Simplicial networks and effective resistance

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Abstract

We introduce the notion of effective resistance for a *simplicial network* (X, R) where X is a simplicial complex and R is a set of resistances for the top simplices, and prove two formulas generalizing previous results concerning effective resistance for resistor networks. Our approach, based on combinatorial Hodge theory, is to assign a unique harmonic class to a *current generator* σ, an extra top-dimensional simplex to be attached to X. We will show that the harmonic class gives rise to the *current* I_σ and the *voltage* V_σ for $X \cup \sigma$, satisfying Thomson's energy-minimizing principle and Ohm's law for simplicial networks.

The effective resistance R_σ of a current generator σ shall be defined as a ratio of the σ-components of V_σ and I_σ. By introducing *potential* for voltage vectors, we present a formula for R_σ via the inverse of the weighted combinatorial Laplacian of X in codimension one. We also derive a formula for R_σ via weighted high-dimensional tree-numbers for X, providing a combinatorial interpretation for R_σ. As an application, we generalize Foster's Theorem, and discuss various high-dimensional examples.

Keywords: effective resistance, simplicial network, combinatorial Laplacians, combinatorial Hodge theory, high-dimensional tree-numbers
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1. Introduction

A *simplicial network* (X, R) consists of a simplicial complex X of dimension d (> 0) and a set R of positive *resistances* for the d-dimensional simplices of X. Additional topological conditions for X will be assumed later as needed. A *current generator* σ is a d-dimensional simplex that is attached to X resulting in a (cell) complex $Y = X \cup \sigma$. The purpose of this paper is to introduce the notion of *effective resistance* R_σ of a current generator σ, and present its formulas and applications. Simplicial networks are a generalization of resistor networks, and the current work aims to extend classical results (see *e.g.* [15, 19]) concerning effective resistance for resistor networks.

Let us outline our approach to R_σ. Suppose a nonzero real number i_σ is assigned to a current generator σ. We will associate a unique cycle I_σ in the chain group $\mathcal{C}_d(Y; \mathbb{R})$, which we call the *current vector* induced by i_σ, as follows. Attach σ to an *acyclization* $\mathcal{A}(X)$ of

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X (see Section 2) to form a complex Z with rank 1 homology group in dimension d (see Section 3 for the definition of Z). By combinatorial Hodge theory [9], there is a unique *harmonic class* for Z determined by i_σ. This harmonic class is the desired I_σ when every element in R equals 1. Otherwise, a similar argument using a *weighted* chain complex for Z will produce I_σ (see Section 3.2). As we shall see, the *energy-minimizing* property of a harmonic class is a high-dimensional analogue of Thomson's Principle for currents in a resistor network. Also, we will define a *voltage vector* $V_\sigma \in \mathcal{C}_d(Y;\mathbb{R})$ by requiring Ohm's law [4] and the orthogonality of current and voltage vectors. In short, we have the current I_σ and voltage V_σ vectors for $Y = X \cup \sigma$ uniquely determined by a given nonzero current i_σ through σ. Now, we shall define R_σ as a ratio of the respective σ-components v_σ and i_σ of V_σ and I_σ.

We will present another definition of R_σ by introducing *potential* for voltage vectors (See (11) in Section 4). For a 1-dimensional potential theory, refer to [3]. Using this definition, we obtain a formula for R_σ via the inverse of the weighted combinatorial Laplacian for X in codimension 1 where the weights are given by the conductances $C = R^{-1}$ (regarding R as a diagonal matrix). This formula generalizes that of effective resistance for 1-dimensional networks via the inverse of the combinatorial Laplacian in dimension zero [18, 15].

We will obtain another formula for R_σ (Theorem 5.2) via weighted high-dimensional tree-numbers for X with the weights $C = R^{-1}$. We refer the readers to [5, 6, 13] for high-dimensional tree-numbers. This formula generalizes a well-known combinatorial interpretation [19] of effective resistance for resistor networks. For its application, we will derive a high-dimensional analogue of Foster's Theorem [8], and compute effective resistance for the standard simplexes (Example 5.5), the complete colorful complexes (Example 5.6), and the hypercubes (Example 5.7).

2. Preliminaries

In this section, we will review definitions regarding simplicial complexes and homology groups. Refer to [17] for further details. We will collect relevant definitions and results concerning combinatorial Hodge theory [7, 9, 12, 16] which are essential for our purpose.

2.1. Simplicial complexes, and boundary and coboundary operators

Let X be an (abstract) simplical complex with an ordered finite vertex set $[n] := \{1,\ldots,n\}$. The dimension of $\sigma \in X$ is $\dim\sigma = |\sigma| - 1$, and the dimension of X is $\dim X = \max\{\dim\sigma \mid \sigma \in X\}$. Let X_i denote the collection of all i-dimensional simplicies (i-faces) of X. The i-th skeleton of X is $X^{(i)} = \bigcup_{0\le j\le i} X_j$. In this paper, we allow X_d ($d = \dim X$) to be a multiset, generalizing the notion of *parallel edges*. (With this condition, X is a cell complex, and we will refer to X simply as a *complex*.)

The i-th *chain group of X with integer coefficients* is a free abelian group $\mathcal{C}_i = \mathcal{C}_i(X) \cong \mathbb{Z}^{|X_i|}$ generated by the *oriented* simplices $[\tau]$ for $\tau \in X_i$. Elements of i-th *chain group of X* called *i-chains*, and an i-chain $x \in \mathcal{C}_i$ may be represented as a formal sum $x = \sum_{\tau\in X_i} n_\tau[\tau]$ or as a column (vector) $x = (n_\tau)_{\tau\in X_i}$ depending on the context. The i-th *boundary operator*

$\partial_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$ is a $|X_{i-1}| \times |X_i|$ matrix given by, for each i-face $\tau = \{v_0, v_1, \ldots, v_i\}$ with $v_0 < v_1 < \cdots < v_i$,

$$\partial_i[\tau] = \sum_{j=0}^{i} (-1)^j [\tau - v_j].$$

Define $\partial_0 : \mathcal{C}_0 \to \mathcal{C}_{-1} \cong \mathbb{R}$ by $\partial_0[v] = 1$ for each $v \in X_0$. We have $\partial_i \partial_{i+1} = 0$ for all i. The collection $\{\mathcal{C}_i, \partial_i\}$ is called an (augmented) chain complex of X. We will write $\{\mathcal{C}_i(X), \partial_{X,i}\}$ to emphasize X. Use $\mathcal{C}_i(X;\mathbb{R})$ for chain groups *with real coefficients*.

The i-th *cochain* group of a simplicial complex X is

$$\mathcal{C}^i = \mathcal{C}^i(X) := \mathrm{Hom}(\mathcal{C}_i(X;\mathbb{Z}), \mathbb{Z}),$$

and its elements called i-*cochains*. Let us denote $\mathcal{C}^i(X;\mathbb{R})$ for chain groups *with real coefficients*. The i-th *coboundary operator* or *alternating difference operator* $\delta_i : \mathcal{C}^i \to \mathcal{C}^i$ is defined by, for $f \in \mathcal{C}^i$,

$$(\delta_i f)[\sigma] = \sum_{j=0}^{i+1} (-1)^j f([\sigma - v_j])$$

where $[\sigma] = [v_0, v_1, \ldots, v_i, v_{i+1}]$. In what follows, we may denote $f([\tau])$ by f_τ for $\tau \in X_i$ for convenience. When $i = 1$ and $i = 2$, the coboundary operators are called *gradient* and *curl*, respectively. Hence, the gradient of $f \in \mathcal{C}^1$ is given by

$$(\mathrm{grad}\ f)([a,b]) := \delta_1 f([a,b]) = f_b - f_a$$

and the curl of $f \in \mathcal{C}^2$ is given by

$$(\mathrm{curl}\ f)([a,b,c]) := \delta_2 f([a,b,c]) = f_{bc} - f_{ac} + f_{ab}.$$

For each i-face τ, we associate an element $[\tau] \in \mathcal{C}_i$ to the characteristic function $\chi_\tau \in \mathcal{C}^i$ defined by for an i-face τ', $\chi_\tau([\tau']) = 1$ if $\tau' = \tau$, and $\chi_\tau([\tau']) = 0$ otherwise. This association induces an isomorphism between $\mathcal{C}_i$ and $\mathcal{C}^i$, and hence we identify their elements by this isomorphism. We may regard $f = \sum_\tau f_\tau \chi_\tau \in \mathcal{C}^i$ as a formal sum $\sum_{\tau \in X_i} f_\tau[\tau]$ or as a column vector $(f_\tau)_{\tau \in X_i}$, which we will denote by f again. Moreover, the i-th coboundary operator δ_i is represented by the transpose $\partial^t_{i+1} : \mathcal{C}_i \to \mathcal{C}_{i+1}$ of the $(i+1)$-th boundary operator ∂_{i+1}. Hence, for our purpose, it will suffice to work with chain groups, boundary operators and their transpose for most of the topological and combinatorial invariants discussed in this paper.

2.2. Homology and cohomology groups, and acyclization

The elements of $\ker \partial_i$ and $\ker \partial^t_{i+1}$ are called i-cycles and i-cocycles, respectively. The i-th reduced homology group and cohomology group with *integer* coefficients $\mathbb{Z}$ are $\tilde{H}_i(X) = \ker \partial_i / \operatorname{im} \partial_{i+1}$, and $\tilde{H}^i(X) = \ker \partial^t_{i+1} / \operatorname{im} \partial^t_i$, respectively. We will write $\tilde{H}_i(X;\mathbb{R})$ ($\tilde{H}^i(X;\mathbb{R})$) for reduced (co)homology with *real* coefficients. Note $\operatorname{rk} \tilde{H}_i(X) = \operatorname{rk} \tilde{H}_i(X;\mathbb{R})$ for all i.

Given a chain complex $\{\mathcal{C}_i(X), \partial_{X,i}\}$ for a simplicial complex X of dimension d, we define an *acyclization* $\mathcal{A}(X)$ of X to be a chain complex $\mathcal{A}(X) = \{\mathcal{A}_i(X), \partial_{\mathcal{A}(X),i}\}$ for $-1 \le i \le d+1$ as follows. For $i \le d$, let $\mathcal{A}_i(X) = \mathcal{C}_i(X)$ and $\partial_{\mathcal{A}(X),i} = \partial_{X,i}$. The $(d+1)$-th chain group $\mathcal{A}_{d+1}(X)$ is free abelian of rank $c := \operatorname{rk} \tilde{H}_d(X)$ with a standard basis, and the $(d+1)$-th boundary operator for $\mathcal{A}(X)$ is an integer matrix of size $|X_d| \times c$ given by

$$\partial_{\mathcal{A}(X),\, d+1} = \begin{bmatrix} z_1 & z_2 & \dots & z_c \end{bmatrix} \tag{1}$$

where $\{z_1, \dots, z_c\}$ is a basis of $\tilde{H}_d(X) = \ker \partial_{X,d}$. Note that $\tilde{H}_i(\mathcal{A}(X)) = \tilde{H}_i(X)$ for $i < d$, and $\tilde{H}_d(\mathcal{A}(X)) = \tilde{H}_{d+1}(\mathcal{A}(X)) = 0$. Also, note that $\ker \partial^t_{\mathcal{A}(X),d+1} = (\ker \partial_{X,d})^\perp$ which we will refer to later. When $\tilde{H}_d(X) = 0$, we define $\mathcal{A}(X)$ to be the same as $\{\mathcal{C}_i(X), \partial_{X,i}\}$.

2.3. Combinatorial Hodge theory

Given a complex X, the i-th combinatorial Laplacian $\Delta_i = \Delta_{X,i} : \mathcal{C}_i(X, \mathbb{R}) \to \mathcal{C}_i(X, \mathbb{R})$ is defined by ([7])

$$\Delta_i = \partial_i^t \partial_i + \partial_{i+1} \partial_{i+1}^t .$$

The i-th *harmonic space* $\mathcal{H}_i(X)$ is $\ker \Delta_i$ and its elements are *i-harmonic classes*.

Regard $\mathcal{C}_i(X, \mathbb{R})$ and $\mathcal{C}^i(X, \mathbb{R})$ as $\mathbb{R}$-vector spaces endowed with a standard inner product $\langle\ ,\ \rangle$ such that the set of all oriented i-faces of X forms an orthonormal basis. From the orthogonal decomposition $\mathcal{C}_i(X, \mathbb{R}) = \mathcal{H}_i(X; \mathbb{R}) \oplus \operatorname{im} \partial_i^t \oplus \operatorname{im} \partial_{i+1}$ (refer to [9, Section 2]), one can deduce

$$\mathcal{H}_i(X) = \ker \partial_i \cap \ker \partial_{i+1}^t . \tag{2}$$

Hence, an i-harmonic class is both an i-cycle and an i-cocycle.

Also from the above decomposition for $\mathcal{C}_i(X; \mathbb{R})$ follows the main result of combinatorial Hodge theory: $\mathcal{H}_i(X)$ is isomorphic to $\tilde{H}_i(X; \mathbb{R})$ (and to $\tilde{H}^i(X; \mathbb{R})$) as $\mathbb{R}$-vector spaces for all i, where the isomorphism maps a harmonic class h to its (co)homology class $\overline{h}$.

The following *energy-minimizing property* of a harmonic class is a consequence of (2): For $h \in \mathcal{H}_i(X)$ and $x \in \overline{h}$,

$$\langle h, h \rangle \le \langle x, x \rangle . \tag{3}$$

This inequality is verified by the following facts: If $x = h + \partial_{i+1} y$ for some $y \in \mathcal{C}_{i+1}(X, \mathbb{R})$, then $\langle h, \partial_{i+1} y \rangle = \langle \partial_{i+1}^t h, y \rangle = 0$ because $h \in \ker \partial_{i+1}^t$. Similarly, if $x = h + \partial_i^t y$ for some $y \in \mathcal{C}^i(X, \mathbb{R})$, then $\langle h, \partial_i^t y \rangle = \langle \partial_i h, y \rangle = 0$ since $h \in \ker \partial_i$.

3. Simplicial networks and effective resistance

In this section, we define simplicial resistor networks as a generalization of resistor networks, and characterize current and voltage vectors for simplicial networks. We will also present the main definition of the paper, the effective resistance of a current generator in a simplicial network.

3.1. Simplicial networks

A *resistor network* is a finite graph where each edge is assigned a positive real number, called a *resistance*, and weighted by the corresponding *conductance*, the reciprocal of resistance. As a generalization, we define a *simplicial network* (X, R) as follows. A *simplicial network* (X, R) consists of a simplicial complex X of dimension d and a set R of *resistances* $r_\tau\,(> 0)$ for the d-faces $\tau \in X_d$. The *resistance matrix*, which we also denote by R, is a diagonal matrix whose diagonal entries are r_τ. We will refer to a simplicial network (X, R) as a *network* for short. The simplicial complex X will be regarded as the weighted simplicial complex each of whose d-faces is weighted by its *conductance* $c_\tau := 1/r_\tau$, and C is the *conductance matrix*, a diagonal matrix whose diagonal entries are c_τ.

Let (X, R) be a network of dimension d with $[n]$ as vertices. A *current generator* is a subset $\sigma \subset [n]$ with $|\sigma| = d + 1$ such that

$$\partial_d[\sigma] = -\partial_{X,d}\,(p) \text{ for some } p \in \mathcal{C}_d(X, \mathbb{R}).$$

Then X together with a current generator σ, which we denote by $Y = X \cup \sigma$, is again a d-dimensional complex with $Y_d = X_d \cup \{\sigma\}$ as a multiset. One can also deduce $\operatorname{rk} \tilde{H}_d(Y, \mathbb{R}) = \operatorname{rk} \tilde{H}_d(X, \mathbb{R}) + 1$ from the fact that $[\sigma] + p$ is a d-dimensional cycle in Y, but not in X.

Eventually, we will define two vectors $I_\sigma \in \ker \partial_{Y,d}$ and $V_\sigma \in (\ker \partial_{Y,d})^\perp$, called *current* and *voltage* vectors for Y, respectively, such that their restrictions I and V to X satisfy the Ohm's law $V = RI$. Then the effective resistance of the current generator σ will be defined as a ratio of the σ-components v_σ and i_σ of V_σ and I_σ, respectively. Details will follow subsequently.

An important characterization of I_σ will be given by an analogue of Thomson's Principle. For a 1-dimensional network (X, R), this principle states that if $I_\sigma = I + i_\sigma[\sigma]$ is a current for $Y = X \cup \sigma$, then I satisfies the *energy-minimizing* property, *i.e.*,

$$I^t R I \leq x^t R x \tag{4}$$

for any cycle of the form $x + i_\sigma[\sigma]$ in Y. This energy-minimizing property will be generalized for I_σ in a network of arbitrary dimension.

3.2. Harmonic class for a network

Given a network (X, R) of dimension d and a current generator σ, we define a chain complex $Z = \{Z_i, \partial_{Z,i}\}$ for $-1 \leq i \leq d + 1$, which represents, intuitively, the union of σ and an acyclization of X. (To avoid confusion concerning the notation Z_i, we will denote the cycle group as the kernel of a boundary operator throughout the paper.) Specifically, we have $Z_i = \mathcal{C}_i(Y) = \mathcal{C}_i(X)$ with $\partial_{Z,i} = \partial_{Y,i} = \partial_{X,i}$ for $i < d$, $Z_d = \mathcal{C}_d(Y) = \mathcal{C}_d(X) \oplus \mathbb{Z}$ with $\partial_{Z,d} = \partial_{Y,d}$, and $Z_{d+1} = \mathcal{A}_{d+1}(X)$ with $\partial_{Z,d+1}$ given by

$$\partial_{Z,\,d+1} = \begin{bmatrix} \partial_{\mathcal{A}(X),\,d+1} \\ 0 \cdots 0 \end{bmatrix} \tag{5}$$

where $\partial_{\mathcal{A}(X),\,d+1} = [z_1 \ldots z_c]$ as in (1), and the last row of 0's is indexed by σ. A routine computation shows $\tilde{H}_d(Z; \mathbb{R}) = \mathbb{R}$. Hence, $\mathcal{H}_d(Z) = \mathbb{R}$ by combinatorial Hodge theory.

Therefore, there is a unique d-harmonic class h for Z up to scalar multiplication. We may call this h the harmonic class of σ with respect to X. Note that h must have a nonzero σ-component. Otherwise, h would be a d-cycle in $\mathcal{A}(X)$, and its homology class $\overline{h}$ would be zero, a contradiction.

Next, we define a *weighted* chain complex of Z incorporating R into the chain complex $\{Z_i, \partial_{Z,i}\}$. Define R' to be the diagonal matrix $R' = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$. Let Q be a diagonal matrix satisfying $Q^2 = R'$. Define the weighted boundary operators $\hat{\partial}_i$ for Z by

$$\hat{\partial}_d = \partial_{Z,d} Q^{-1} \qquad \hat{\partial}_{d+1} = Q \partial_{Z,d+1}$$

and $\hat{\partial}_i = \partial_{Z,i}$ for all other i. Then we have $\hat{\partial}_i \hat{\partial}_{i+1} = 0$ for all i, and $\{Z_i, \hat{\partial}_i\}$ is the desired weighted chain complex. By (2), we obtain

$$\hat{\mathcal{H}}_d(Z) = \ker \hat{\partial}_d \cap \ker \hat{\partial}_{d+1}^t \tag{6}$$

where $\hat{\mathcal{H}}_d(Z)$ is the kernel of $\hat{\Delta}_{Z,d} := \hat{\partial}_d^t \hat{\partial}_d + \hat{\partial}_{d+1} \hat{\partial}_{d+1}^t$. Hence, every element $h \in \hat{\mathcal{H}}_d(Z)$ is of the form $h = Qw$ for a unique $w \in \ker \partial_{Z,\,d}$. Since $Z_d = \mathcal{C}_d(Y)$, we also have $h, w \in \mathcal{C}_d(Y; \mathbb{R})$.

Define $\hat{H}_d(Z; \mathbb{R}) := \ker \hat{\partial}_d / \operatorname{im} \hat{\partial}_{d+1}$. Then we have $\operatorname{rk} \tilde{H}_d(Z; \mathbb{R}) = \operatorname{rk} \hat{H}_d(Z; \mathbb{R}) = \operatorname{rk} \hat{\mathcal{H}}_d(Z)$, where the first equality follows from $\operatorname{rk} \hat{\partial}_i = \operatorname{rk} \partial_{Z,i}$ for all i and the second from combinatorial Hodge theory. Hence, $\hat{\mathcal{H}}_d(Z) = \mathbb{R}$, and its generator must have a nonzero σ-component by a similar reasoning as above.

3.3. Current and voltage vectors for a network

Let h_σ be the unique generator of $\hat{\mathcal{H}}_d(Z)$ with a given σ-component i_σ. This h_σ does not depend on the choice of an acyclization of X since a new acyclization is obtained by a change of basis of $\tilde{H}_d(X)$ and $\ker \hat{\partial}_{d+1}^t$ is invariant under this change. We define the current vector I_σ for $Y = X \cup \sigma$ to be the unique d-cycle in $\mathcal{C}_d(Y; \mathbb{R})$ satisfying

$$h_\sigma = Q I_\sigma \,.$$

If I denotes the restriction of I_σ to X so that $I_\sigma = I + i_\sigma[\sigma]$, then I_σ is characterized by

$$I_\sigma \in \ker \partial_{Y,d} \quad \text{and} \quad RI \in \ker \partial_{\mathcal{A}(X),\,d+1}^t = (\ker \partial_{X,\,d})^\perp \tag{7}$$

where the first condition follows from $\partial_{Z,\,d} = \partial_{Y,d}$, and the second from (6) and (5).

To justify the definition of I_σ, we check that I_σ satisfies a *generalized Thomson's Principle.* Given a cycle $w = x + i_\sigma[\sigma] \in \ker \partial_{Y,d}$, note that $I_\sigma - w = I - x$ is a d-cycle in X, and, therefore, an element of $\operatorname{im} \partial_{\mathcal{A}(X),d+1}$. By the definition of $\partial_{Z,d+1}$, we have $I_\sigma - w \in \operatorname{im} \partial_{Z,d+1}$. Since $\hat{\partial}_{d+1} = Q\partial_{Z,d+1}$, we also have $h_\sigma - Qw = Q(I_\sigma - w) \in \operatorname{im} \hat{\partial}_{d+1}$. Hence, Qw is in the homology class of h_σ, and $\langle h_\sigma, h_\sigma \rangle \leq \langle Qw, Qw \rangle$ by (3). From this inequality, one can easily deduce $I^t RI \leq x^t Rx$, which is the desired generalization of (4).

Next, we define the voltage vector V_σ for $Y = X \cup \sigma$ as follows. First we define the voltage V for X by the generalized Ohm's law $V = RI$. We define V_σ for Y to be the extension of V that is orthogonal to I_σ. Hence, $V_\sigma = V + v_\sigma[\sigma]$ is the unique vector satisfying

$$V = RI \quad \text{and} \quad V_\sigma \perp I_\sigma.$$

Note that v_σ is obtained from the following consequence of these conditions:

$$I^t RI + i_\sigma v_\sigma = 0\,. \tag{8}$$

Also from the above conditions for V_σ, one can deduce

$$V_\sigma = V + v_\sigma[\sigma] \in \ker \partial^t_{\mathcal{A}(Y),\, d+1} = (\ker \partial_{Y,d})^\perp,$$

which we will also verify after introducing *potential* in Section 4.

3.4. Effective resistance of a current generator

Now, we present the main definition of the paper. For a network (X, R) and a current generator σ, let $I_\sigma = I + i_\sigma[\sigma]$ and $V_\sigma = V + v_\sigma[\sigma]$ be the current and voltage vectors for $Y = X \cup \sigma$, respectively, determined by a non-zero current i_σ through σ.

Definition 3.1. We define the *effective resistance* R_σ of σ to be

$$R_\sigma = \left| \frac{v_\sigma}{i_\sigma} \right|.$$

Remarks: This is well-defined because changing i_σ to $m i_\sigma$ $(m \neq 0)$ results in mI_σ and mV_σ. Note from (8) that the product $i_\sigma v_\sigma$ is always negative. Hence, we may define $R_\sigma = -v_\sigma / i_\sigma$. We also note in passing that if $|i_\sigma| = 1$, then $R_\sigma = I^t RI$, i.e., *the effective resistance of a current generator σ is the energy created by a unit flow through σ.*

4. Effective resistance via simplicial potential

In this section, we introduce the notion of potential for voltage vectors, and present a formula for effective resistance R_σ for a current generator σ in a network (X, R) of dimension d via the inverse of the combinatorial Laplacian in codimension 1. A *potential* of an element $x \in \mathcal{C}^i(X; \mathbb{R})$ is an element $\phi \in \mathcal{C}^{i-1}(X; \mathbb{R})$ such that $x = \partial_i^t \phi$. For example, refer to [3] for 1-dimensional potential theory.

4.1. Potential for voltage vectors

Since $V = RI \in \ker \partial^t_{\mathcal{A}(X),\, d+1}$ and $\tilde{H}^d(\mathcal{A}(X); \mathbb{R}) = 0$, there is a $(d-1)$-cochain $\phi \in \mathcal{C}^{d-1}(X; \mathbb{R}) = \mathcal{C}^{d-1}(Y; \mathbb{R})$ such that

$$V = \partial^t_{X,\,d} \phi.$$

Hence, ϕ is a potential for V. It is important to note that a potential for V is also a potential for V_σ , i.e.,

$$V_\sigma = \partial^t_{Y,\,d} \phi \quad \text{whenever} \quad V = \partial^t_{X,\,d} \phi\,. \tag{9}$$

Indeed, the restriction of $\partial^t_{Y,d}\phi$ to X is $\partial^t_{X,d}\phi$ which equals V, and $\partial^t_{Y,d}\phi$ is orthogonal to I_σ because $\partial^t_{Y,d}\phi \in \ker \partial^t_{\mathcal{A}(Y),d+1} = (\ker \partial_{Y,d})^\perp$ and $I_\sigma \in \ker \partial_{Y,d}$. Hence $\partial^t_{Y,d}\phi = V_\sigma$ by the definition of V_σ. In particular, we have also shown

$$V_\sigma = V + v_\sigma[\sigma] \in \ker \partial^t_{\mathcal{A}(Y),d+1}, \tag{10}$$

which we will refer to later.

A potential $\phi \in \mathcal{C}^{d-1}(X;\mathbb{R})$ of V gives rise to another expression for the effective resistance R_σ of a current generator σ. Let ∂_σ denote $\partial_{Y,d}[\sigma]$, the column of $\partial_{Y,d}$ indexed by σ. Then, we obtain $v_\sigma = \partial^t_\sigma\phi$ from $V_\sigma = V + v_\sigma[\sigma] = \partial^t_{Y,d}\phi$. Further, suppose $[\sigma] = [v_0, \dots, v_d]$, and let $\sigma_j := \sigma - \{v_j\}$ for each j. Then we have $\partial^t_\sigma\phi = \sum_{j=0}^d (-1)^j \phi_{\sigma_j}$. Hence, for a nonzero i_σ, we obtain

$$R_\sigma = -\frac{v_\sigma}{i_\sigma} = -\frac{\partial^t_\sigma\phi}{i_\sigma} = -\frac{\sum_{j=0}^d (-1)^j \phi_{\sigma_j}}{i_\sigma}. \tag{11}$$

Note that this expression generalizes a definition of effective resistance by *potential difference*.

The following lemma characterizes a potential for V via a *generalized Kirchhoff's equation*. Define the weighted Laplacian for (X,R) in codimension 1 with the weights $C = R^{-1}$ to be

$$\hat{L} = \hat{L}_{d-1} = \partial_{X,d} C \partial^t_{X,d}$$

as an operator on $\mathcal{C}^{d-1}(X;\mathbb{R})$. Now, we define a generalized Kirchhoff's equation to be

$$\hat{L}\phi = -i_\sigma\, \partial_\sigma. \tag{12}$$

Lemma 4.1. *Let V be the voltage vector induced by a nonzero current i_σ through the current generator σ. Then a $(d-1)$-cochain $\phi \in \mathcal{C}^{d-1}$ is a potential for V if and only if ϕ is a solution of a generalized Kirchhoff's equation (12).*

Proof. Let $\phi \in \mathcal{C}^{d-1}$ be a potential for V. From $I_\sigma = I + i_\sigma[\sigma] \in \ker \partial_{Y,d}$, we get $\partial_{Y,d} I = -i_\sigma\, \partial_\sigma$. Now, (12) follows from $CV = I$ and $V = \partial^t_{X,d}\phi$ together with the fact that $\partial_{Y,d}$ restricts to $\partial_{X,d}$ on X. Conversely, suppose an element $\phi \in \mathcal{C}^{d-1}$ satisfies $\hat{L}\phi = -i_\sigma\, \partial_\sigma$. We may rewrite this equation as $\partial_{Y,d}(C\partial^t_{X,d}\phi + i_\sigma[\sigma]) = 0$, or $C\partial^t_{X,d}\phi + i_\sigma[\sigma] \in \ker \partial_{Y,d}$. Since $\partial^t_{\mathcal{A}(X),d+1}\partial^t_{X,d} = 0$, we see $R(C\partial^t_{X,d}\phi) = \partial^t_{X,d}\phi \in \ker \partial^t_{\mathcal{A}(X),d+1}$. Therefore, by (7), we conclude that $C\partial^t_{X,d}\phi + i_\sigma[\sigma]$ is equal to $I_\sigma = I + i_\sigma[\sigma]$, the current vector induced by i_σ. Consequently, we have $\partial^t_{X,d}\phi = RI = V$, and ϕ is a potential of V. □

We also note that a generalized Thomson's principle (in Section 3.3) can be stated in terms of potential as follows: Let Φ be the set of all $\phi \in \mathcal{C}^{d-1}$ such that $\partial^t_\sigma\phi = 1$, and $C_\sigma := 1/R_\sigma$. Then $\phi \in \Phi$ is a solution for $\hat{L}\phi = C_\sigma\partial_\sigma$ if and only if $\phi \in \Phi$ satisfies $\phi^t\hat{L}\phi = \inf_{\phi' \in \Phi}\{\phi'^t \hat{L}\phi'\}$. Note that the infimum is C_σ.

4.2. Main Theorem: a formula for R_σ via Green's function

The weighted combinatorial Laplacian $\hat{\Delta}_{X,\,d-1}$ for a d-dimensional network (X,R) with the weights $C=R^{-1}$ is a symmetric operator on $\mathcal{C}^{d-1}$ defined by

$$\hat{\Delta}=\hat{\Delta}_{X,\,d-1}=\hat{L}_{d-1}+\hat{J}_{d-1}$$

where $\hat{L}=\hat{L}_{d-1}=\partial_{X,d}C\partial_{X,d}^t$ as before and $\hat{J}=\hat{J}_{d-1}=\partial_{X,d-1}^t\partial_{X,d-1}$.

If X satisfies $\tilde{H}_{d-1}(X;\mathbb{R})=0$, then $\hat{\Delta}$ is invertible by combinatorial Hodge theory, and we call the inverse $\hat{\Delta}^{-1}=\hat{\Delta}_{X,d-1}^{-1}$ the *combinatorial Green's function* of X. Its rows and columns are indexed by the set X_{d-1} of all $(d-1)$-simplices of X, and we may write $\hat{\Delta}^{-1}=(g_{\nu,\nu'})_{\nu,\nu'\in X_{d-1}}$ to specify its entries. For a *connected* 1-dimensional network (X,R), the effective resistance R_{ab} between two distinct vertices a and b is given via $\hat{\Delta}_{X,0}^{-1}=(g_{ab})_{a,b\in V(X)}$ as follows [18, 15]:

$$R_{ab}=g_{aa}+g_{bb}-g_{ab}-g_{ba}\,.$$

The following theorem, which generalizes this formula, is a main result of the paper.

Theorem 4.2. *Let (X,R) be a network of dimension d (>0) with $\tilde{H}_{d-1}(X;\mathbb{R})=0$, and $\hat{\Delta}^{-1}=(g_{\nu,\nu'})_{\nu,\nu'\in X_{d-1}}$ its combinatorial Green's function. Let σ be a current generator. Then, the effective resistance R_σ of σ is*

$$R_\sigma=\partial_\sigma^t\,\hat{\Delta}^{-1}\,\partial_\sigma=\sum_{j,j'=0}^{d}(-1)^{j+j'}g_{\sigma_j,\sigma_{j'}}$$

where $[\sigma]=[v_0,v_1,\cdots,v_d]$ and $\sigma_j=\sigma-\{v_j\}$ for each j.

Proof. Since R_σ is independent of the current i_σ through σ, we will assume $i_\sigma=-1$ for convenience. Let $V_\sigma=V+v_\sigma[\sigma]$ be the voltage vector induced by $i_\sigma=-1$. By (11), we have $R_\sigma=\partial_\sigma^t\phi$ for any potential ϕ of V. Hence, we will prove the first equality of the theorem by showing that the element $\phi:=\hat{\Delta}^{-1}\partial_\sigma$ is a potential for V. By Lemma 4.1, it suffices to prove $\hat{L}\phi=\partial_\sigma$. In fact, $\hat{L}\phi=(\hat{\Delta}-\hat{J})\phi=\partial_\sigma-\hat{J}\phi$ by the definitions of $\hat{L}$ and ϕ. Hence, the proof reduces to showing $\hat{J}\phi=0$.

To that end, note $\hat{J}\hat{L}=\partial_{X,d-1}^t\partial_{X,d-1}\partial_{X,d}C\partial_{X,d}^t=0$ and $\hat{J}\partial_\sigma=\partial_{X,d-1}^t\partial_{X,d-1}\partial_{Y,d}[\sigma]=0$ where the second equation follows from $\partial_{X,d-1}=\partial_{Y,d-1}$. From these equations, we see that $\hat{J}^2\phi=\hat{J}(\hat{J}+\hat{L})\phi=\hat{J}\hat{\Delta}\hat{\Delta}^{-1}\partial_\sigma=\hat{J}\partial_\sigma=0$. Since $\hat{J}$ is symmetric, $\langle\hat{J}\phi,\hat{J}\phi\rangle=\phi^t\hat{J}^2\phi=0$, from which $\hat{J}\phi=0$ follows.

For the second equality of the theorem, note that the σ_j-component ϕ_{σ_j} of $\phi=\hat{\Delta}^{-1}\partial_\sigma$ equals $\phi_{\sigma_j}=\sum_{j'=0}^{d}(-1)^{j'}g_{\sigma_j,\sigma_{j'}}$ for each $j\in[0,n]$. Since we have $R_\sigma=\sum_{j=0}^{d}(-1)^j\phi_{\sigma_j}$ by (11), the result follows. $\square$

Example 4.3. Let X be the d-skeleton of a standard $(n-1)$-simplex on $[n]:=\{1,2,\cdots,n\}$ with unit resistance for each d-simplex. A routine verification shows that $\Delta_{X,d-1}=n\cdot\mathrm{id}$ where id is an identity matrix. Hence, for each $\sigma\in\binom{[n]}{d+1}$, we have

$$R_\sigma=\sum_{j,j'=0}^{d}(-1)^{j+j'}g_{\sigma_j,\sigma_{j'}}=\frac{d+1}{n}.$$

This example will be revisited in Example 5.5.

5. Effective resistance via high-dimensional tree-numbers

For a 1-dimensional network, effective resistance can be expressed in terms of spanning trees [13, 19]. In this section, we will establish a high-dimensional analogue of this expression. For that purpose, we will review high-dimensional tree-numbers (refer to [5, 6, 14]). In this section, we assume $\tilde{H}_{d-1}(X) = 0$ where $d = \dim X$ (> 0) .

5.1. High-dimensional trees

For a non-empty subset $T \subset X_i$, define $X_T = T \cup X^{(i-1)}$, regarded as a subcomplex of X. For $i \in [0, d]$, the i-dimensional subcomplex X_T of X is an *i-dimensional spanning tree*, or *i-tree* for short, if

(i) $\tilde{H}_i(X_T) = 0$, and

(ii) $\operatorname{rk} \tilde{H}_{i-1}(X_T) = 0$.

Let $\mathcal{T}_i = \mathcal{T}_i(X) := \{ T \mid X_T \text{ is an } i\text{-tree} \}$. Note that $\mathcal{T}_i(X) \neq \emptyset$ iff $\operatorname{rk} \tilde{H}_{i-1}(X) = 0$. Keeping in mind that $|\tilde{H}_{i-1}(X_T)|$ is finite for an i-tree X_T, define the i-th *tree-number* $k_i(X)$ of X (by Kalai [13]) to be

$$k_i(X) := \sum_{T \in \mathcal{T}_i} |\tilde{H}_{i-1}(X_T)|^2.$$

This definition generalizes the tree-number of a connected graph. Indeed, one can easily show that $k_1(X)$ equals the number of spanning trees in $X^{(1)}$ as a graph. Moreover, if X_T is an i-tree, then $|T| = \operatorname{rk} \partial_i$.

Proposition 5.1. *Let $\mathcal{A}(X)$ be an acyclization of X, and $\partial = \partial_{\mathcal{A}(X), d+1}$. For $T \subset X_d$, let $\partial_{\overline{T}}$ be the submatrix of ∂ obtained by deleting the rows indexed by T. Then $\partial_{\overline{T}}$ is a non-singular square matrix iff $T \in \mathcal{T}_d(X)$, and in that case, $|\det \partial_{\overline{T}}| = |\tilde{H}_{d-1}(X_T)|$.*

Proof. See [5, Proposition 4.1] or [14, Theorem 6]. □

As a consequence, we obtain a determinantal formula for $k_d(X)$ with $\partial = \partial_{\mathcal{A}(X), d+1}$:

$$k_d(X) = \sum_{T \in \mathcal{T}_d(X)} (\det \partial_{\overline{T}})^2 = \det \partial^t \partial \tag{13}$$

where the second equality follows from the Cauchy-Binet Theorem (refer to [10]).

Recall that each top-dimensional simplex τ of a network (X, R) is weighted by its conductance $c_\tau = r_\tau^{-1}$. For non-empty $T \subset X_d$, let $c_T = \prod_{\tau \in T} c_\tau$. We define the *weighted* tree-number $\hat{k}_d(X)$ of (X, R) to be

$$\hat{k}_d(X) := \sum_{T \in \mathcal{T}_d(X)} c_T |\tilde{H}_{i-1}(X_T)|^2 = \det R^{-1} \cdot \det \partial^t R \partial \tag{14}$$

where the second equality is an easy consequence of (13).

5.2. A combinatorial formula for R_σ

For a current generator σ, we assign weight 1 to σ. Note that, under the assumption $\tilde{H}_{d-1}(X) = 0$, if σ is a subset $\sigma \subset [n]$ with $|\sigma| = d+1$ such that the collection of all proper subsets is a subcomplex of X, then σ is a current generator of X. That is because the facts $\partial_d[\sigma] \in \ker \partial_{X,d-1}$ and $\tilde{H}_{d-1}(X) = 0$ imply $\partial_d[\sigma] \in \operatorname{im} \partial_{X,d}$.

In order to give a combinatorial formula for R_σ, via high-dimensional tree-numbers, we need a generalization of the tree-number of an edge-contracted graph. Let σ be a current generator of X, and $Y = X \cup \sigma$ as before. Define $\mathcal{T}_d(X)_\sigma = \{\, T \in \mathcal{T}_d(Y) \mid \sigma \in T \,\}$. If X is a connected graph, then $\mathcal{T}_1(X)_\sigma$ corresponds bijectively to the set of all spanning trees in the contraction Y/σ. Hence, we will regard $\mathcal{T}_d(X)_\sigma$ as a high-dimensional analogue of the former for enumeration purposes. Thus, we define

$$k_d(X)_\sigma := \sum_{T \in \mathcal{T}_d(X)_\sigma} |\tilde{H}_{d-1}(Y_T)|^2,$$

as a generalization of the tree-number of an edge-contracted graph.

By a completely analogous manner to the case of a simplicial complex, we may define $\mathcal{T}_d(Y/\sigma)$ with Y/σ as a cell complex, and the map from $\mathcal{T}_d(X)_\sigma$ to $\mathcal{T}_d(Y/\sigma)$ induced by contracting σ to a point is shown to be a torsion-preserving bijection [11, Corollary 2.11]. Therefore, we also note

$$k_d(Y/\sigma) = k_d(X)_\sigma \,.$$

Let $\mathcal{A}(Y)$ be an acyclization of Y, and let $D = \partial_{\mathcal{A}(Y),\, d+1}$. Note that the rows of D are indexed by $X_d \cup \{\sigma\}$. Let $\tilde{D}$ be obtained from D by deleting the row indexed by σ. Applying Proposition 5.1, we see that $D_{\overline{T}}$ is a non-singular submatrix of $\tilde{D}$ iff $T \in \mathcal{T}_d(Y)$ and $\sigma \in T$, *i.*e., iff $T \in \mathcal{T}_d(X)_\sigma$, and in that case, $|\det D_{\overline{T}}| = |\tilde{H}_{d-1}(Y_T)|$. Hence, we have

$$k_d(X)_\sigma = \sum_{T \in \mathcal{T}_d(X)_\sigma} (\det D_{\overline{T}})^2 = \det \tilde{D}^t \tilde{D} \tag{15}$$

where the second equality follows again from the Cauchy-Binet Theorem. For a network (X, R), we define

$$\hat{k}_d(X)_\sigma := \sum_{T \in \mathcal{T}_d(X)_\sigma} c_T |\tilde{H}_{d-1}(Y_T)|^2 = \det R^{-1} \cdot \det \tilde{D}^t R \tilde{D}. \tag{16}$$

Theorem 5.2. *For a d-dimensional simplicial network (X, R) with $\tilde{H}_{d-1}(X) = 0$ and a current generator σ,*

$$R_\sigma = \frac{\hat{k}_d(X)_\sigma}{\hat{k}_d(X)} = \frac{\hat{k}_d((X \cup \sigma)/\sigma)}{\hat{k}_d(X)}.$$

Proof. Since $R_\sigma = -v_\sigma / i_\sigma$, we will show $i_\sigma = \hat{k}_d(X)/\hat{k}_d(X)_\sigma$ when $v_\sigma = -1$. A main ingredient of the proof is $D := \partial_{\mathcal{A}(Y),\, d+1}$ whose columns form a basis for $\tilde{H}_d(Y)$. Since

d-cycles, $[\sigma]+p \in \tilde{H}_d(Y)$ together with the basis of $\tilde{H}_d(X)$, define a basis for $\tilde{H}_d(Y)$, we may take

$$D := \partial_{\mathcal{A}(Y),d+1} = \begin{bmatrix} \partial_{\mathcal{A}(X),d+1} & p \\ 0 \cdots 0 & 1 \end{bmatrix}$$

where the last row is indexed by σ. Recall that $I_\sigma = I + i_\sigma[\sigma]$ is a d-cycle in Y, *i.e.*, $I_\sigma \in \ker \partial_{Y,d}$ (refer to (7)). Since $\ker \partial_{\mathcal{A}(Y),\,d}/\operatorname{im} \partial_{\mathcal{A}(Y),\,d+1} = \tilde{H}_d(\mathcal{A}(Y)) = 0$, we have $I_\sigma = Dy$ for some $y \in \mathcal{A}_{d+1}(Y)$. From the expression for D, it follows that i_σ equals the last component of y, which we denote by y_σ. We will show $y_\sigma = \hat{k}_d(X)/\hat{k}_d(X)_\sigma$ to complete the proof.

By (10), we have $D^t V_\sigma = 0$, *i.e.*, V_σ is orthogonal to each column of D. Then by the assumption $v_\sigma = -1$, we obtain

$$0 = \langle p + [\sigma], V_\sigma \rangle = \langle p + [\sigma], V + v_\sigma[\sigma] \rangle, \text{ or, } \langle p, V \rangle = 1.$$

It follows that

$$(0, \ldots, 0, 1)^t = \tilde{D}^t V = \tilde{D}^t R I = \tilde{D}^t R \tilde{D} y\,.$$

Since $\det \tilde{D}^t R \tilde{D} = \det R \cdot \hat{k}_d(X)_\sigma$ is non-zero, we see that $(\tilde{D}^t R \tilde{D})^{-1}$ exists, and it follows that y_σ equals its lower-right corner entry. Also, the cofactor of the lower-right corner entry of $\tilde{D}^t R \tilde{D}$ equals $\det(\partial^t R \partial) = \det R \cdot \hat{k}_d(X)$. Therefore, we conclude

$$R_\sigma = \frac{1}{y_\sigma} = \frac{\det \tilde{D}^t R \tilde{D}}{\det(\partial^t R \partial)} = \frac{\hat{k}_d(X)_\sigma}{\hat{k}_d(X)}.$$

□

5.3. High-dimensional Foster's theorem

Based on our combinatorial formula for simplicial effective resistance (Theorem 5.2), we presents a high-dimensional analogue of Foster's theorem [8].

Theorem 5.3. *Let (X, R) be a d-dimensional simplicial network with $\tilde{H}_{d-1}(X) = 0$ and* $\operatorname{rk} \partial_{X,d} = \gamma_d$*. Then*

$$\sum_{\tau \in X_d} c_\tau R_\tau = \gamma_d.$$

Proof. Summing $c_T |\tilde{H}_{d-1}(X_T)|^2$ over $\mathcal{S} := \{\, (T, \tau) \mid T \in \mathcal{T}_d(X) \text{ and } \tau \in T\}$ and changing the order of summation yields

$$\hat{k}_d(X)\gamma_d = \sum_{(T,\tau)\in\mathcal{S}} c_T |\tilde{H}_{d-1}(X_T)|^2 = \sum_{\tau \in X_d} c_\tau \hat{k}_d(X)_\tau$$

where the first equality follows from the fact $|T| = \gamma_d$ for $T \in \mathcal{T}_d(X)$ and (14), and the second equality from (16). The result is immediate from Theorem 5.2. □

A simplicial complex X is called *facet-trasitive* if X has an automorphism taking any facet to any other facet. In a facet-transitive complex, effective resistance is clearly constant on facets, and hence the effective resistance follows from the theorem.

Corollary 5.4. *Suppose that X is a d-dimensional simplicial complex each of whose d-face has a unit resistance. If X is facet-transitive, the effective resistance R_σ for $\sigma \in X_d$ equals*

$$R_\sigma = \frac{\gamma_d}{|X_d|}.$$

We apply the corollary to obtain effective resistances in the following three complexes: skeleta of standard simplexes, complete colorful complexes, and hypercubes.

Example 5.5. Let X be the d-skeleton of a standard $(n-1)$-simplex on $[n] := \{1, 2, \dots, n\}$. The collection $T = \{\tau \in X_d \mid n \in \tau\}$ is a d-tree in X, and we have $\gamma_d = |T| = \binom{n-1}{d}$. For every $\sigma \in \binom{[n]}{d+1}$, its effective resistance R_σ is equal to

$$R_\sigma = \frac{\binom{n-1}{d}}{\binom{n}{d+1}} = \frac{d+1}{n}.$$

Example 5.6. For each d-face σ of a *complete colorful complex*, we will compute R_σ. For disjoint vertex sets $V_1, \dots, V_r$ ("color classes") with $|V_1| = n_1, \dots, |V_r| = n_r$, a complete colorful complex $K := K(n_1, n_2, \dots, n_r)$ is defined to be a simplicial complex each of whose faces is a set of vertices with no more than one vertex of each color.

The number of d-faces in K is $e_{d+1}(n_1, \dots, n_r)$, and $\gamma_d = \sum_{j=0}^{d} \binom{r-j-1}{r-d-1} e_j(n_1-1, \dots, n_r-1)$ [2, Proposition 1.2], where e_j is the j-th elementary symmetric function. For each d-face σ,

$$R_\sigma = \Big(\sum_{j=0}^{d} \binom{r-j-1}{r-d-1} e_j(n_1-1, \dots, n_r-1)\Big) \Big/ \Big(e_{d+1}(n_1, \dots, n_r)\Big).$$

Example 5.7. We apply the idea of the proof of Theorem 5.3 to a hypercube. For a definition of hypercubes, we refer the readers to [6]. A hypercube Q_n is the n-fold product $[0,1] \times \cdots \times [0,1]$, where $[0,1]$ is a cell complex with two 0-cells, 0 and 1, and one 1-cell, $(0,1)$. The number of d-cells in Q_n is $\binom{n}{d} 2^{n-d}$, and

$$\gamma_d = \sum_{j=d}^{n} \binom{n}{j} \binom{j-1}{d-1}$$

[1, Theorem 1.5]. We have $R_\sigma = k_d(Q_n)_\sigma / k_d(Q_n)$ for a d-cell σ in Q_n. Then R_σ equals

$$R_\sigma = \Big(\sum_{j=d}^{n} \binom{n}{j} \binom{j-1}{d-1}\Big) \Big/ \Big(\binom{n}{d} 2^{n-d}\Big).$$

Moreover, since $k_d(Q_n) = \prod_{j=d+1} (2j)^{\binom{n}{j}\binom{j-2}{d-1}}$ [6, Corollary 3.5], we have

$$k_d(Q_n)_\sigma = \Big(\sum_{j=d}^{n} \binom{n}{j} \binom{j-1}{d-1}\Big) \Big/ \Big(\binom{n}{d} 2^{n-d}\Big) \cdot \prod_{j=d+1} (2j)^{\binom{n}{j}\binom{j-2}{d-1}}.$$

Finally, we end the paper with the following intriguing identity for R_σ as a high-dimensional analogue of [15, Theorem 4]. By Theorem 4.2 and Theorem 5.2, we obtain

$$\sum_{j,j'=0}^{d} (-1)^{j+j'} g_{\sigma_j,\sigma_{j'}} = \frac{\hat{k}_d(X)_\sigma}{\hat{k}_d(X)}. \tag{17}$$

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