THE BOUNDEDNESS BELOW OF 2×2 UPPER TRIANGULAR OPERATOR MATRICES

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When $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are given we denote by M_C an operator acting on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$ of the form

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. In this paper we characterize the boundedness below of M_C. Our characterization is as follows: M_C is bounded below for some $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ if and only if A is bounded below and $\alpha(B) \le \beta(A)$ if $R(B)$ is closed; $\beta(A) = \infty$ if $R(B)$ is not closed, where $\alpha(\cdot)$ and $\beta(\cdot)$ denote the nullity and the deficiency, respectively. In addition, we show that if $\sigma_{ap}(\cdot)$ and $\sigma_d(\cdot)$ denote the approximate point spectrum and the defect spectrum, respectively, then the passage from $\sigma_{ap}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ to $\sigma_{ap}(M_C)$ can be described as follows:

$$\sigma_{ap}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_{ap}(M_C) \cup W \quad \text{for every } C \in \mathcal{L}(\mathcal{K}, \mathcal{H}),$$

where W lies in certain holes in $\sigma_{ap}(A)$, which happen to be subsets of $\sigma_d(A) \cap \sigma_{ap}(B)$.

1 Introduction

The study of upper triangular operator matrices arises naturally from the following fact: if T is a Hilbert space operator and $\mathcal{H}$ is an invariant subspace for T then T has the following 2×2 upper triangular operator matrix representation:

$$T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : \mathcal{H} \oplus \mathcal{H}^\perp \longrightarrow \mathcal{H} \oplus \mathcal{H}^\perp,$$

and one way to study operators is to see them as entries of simpler operators. The upper triangular operator matrices (more generally, block operator matrices) have been studied

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by numerous authors. This paper is concerned with the boundedness below of 2×2 upper triangular operator matrices.

Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional separable Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ denote the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, and abbreviate $\mathcal{L}(\mathcal{H}, \mathcal{H})$ to $\mathcal{L}(\mathcal{H})$. If $A \in \mathcal{L}(\mathcal{H})$ write $\sigma(A)$ for the spectrum of A. If $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ write $N(A)$ for the null space of A; $R(A)$ for the range of A; $\alpha(A)$ for the nullity of A, i.e., $\alpha(A) := \dim N(A)$; $\beta(A)$ for the deficiency of A, i.e., $\beta(A) := \dim \overline{R(A)}^{\perp}$. Recall that an operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is said to be *bounded below* if there exists $k > 0$ for which $||x|| \le k\, ||Ax||$ for each $x \in \mathcal{H}$. If $A \in \mathcal{L}(\mathcal{H})$ then the approximate point spectrum, $\sigma_{ap}(A)$, and the defect spectrum, $\sigma_d(A)$, of A are defined by

$$\sigma_{ap}(A) := \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not bounded below}\};$$
$$\sigma_{d}(A) := \{\lambda \in \mathbb{C} : \ A - \lambda \text{ is not onto}\}.$$

If $\mathfrak{S}$ is a compact subset of $\mathbb{C}$, write $\operatorname{int} \mathfrak{S}$ for the interior points of $\mathfrak{S}$; $\operatorname{iso} \mathfrak{S}$ for the isolated points of $\mathfrak{S}$; $\operatorname{acc} \mathfrak{S}$ for the accumulation points of $\mathfrak{S}$; $\partial\, \mathfrak{S}$ for the topological boundary of $\mathfrak{S}$. When $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are given we denote by M_C an operator acting on $\mathcal{H} \oplus \mathcal{K}$ of the form

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. The invertibility, the spectrum and Weyl's theorem of M_C were considered in [DJ],[HLL], and [Le]. In this paper we characterize the boundedness below of M_C. Our characterization is as follows:

Theorem 1. *An 2×2 operator matrix $M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is bounded below for some $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ if and only if A is bounded below and*

$$\begin{cases} \alpha(B) \le \beta(A) & \text{if } R(B) \text{ is closed,} \\ \beta(A) = \infty & \text{if } R(B) \text{ is not closed.} \end{cases}$$

In Section 1 we give the proof of Theorem 1. In Section 2 we give a description of the passage from $\sigma_{ap}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ to $\sigma_{ap}(M_C)$.

1 Proof of Theorem 1

If $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ then the *reduced minimum modulus* of T is defined by (cf. [Ap])

$$\gamma(T) = \begin{cases} \inf\big\{||Tx|| : \ \operatorname{dist}(x,\, N(T)) = 1\big\} & \text{if } T \ne 0 \\ 0 & \text{if } T = 0. \end{cases}$$

Thus $\gamma(T) > 0$ if and only if T has closed non-zero range (cf. [Ap],[Go]). If $T \in \mathcal{L}(\mathcal{H})$ is a non-zero operator then we can see ([Ap]) that $\gamma(T) = \inf\big(\sigma(|T|) \setminus \{0\}\big)$, where $|T|$ denotes

$(T^*T)^{\frac{1}{2}}$. Thus we have that $\gamma(T) = \gamma(T^*)$. From the definition we can also see that if T is bounded below then $||x|| \le \frac{1}{\gamma(T)} ||Tx||$ for each $x \in \mathcal{H}$.

If $M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ write

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}. \tag{1}$$

Recall ([Ha1, Theorem 3.3.2]) that if $S \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ then

$$S, T \text{ bounded below} \Longrightarrow ST \text{ bounded below} \Longrightarrow T \text{ bounded below.} \tag{2}$$

Since $\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ is invertible for every $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, applying (2) to (1) gives

$$A, B \text{ bounded below} \Longrightarrow M_C \text{ bounded below} \Longrightarrow A \text{ bounded below.} \tag{3}$$

To prove Theorem 1 we establish an auxiliary lemma, which is a result of independent interest.

Lemma 1. *Let $T \in \mathcal{L}(\mathcal{H})$ and $T \neq 0$. Then T satisfies one of the following two conditions:*
(i) *There exists a unit vector x in $N(T)^\perp$ such that $||Tx|| = \gamma(T)$;*
(ii) *There exists an orthonormal sequence $\{x_n\}$ in $N(T)^\perp$ such that $||Tx_n|| \to \gamma(T)$.*
In particular, if $R(T)$ is not closed then T must satisfy the condition (ii) with $\gamma(T) = 0$.

Proof. Suppose $T \neq 0$ and write $\alpha := \gamma(T) = \inf\big(\sigma(|T|) \setminus \{0\}\big)$. Let E be the spectral measure on the Borel subsets of $\sigma(|T|)$ such that $|T| = \int z\, dE(z)$. There are two cases to consider.

Case 1: $\alpha \in acc\big(\sigma(|T|) \setminus \{0\}\big)$. In this case, there exists a strictly decreasing sequence $\{\alpha_n\}$ of elements in $\sigma(|T|) \setminus \{0\}$ such that $\alpha_n \to \alpha$. Since the α_n's are distinct, there exists a sequence $\{U_n\}$ of mutually disjoint open intervals such that $\alpha_n \in U_n$ for all $n \in \mathbb{Z}^+$. Define $F_n := U_n \cap \sigma(|T|)$ $(n \in \mathbb{Z}^+)$. Then the F_n's are nonempty relatively open subsets of $\sigma(|T|)$. Thus $E(F_n)\mathcal{H} \neq \{0\}$ for each $n \in \mathbb{Z}^+$. For each $n \in \mathbb{Z}^+$, choose a unit vector x_n in $E(F_n)\mathcal{H}$. Since the F_n's are mutually disjoint, it follows that $\{x_n\}$ is an orthonormal sequence. We will show that $x_n \in N(T)^\perp$ $(n \in \mathbb{Z}^+)$. If $|T|$ is invertible then $N(T)^\perp = N(|T|)^\perp = \mathcal{H}$, so evidently, $x_n \in N(T)^\perp$. Now suppose $|T|$ is not invertible. Since $|T|$ is a normal operator, $|T|$ is unitarily equivalent to a multiplication operator M_φ. But since our argument below depends only on the inner product, we may assume without loss of generality that $|T|$ is a multiplication operator. Let $|T| := M_\varphi$. If $F_0 := \{0\}$ then $E(F_0)$ is the multiplication by $\chi_{\varphi^{-1}(0)}$. Thus if $f \in N(|T|)$ then $\varphi f = 0$ and hence

$$\big(\chi_{\varphi^{-1}(0)} f\big)(x) = \begin{cases} 0 & \text{if } f(x) = 0, \\ f(x) & \text{if } f(x) \neq 0, \end{cases}$$

which shows that $E(F_0)f = f$. Therefore if $f \in N(|T|)$ then for each $n \in \mathbb{Z}^+$,

$$(f,\, x_n) = (E(F_0)f,\, E(F_n)x_n) = (f,\, E(F_0 \cap F_n)x_n) = (f, 0) = 0,$$

which shows that $x_n \in N(|T|)^\perp$ for all $n \in \mathbb{Z}^+$. It thus follows that $x_n \in N(T)^\perp$. On the other hand, for each $n \geq 2$,

$$\begin{aligned} ||Tx_n||^2 &= (T^*Tx_n,\, x_n) \leq ||(T^*T)|_{E(F_n)\mathcal{H}}|| = r\big((T^*T)|_{E(F_n)\mathcal{H}}\big) \\ &\leq \big(\sup F_n\big)^2 \leq \big(\sup U_n\big)^2 \leq \alpha_{n-1}^2, \end{aligned}$$

where $r(\cdot)$ denotes the spectral radius. Therefore we have that $\alpha \leq ||Tx_n|| \leq \alpha_{n-1}$ $(n \geq 2)$, which implies that $||Tx_n|| \to \alpha = \gamma(T)$.

Case 2: $\alpha \in iso\,\big(\sigma(|T|) \setminus \{0\}\big)$. Let $\mathfrak{L} := E(\{\alpha\})$ and $\mathfrak{M} := E\big(\sigma(|T|) \setminus \{\alpha\}\big)$. Then $\mathcal{H}$ can be decomposed as $\mathcal{H} = \mathfrak{L} \oplus \mathfrak{M}$, where $\mathfrak{L}$ and $\mathfrak{M}$ are $|T|$-invariant subspaces, $\sigma(|T|\,|_{\mathfrak{L}}) = \{\alpha\}$ and $\sigma(|T|\,|_{\mathfrak{M}}) = \sigma(|T|) \setminus \{\alpha\}$: more precisely, we can write

$$|T| = \begin{pmatrix} \alpha & 0 \\ 0 & |T|\,|_{\mathfrak{M}} \end{pmatrix} :\ \mathfrak{L} \oplus \mathfrak{M} \longrightarrow \mathfrak{L} \oplus \mathfrak{M}.$$

But since $||Tx|| = ||\,|T|x||$ for all $x \in \mathcal{H}$, it follows that for every unit vector x_0 in $\mathfrak{L}$, $||Tx_0|| = ||\,|T|x_0|| = ||\alpha x_0|| = \alpha$.

For the second assertion suppose $\gamma(T) = 0$ and $T \neq 0$. If T satisfies the condition (i) then there exists a unit vector $x \in N(T)^\perp$ such that $Tx = 0$, giving a contradiction. This shows that T must satisfy the condition (ii). □

Proof of Theorem 1. We first claim that if A is bounded below and $R(B)$ is closed, then

$$\text{(4)} \qquad \alpha(B) \leq \beta(A) \iff M_C \text{ is bounded below for some } C \in \mathcal{L}(\mathcal{K}, \mathcal{H}).$$

To show this suppose $\alpha(B) \leq \beta(A)$. Since $\dim N(B) \leq \dim R(A)^\perp$, there exists a isometry $J : N(B) \to R(A)^\perp$. Define an operator $C : \mathcal{K} \to \mathcal{H}$ by

$$C := \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} N(B) \\ N(B)^\perp \end{pmatrix} \to \begin{pmatrix} R(A)^\perp \\ R(A) \end{pmatrix}.$$

Then M_C is one-one. Assume to the contrary that M_C is not bounded below. Then there exists a sequence $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ of unit vectors in $\mathcal{H} \oplus \mathcal{K}$ for which

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} Ax_n + Cy_n \\ By_n \end{pmatrix} \longrightarrow 0.$$

Write $y_n := \alpha_n + \beta_n$ for $n \in \mathbb{Z}^+$, where $\alpha_n \in N(B)$ and $\beta_n \in N(B)^\perp$. Since $\gamma(B) > 0$ and $By_n \to 0$, it follows that $\beta_n \to 0$. Also by the definition of C, $Cy_n = C(\alpha_n + \beta_n) = C\alpha_n \to 0$ and hence $\alpha_n \to 0$. Therefore $y_n \to 0$ and $||x_n|| \to 1$. But since $Ax_n \to 0$, it follows that A is not bounded below, giving a contradiction. This proves that M_C is bounded below.

Conversely, suppose M_C is bounded below for some $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. Write M_C as in (1). Since $\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ have closed ranges, it follows from an index theorem of R. Harte [Ha2],[Ha3] that

$$\begin{aligned} &N \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \bigoplus N \left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \right) \bigoplus R(M_C)^\perp \\ &\qquad \cong N(M_C) \bigoplus R \left(\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \right)^\perp \bigoplus R \left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \right)^\perp, \end{aligned}$$

which implies that $\alpha(B) + \beta(M_C) = \beta(A) + \beta \left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \right)$. Since

$$\beta(M_C) \geq \beta \left(\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \right),$$

it follows that $\alpha(B) \leq \beta(A)$. This proves (4). We next claim that if A is bounded below and $R(B)$ is not closed, then

$$\text{(5)} \qquad \beta(A) = \infty \Longleftrightarrow M_C \text{ is bounded below for some } C \in \mathcal{L}(\mathcal{K}, \mathcal{H}).$$

To show this suppose $\beta(A) = \infty$. Then with no restriction on $R(B)$, M_C is bounded below for some $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. To see this, observe $\dim R(A)^\perp = \infty$, so there exists an isomorphism $C_0 : \mathcal{K} \to R(A)^\perp$. Define an operator $C : \mathcal{K} \to \mathcal{H}$ by

$$C := \begin{pmatrix} C_0 & 0 \end{pmatrix} : \mathcal{K} \to \begin{pmatrix} R(A)^\perp \\ R(A) \end{pmatrix}.$$

Then a straightforward calculation shows that M_C is one-one and

$$\begin{aligned} \gamma(M_C) &= \inf_{||x||^2 + ||y||^2 = 1} || \begin{pmatrix} Ax + Cy \\ By \end{pmatrix} || \\ &\geq \inf_{||x||^2 + ||y||^2 = 1} \left(||Ax||^2 + ||Cy||^2 \right)^{\frac{1}{2}} \\ &\geq \inf_{||x||^2 + ||y||^2 = 1} \left(\gamma(A)^2 ||x||^2 + ||y||^2 \right)^{\frac{1}{2}} \\ &\geq \min \{1, \ \gamma(A)\} > 0, \end{aligned}$$

which implies that M_C is bounded below. For the converse, assume $\beta(A) = N < \infty$. Since $R(B)$ is not closed it follows from Lemma 1 that there exists an orthonormal sequence $\{y_n\}$ in $N(B)^\perp$ such that $By_n \to 0$. But since M_C is bounded below we have

$$\inf_{||x||^2 + ||y||^2 = 1} || \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} || = \inf_{||x||^2 + ||y||^2 = 1} || \begin{pmatrix} Ax + Cy \\ By \end{pmatrix} || > 0.$$

We now argue that there exist $\epsilon > 0$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ for which

$$\text{(6)} \qquad \text{dist} \left(R(A), \, Cy_{n_k} \right) > \epsilon \quad \text{for all } k \in \mathbb{Z}^+.$$

Indeed, assume to the contrary that dist $\big(R(A), Cy_n\big) \to 0$ as $n \to \infty$. Thus there exists a sequence $\{x_n\}$ in $\mathcal{H}$ such that dist $\big(Ax_n, Cy_n\big) \to 0$. Let $z_n := ||\begin{pmatrix} x_n \\ y_n \end{pmatrix}||^{-1} x_n$ and $w_n := ||\begin{pmatrix} x_n \\ y_n \end{pmatrix}||^{-1}(-y_n)$. Then $||\begin{pmatrix} z_n \\ w_n \end{pmatrix}|| = 1$ and $||\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\begin{pmatrix} z_n \\ w_n \end{pmatrix}|| = ||\begin{pmatrix} Az_n+Cw_n \\ Bw_n \end{pmatrix}|| \longrightarrow 0$, giving a contradiction. This proves (6). There is no loss in simplifying the notation and assuming that

$$\text{(7)} \qquad \text{dist}\,\big(R(A), Cy_n\big) > \epsilon \quad \text{for all } n \in \mathbb{Z}^+.$$

Since $\beta(A) = N$, there exists an orthonormal basis $\{e_1, \cdots, e_N\}$ for $R(A)^\perp$. Let P_m be the projection from $\mathcal{H}$ to $\vee\{e_m\}$ for $m = 1, \cdots, N$, where $\vee(\cdot)$ denotes the closed linear span. If we let $Cy_n := \alpha_n + \beta_n$ $(n \in \mathbb{Z}^+)$, where $\alpha_n \in R(A)$ and $\beta_n \in R(A)^\perp$, then by (7), $||\beta_n|| > \epsilon$ for all $n \in \mathbb{Z}^+$. Observe that $\sum_{n=1}^\infty ||\frac{1}{n}\beta_n|| = \infty$ and hence $||\sum_{n=1}^\infty P_{m_0}(\frac{1}{n}\beta_n e^{i\theta_n})|| = \infty$ for some $m_0 \in \{1, \cdots, N\}$ and for some $\theta_n \in [0, 2\pi)$ $(n \in \mathbb{Z}^+)$. Now if we write $y := \sum_{n=1}^\infty \frac{1}{n} y_n e^{i\theta_n}$, then $||y||^2 = \sum_{n=1}^\infty \frac{1}{n^2} < \infty$ and hence $y \in \mathcal{K}$. But

$$||Cy|| \ge ||P_{m_0}(Cy)|| = ||\sum_{n=1}^{\infty} P_{m_0} C(\frac{1}{n} y_n e^{i\theta_n})|| = ||\sum_{n=1}^{\infty} P_{m_0}(\frac{1}{n}\beta_n e^{i\theta_n})|| = \infty,$$

giving a contradiction. Therefore we must have that $\beta(A) = \infty$. This proves (5). Now Theorem 1 follows from (3), (4) and (5). □

The following corollary is immediate from Theorem 1.

Corollary 1. *For a given pair (A, B) of operators we have*

$$\bigcap_{C \in \mathcal{L}(\mathcal{K},\mathcal{H})} \sigma_{ap}(M_C) = \sigma_{ap}(A) \bigcup \{\lambda \in \mathbb{C} : R(B-\lambda) \text{ is closed and } \beta(A-\lambda) < \alpha(B-\lambda)\}$$
$$\bigcup \{\lambda \in \mathbb{C} : R(B-\lambda) \text{ is not closed and } \beta(A-\lambda) < \infty\}.$$

The following is the dual statement of Corollary 1.

Corollary 2. *For a given pair (A, B) of operators we have*

$$\bigcap_{C \in \mathcal{L}(\mathcal{K},\mathcal{H})} \sigma_d(M_C) = \sigma_d(B) \bigcup \{\lambda \in \mathbb{C} : R(A-\lambda) \text{ is closed and } \alpha(B-\lambda) < \beta(A-\lambda)\}$$
$$\bigcup \{\lambda \in \mathbb{C} : R(A-\lambda) \text{ is not closed and } \alpha(B-\lambda) < \infty\}.$$

Combining Corollaries 1 and 2 gives:

Corollary 3 ([DJ, Theorem 2]). *For a given pair (A, B) of operators we have*

$$\bigcap_{C\in\mathcal{L}(\mathcal{K},\mathcal{H})} \sigma(M_C) = \sigma_{ap}(A) \bigcup \sigma_d(B) \bigcup \{\lambda \in \mathbb{C} : \ \alpha(B-\lambda) \neq \beta(A-\lambda)\}.$$

Remark. In many applications, the entries of block operator matrices are unbounded operators. Section 1 deals only with the bounded case. We expect that an analogue of Theorem 1 holds for the unbounded case.

2 The passage from $\sigma_{ap}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ to $\sigma_{ap}(M_C)$

In [HLL], it was shown that the passage from $\sigma\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ to $\sigma(M_C)$ is accomplished by removing certain open subsets of $\sigma(A)\cap\sigma(B)$ from the former, that is, there is equality

$$\sigma\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma(M_C) \cup W, \tag{8}$$

where W is the union of certain of the holes in $\sigma(M_C)$ which happen to be subsets of $\sigma(A)\cap\sigma(B)$. However we need not expect the case for the approximate point spectrum (see Examples 1 and 2 below). The passage from $\sigma_{ap}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ to $\sigma_{ap}(M_C)$ is more delicate.

Theorem 2. *For a given pair (A, B) of operators we have that for every $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$,*

$$\eta\big(\sigma_{ap}(A)\cup\sigma_{ap}(B)\big) = \eta\big(\sigma_{ap}(M_C)\big), \tag{9}$$

where $\eta(\cdot)$ denotes the "polynomially-convex hull". More precisely,

$$\sigma_{ap}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_{ap}(M_C) \cup W, \tag{10}$$

where W lies in certain holes in $\sigma_{ap}(A)$, which happen to be subsets of $\sigma_d(A)\cap\sigma_{ap}(B)$. Hence, in particular, $r_{ap}(M_C)$ is a constant, and furthermore for every $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$,

$$r\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r_{ap}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = r_{ap}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \tag{11}$$

where $r(\cdot)$ and $r_{ap}(\cdot)$ denote the spectral radius and the "approximate point spectral radius".

Proof. First, observe that for a given pair (A, B) of operators we have that for every $C \in \mathcal{L}(\mathcal{K},\mathcal{H})$,

$$\sigma_{ap}(A) \subseteq \sigma_{ap}(M_C) \subseteq \sigma_{ap}(A)\cup\sigma_{ap}(B) = \sigma_{ap}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}: \tag{12}$$

the first and the second inclusions follow from (3) and the last equality is obvious. We now claim that for every $T \in \mathcal{L}(\mathcal{H})$,

$$\eta\big(\sigma(T)\big) = \eta\big(\sigma_{ap}(T)\big). \tag{13}$$

Indeed since $\operatorname{int}\sigma_{ap}(T) \subseteq \operatorname{int}\sigma(T)$ and $\partial\,\sigma(T) \subseteq \sigma_{ap}(T)$, we have that $\partial\,\sigma(T) \subseteq \partial\,\sigma_{ap}(T)$, which implies that the passage from $\sigma_{ap}(T)$ to $\sigma(T)$ is filling in certain holes in $\sigma_{ap}(T)$, proving (13). Now suppose $\lambda \in \big(\sigma_{ap}(A) \cup \sigma_{ap}(B)\big) \setminus \sigma_{ap}(M_C)$. Thus by (12), $\lambda \in \sigma_{ap}(B) \setminus \sigma_{ap}(A)$. Since $M_C - \lambda$ is bounded below it follows from Theorem 1 that if $R(B-\lambda)$ is not closed then $\beta(A-\lambda) = \infty$, and if instead $R(B-\lambda)$ is closed then $\beta(A-\lambda) \geq \alpha(B-\lambda) > 0$, where the last inequality comes from the fact that $B - \lambda$ is not one-one since $B - \lambda$ is not bounded below. Therefore $\lambda \in \sigma_d(A)$. On the other hand, λ should be in one of the holes in $\sigma_{ap}(A)$: for if this were not so then by (13), $A - \lambda$ would be invertible, a contradiction. This proves (9) and (10). The equality (11) follows at once from (9) and (13). □

Recall ([Pe, Definition 4.8]) that an operator $A \in \mathcal{L}(\mathcal{H})$ is *quasitriangular* if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of projections of finite rank in $\mathcal{L}(\mathcal{H})$ that converges strongly to the identity and satisfies $||P_nAP_n - AP_n|| \to 0$. Also recall that an operator $A \in \mathcal{L}(\mathcal{H})$ is called *left-Fredholm* if A has closed range and $\alpha(A) < \infty$ and *right-Fredholm* if A has closed range and $\beta(A) < \infty$. If A is both left- and right-Fredholm, we call it *Fredholm*. The *index*, $\operatorname{ind} A$, of a left- or right-Fredholm operator A is defined by $\operatorname{ind} A = \alpha(A) - \beta(A)$. If $A \in \mathcal{L}(\mathcal{H})$ then the left essential spectrum, $\sigma_e^+(A)$, the right essential spectrum, $\sigma_e^-(A)$, and the essential spectrum, $\sigma_e(A)$, of A are defined by

$$\begin{aligned}
\sigma_e^+(A) &= \{\lambda \in \mathbb{C} : A - \lambda \text{ is not left-Fredholm}\};\\
\sigma_e^-(A) &= \{\lambda \in \mathbb{C} : A - \lambda \text{ is not right-Fredholm}\};\\
\sigma_e(A) &= \{\lambda \in \mathbb{C} : A - \lambda \text{ is not Fredholm}\}.
\end{aligned}$$

Recall ([Pe, Definition 1.22]) that the *spectral picture* of an operator $A \in \mathcal{L}(\mathcal{H})$, denoted $\mathcal{SP}(A)$, is the structure consisting of the set $\sigma_e(A)$, the collection of holes and pseudoholes in $\sigma_e(A)$, and the indices associated with these holes and pseudoholes, where a *hole* in $\sigma_e(A)$ is a nonempty bounded component of $\mathbb{C} \setminus \sigma_e(A)$ and a *pseudohole* in $\sigma_e(A)$ is a nonempty component of $\sigma_e(A) \setminus \sigma_e^+(A)$ or of $\sigma_e(A) \setminus \sigma_e^-(A)$. From the work of Apostol, Foias and Voiculescu ([Pe, Theorem 1.31]), we have that A is quasitriangular if and only if the spectral picture of A contains no hole or pseudohole associated with a negative number. We now have:

Corollary 4. *If A is a quasitriangular operator (e.g., A is either compact or cohyponormal) then for every $B \in \mathcal{L}(\mathcal{K})$ and $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$,*

$$\sigma_{ap}(M_C) = \sigma_{ap}(A) \cup \sigma_{ap}(B).$$

Proof. The inclusion $\subseteq$ is the second inclusion in (12). For the reverse inclusion suppose $\lambda \in \sigma_{ap}(A) \cup \sigma_{ap}(B)$. If $\lambda \in \big(\sigma_{ap}(A) \cup \sigma_{ap}(B)\big) \setminus \sigma_{ap}(M_C)$ then by Theorem 2, $\lambda \in \sigma_d(A) \cap \sigma_{ap}(B)$ and $A - \lambda$ is bounded below. But since A is quasitriangular, we have that $\beta(A-\lambda) \leq \alpha(A-\lambda) = 0$. Therefore $A - \lambda$ is invertible, a contradiction. □

We conclude with three examples. We first recall the definition of Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\,\mathbb{D}$ in the complex plane. Recall that the

Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \ldots\}$. Write $C(\mathbb{T})$ for the set of all continuous complex–valued functions on $\mathbb{T}$ and $H^\infty(\mathbb{T}) := L^\infty \cap H^2$. If P denotes the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$ the operator T_φ defined by $T_\varphi g := P(\varphi g)$ $(g \in H^2(\mathbb{T}))$ is called the *Toeplitz operator* with symbol φ. For the basic theory of Toeplitz operators, see [Do1], [Do2], [GGK1], [GGK2], and [Ni].

Example 1. One might expect that in Theorem 2, W is the union of certain of the holes in $\sigma_{ap}(M_C)$ together with the closure of some isolated points of $\sigma_{ap}(B)$. But this is not the case. To see this, let $\varphi \in H^\infty$ be an inner function (i.e., $|\varphi| = 1$ a.e.) with $\dim (\varphi H^2)^\perp = \infty$ (e.g., $\varphi(z) = \exp\left(\frac{z+\lambda}{z-\lambda}\right)$ with $|\lambda| = 1$), let ψ be any function in $C(\mathbb{T})$ with $||\psi||_\infty < 1$, and let J be an isometry from H^2 to $(\varphi H^2)^\perp$. Define

$$M_J := \begin{pmatrix} T_\varphi & J \\ 0 & T_\psi \end{pmatrix}.$$

Note that T_φ is a non–normal isometry and hence $\sigma_{ap}(T_\varphi) = \mathbb{T}$. Since $R(T_\varphi) \perp R(J)$, it follows that $||M_J \begin{pmatrix} x \\ y \end{pmatrix}|| \geq || \begin{pmatrix} x \\ y \end{pmatrix} ||$ for all $\begin{pmatrix} x \\ y \end{pmatrix} \in H^2 \oplus H^2$, which says that M_J is bounded below. Observe

$$\gamma(M_J) = \inf_{||\begin{pmatrix} x_n \\ y_n \end{pmatrix}||=1} ||M_J \begin{pmatrix} x_n \\ y_n \end{pmatrix}|| \geq 1.$$

Thus by [Go, Theorem V.1.6], we have that for all $|\lambda| < 1 (\leq \gamma(M_J))$,

(i) $M_J - \lambda$ is semi-Fredholm;
(ii) $\alpha(M_J - \lambda) \leq \alpha(M_J) = 0$,

which implies that $M_J - \lambda$ is bounded below for all $|\lambda| < 1$. But since $\sigma(T_\psi)$ is contained in the polynomially–convex hull of the range of ψ, it follows from our assumption that $\sigma_{ap}(T_\psi) \subseteq \mathbb{D}$. Thus by Theorem 2 we have that $\sigma_{ap}(M_J) = \mathbb{T}$. Note that $\sigma_{ap}(T_\psi)$ has disappeared in the passage from $\sigma_{ap}\begin{pmatrix} T_\varphi & 0 \\ 0 & T_\psi \end{pmatrix}$ to $\sigma_{ap}(M_J)$.

Example 2. We need not expect a general information for removing in the passage from $\sigma_{ap}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ to $\sigma_{ap}(M_C)$. To see this, let T_φ, T_ψ, and J be given as in Example 1. Also let ζ be a function in $C(\mathbb{T})$ such that $\sigma_{ap}(T_\zeta)$ is a compact subset σ of $\sigma_{ap}(T_\psi)$. We define, on $H^2 \oplus H^2$, $A := T_\varphi \oplus T_\varphi$, $B := T_\psi \oplus T_\zeta$, $C := J \oplus 0$ and in turn

$$M_C := \begin{pmatrix} T_\varphi & 0 & J & 0 \\ 0 & T_\varphi & 0 & 0 \\ 0 & 0 & T_\psi & 0 \\ 0 & 0 & 0 & T_\zeta \end{pmatrix}.$$

A straightforward calculation shows

$$\sigma_{ap}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_{ap}(A) \cup \sigma_{ap}(B) = \mathbb{T} \cup \sigma_{ap}(T_\psi).$$

On the other hand, M_C is unitarily equivalent to the operator

$$\begin{pmatrix} T_\varphi & J \\ 0 & T_\psi \end{pmatrix} \bigoplus \begin{pmatrix} T_\varphi & 0 \\ 0 & T_\zeta \end{pmatrix}.$$

By Example 1 above, $\sigma_{ap} \begin{pmatrix} T_\varphi & J \\ 0 & T_\psi \end{pmatrix} = \mathbb{T}$. It therefore follows that

$$\sigma_{ap}(M_C) = \sigma_{ap} \begin{pmatrix} T_\varphi & J \\ 0 & T_\psi \end{pmatrix} \bigcup \sigma_{ap} \begin{pmatrix} T_\varphi & 0 \\ 0 & T_\zeta \end{pmatrix} = \mathbb{T} \cup \sigma.$$

Example 3. One might conjecture that if M_C is bounded below then $R(B)$ is closed. But this is not the case. For example, in Example 1, take a function $\psi \in C(\mathbb{T})$ whose range includes 0, and consider M_C.

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