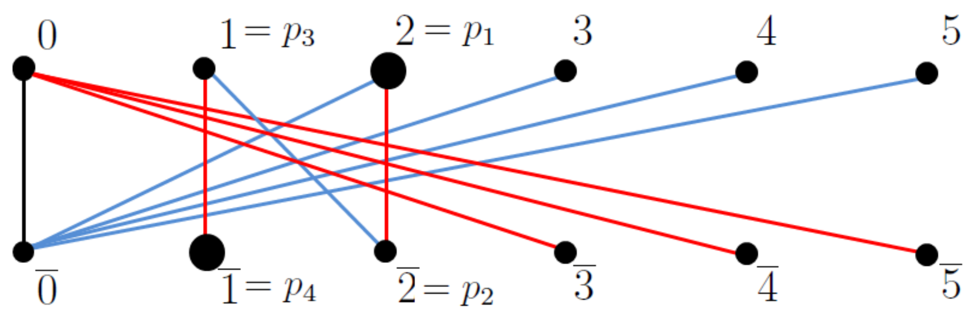


Figure 5: the spanning tree T_P in $S_{F,\mathbf{b},\mathbf{e}}$

Theorem 5.5. *The action of $S_m \times S_n$ as a subgroup of $S_{m+1} \times S_{n+1}$ on $\tilde{H}_{m+n}(I(K_{m+1,n+1}))$ is isomorphic to the action on $\mathcal{F}_e^B(K_{m,n})$ tensored with the sign representation* sgn.

Proof. Take $\sigma \in S_m \times S_n$. Since

$$\begin{aligned}
\sigma(z_{F,\mathbf{b},\mathbf{e}}) &= \pm\sigma(z_{\mathbf{b}}) \wedge \big(\bigwedge_{e\in\mathbf{e}} \sigma(z_e)\big) \wedge \big(\bigwedge_{e\in E(F)\setminus(\mathbf{e}\cup P)} \sigma(z_e)\big) \\
&= \pm z_{\sigma(\mathbf{b})} \wedge \big(\bigwedge_{e\in\sigma(\mathbf{e})} z_e\big) \wedge \big(\bigwedge_{e\in E(\sigma(F))\setminus(\sigma(\mathbf{e})\cup\sigma(P))} z_e\big)\,,
\end{aligned}$$

we have $\sigma(z_{F,\mathbf{b},\mathbf{e}}) = \pm z_{\sigma(F,\mathbf{b},\mathbf{e})}$. Moreover, we see that

$$\begin{aligned}
\sigma(t_P) &= \sigma\Big(0\overline{0} \wedge \big(1v_P(1) \wedge \cdots \wedge mv_P(m)\big) \wedge \big(\overline{1}u_P(\overline{1}) \wedge \cdots \wedge \overline{n}u_P(\overline{n})\big)\Big) \\
&= 0\overline{0} \wedge \big(\sigma(1)\sigma(v_P(1)) \wedge \cdots \wedge \sigma(m)\sigma(v_P(m))\big) \wedge \big(\sigma(\overline{1})\sigma(u_P(\overline{1})) \wedge \cdots \wedge \sigma(\overline{n})\sigma(u_P(\overline{n}))\big) \\
&= \mathrm{sgn}(\sigma)(0\overline{0} \wedge \big(1v_{\sigma(P)}(1) \wedge \cdots \wedge mv_{\sigma(P)}(m)\big) \wedge \big(\overline{1}u_{\sigma(P)}(\overline{1}) \wedge \cdots \wedge \overline{n}u_{\sigma(P)}(\overline{n})\big) \\
&= \mathrm{sgn}(\sigma)t_{\sigma(P)}
\end{aligned}$$

where the third equality follows because if $\sigma(x) = y$, then $\sigma(v_P(x)) = v_{\sigma(P)}(y)$ and $\sigma(u_P(x)) = u_{\sigma(P)}(y)$. Therefore, we conclude that $\sigma(z_{F,\mathbf{b},\mathbf{e}}) = \mathrm{sgn}(\sigma)\ z_{\sigma(F,\mathbf{b},\mathbf{e})}$. □

6. Möbius coinvariants of bi-coned graphs

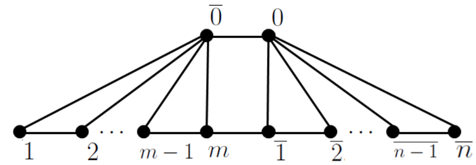
Our combinatorial interpretation for $\mu^\perp(K_{m+1,n+1})$ can be applied to the Möbius coinvariant of *bi-coned graphs*, a bipartite analogue of *coned graphs*. A bi-coned graph is defined as follows. Let $G = (V(G), E(G))$ be a graph with $n-1$ vertices, and $V(G) = U \uplus \overline{U}$ with $0 < |U| < n-1$. The *bi-coned graph* G^U on G is a graph obtained from G by adding two vertices 0 and $\overline{0}$ and edges $0\overline{u}$ for $\overline{u} \in \overline{U}$ and $\overline{0}u$ for $u \in U$. Note that G^U is a connected graph with $n+1$ vertices. For example, $(K_{m,n})^U$ with $U = [m]$ is $K_{m+1,n+1}$.

Define a bi-rooted tree in G (with $V(G) = U \uplus \overline{U}$ as above) to be a tree with two root vertices, one in U and the other in $\overline{U}$. Now define a B-edge-rooted forest in G to be a spanning forest in G with exactly one component bi-rooted and the rest edge-rooted. One can check that the proof of Theorem 3.4 applies to a bi-coned graph G^U, showing that $\mu^\perp(G^U)$ is the number of B-edge-rooted forests in G.

For example, we will compute the Möbius coinvariant of the bi-coned graph on a path. Let P_{m+n-1} be a path of length $m+n-1$, its vertices labeled by $[m] \uplus [\bar{n}]$ (Figure 6(a)), and $(P_{n+m-1})^{[m]}$ its bi-coned graph (Figure 6(b)). Note that a bi-rooted tree in P_{m+n-1} is a path containing the edge $m\overline{1}$. Hence, a B-edge-rooted forest in P_{m+n-1} consists of one component that is a bi-rooted path from $a \in [m]$ to $\overline{n-b+1} \in [n]$ (for some $1 \le a \le m$ and $1 \le b \le n$) and other components that are edge-rooted paths contained in the path from 1 to $a-1$ or in the path from $\overline{n-b+2}$ to $\overline{n}$ (if $3 \le a \le m$ and $3 \le b \le n$). Note that $a \neq 2$ and $b \neq 2$ because otherwise edge-rooted paths would be needed, but cannot be defined. Since the number of edge-rooted forests in a path of length l is 2^{l-1} [14], we have

$$\begin{aligned}
\mu^\perp((P_{n+m-1})^{[m]}) &= \big(m + \sum_{a=3}^{m}(m-a+1)2^{a-3}\big)\big(n + \sum_{b=3}^{n}(n-b+1)2^{b-3}\big) \\
&= 2^{m-1}2^{n-1}.
\end{aligned}$$

(a) a path P_{m+n-1}



(b) a bi-coned graph $(P_{n+m-1})^{[m]}$

Figure 6: A path P_{m+n-1} and its bi-coned graph $(P_{n+m-1})^{[m]}$ for $[m]$

Finally, we suggest the following future works. One may generalize the definition of a bi-coned graph on G to allow $V(G) = U_1 \cup U_2$ where $U_1 \cap U_2$ may be *non-empty*. This generalization includes complete multipartite graphs. One may consider finding a combinatorial interpretation of the Möbius coninvariants for these graphs, for which a more careful study of the internal activities of spanning trees is needed. Also we leave a study of their h-vectors and Stanley's M-vector conjecture for these graphs as a future work. Refer to [15, 16] for related results concerning coned graphs.

Acknowledgement

The authors would like to thank Prof. A. Postnikov and Prof. B. Sturmfels for encouraging conversations and anonymous referees for their careful reading of our manuscript. The authors are partially supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(0450-20160023), and by Samsung Science and Technology Foundation under Project Number SSTF-BA1402-08.

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