ON ALUTHGE TRANSFORMS OF p-HYPONORMAL OPERATORS

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Dedicated to Professor Tadasi Huruya on the occasion of his sixtieth birthday

Abstract. In this note we give an example of an ∞-hyponormal operator T whose Aluthge transform $\widetilde{T}$ is not $(1+\varepsilon)$-hyponormal for any $\varepsilon > 0$ and show that the sequence $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ of interated Aluthge transforms of T need not converge in the weak operator topology, which solve two problems in [6].

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry satisfying $\ker U = \ker T$ and $\ker U^* = \ker T^*$. If $T = U|T|$ then the *Aluthge transform* of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ (cf. [1],[6]). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if

$$(T^*T)^p - (TT^*)^p \geq 0, \qquad p \in (0, \infty). \tag{1.1}$$

If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is *semi-hyponormal.* It is well known that q-hyponormal operators are p-hyponormal operators for $p \leq q$. In particular, T is said to be *∞-hyponormal* if (1.1) holds for every $p > 0$. Notice that the subnormality for operators is different from the ∞-hyponormality (cf. [7]). In fact, D. Xia ([8]) introduced the notion of semi-hyponormal operators, which was generalized to p-hyponormal operators (cf. [3],[4]). It is well known ([1]) that if T is p-hyponormal, then $\widetilde{T}$ is $(p+\frac{1}{2})$-hyponormal for $0 < p < \frac{1}{2}$; hyponormal for $\frac{1}{2} \leq p \leq 1$. In this note we show that this result is extremal, in the sense that there is a hyponormal operator T whose Aluthge transform $\widetilde{T}$ is not $(1+\varepsilon)$-hyponormal for any $\varepsilon > 0$; this answers a question in [6, Problem 1.27] in the negative. In addition, we show that the sequence $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}{}_{n=1}^{\infty}$ of iterated Aluthge transforms of T need not converge in the weak operator topology; this answers [6, Conjecture 1.11] in the negative.

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2. The Hyponormality by Aluthge Transform

We adopt an idea of [2]. Let $\mathcal{H} \equiv \oplus_{i=-\infty}^{\infty} \mathcal{H}_i$, where $\mathcal{H}_i = \mathbb{C}^2$, and define T on $\mathcal{H}$ by

$$T := \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & C & 0 & & & & \\ & & C & [0] & & & \\ & & & D & 0 & & \\ & & & & D & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix}, \tag{2.1}$$

where C and D are positive semidefinite matrices in $M_2(\mathbb{C})$. Then we have $T = U|T|$, where $U = B \otimes I_2$ with the bilateral unweighted shift B on $\ell_2(\mathbb{Z})$, where $\mathbb{Z}$ is the set of integers, and the 2×2 identity matrix I_2, and so

$$\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & C & 0 & & & & \\ & & D^{\frac{1}{2}} C^{\frac{1}{2}} & [0] & & & \\ & & & D & 0 & & \\ & & & & D & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix}.$$

Therefore one can get the following lemma.

Lemma 2.1. *Let T be an operator defined as (2.1). Then the following assertions hold.*
(i) *T is p-hyponormal $\iff D^{2p} \geq C^{2p}$ for any $p > 0$.*
(ii) *$\widetilde{T}$ is p-hyponormal $\iff (C^{\frac{1}{2}} D C^{\frac{1}{2}})^p \geq C^{2p}$ and $D^{2p} \geq (D^{\frac{1}{2}} C D^{\frac{1}{2}})^p$ for any $p > 0$.*

Proof. Immediate from a straight forward calculation. □

In [6, Problem 1.27] the following question was addressed: If $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator, is $\widetilde{T}$ necessarily $(1+\varepsilon)$-hyponormal for some $\varepsilon > 0$? We now answer this question in the negative. In fact we can prove more:

Theorem 2.2. *There is an ∞-hyponormal operator T whose Aluthge transform $\widetilde{T}$ is not $(1+\varepsilon)$-hyponormal for any $\varepsilon > 0$.*

Proof. Let T_x be defined by (2.1) with

$$C = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \qquad (x \geq 1).$$

Then we have $D^{2p} \geq C^{2p}$ for any $p > 0$, and so T_x is ∞-hyponormal for any $x \geq 1$. Observe that

$$D^{\frac{1}{2}} C D^{\frac{1}{2}} = \frac{1}{2} \begin{pmatrix} x & \sqrt{x} \\ \sqrt{x} & 1 \end{pmatrix}.$$

By the mathematical induction we can see that for $n = 0, 1, 2, ...,$

$$\left(D^{\frac{1}{2}}CD^{\frac{1}{2}}\right)^{\frac{1}{2^n}} = 2^{-\frac{1}{2^n}}\left(\frac{1}{1+x}\right)^{\frac{2^n-1}{2^n}}\begin{pmatrix} x & \sqrt{x} \\ \sqrt{x} & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} & D^{2(1+\frac{1}{2^n})} - (D^{\frac{1}{2}}CD^{\frac{1}{2}})^{1+\frac{1}{2^n}} \\ = & \begin{pmatrix} x^{2(1+\frac{1}{2^n})} & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} x & \sqrt{x} \\ \sqrt{x} & 1 \end{pmatrix}\cdot 2^{-\frac{1}{2^n}}\left(\frac{1}{1+x}\right)^{\frac{2^n-1}{2^n}}\begin{pmatrix} x & \sqrt{x} \\ \sqrt{x} & 1 \end{pmatrix} \\ = & \begin{pmatrix} * & * \\ * & 1-(\frac{1}{2})^{1+\frac{1}{2^n}}(1+x)^{\frac{1}{2^n}} \end{pmatrix}. \end{aligned}$$

Note that for any $n \in \mathbb{N}$ we can choose a strictly increasing sequence $\{x_n\}$ of real numbers such that

$$1-\left(\frac{1}{2}\right)^{1+\frac{1}{2^n}}(1+x_n)^{\frac{1}{2^n}} < 0. \tag{2.2}$$

Therefore $D^{2(1+\frac{1}{2^n})} - (D^{\frac{1}{2}}CD^{\frac{1}{2}})^{1+\frac{1}{2^n}}$ is not positive semidefinite for x_n satisfying (2.2). Thus by Lemma 2.1, $\widetilde{T}_{x_n}$ is not $(1+\frac{1}{2^n})$-hyponormal. On the other hand, note that $\|T_x\| = x$. Let $\mathcal{K} \equiv \oplus_{i=1}^{\infty}\mathcal{H}_i$, where $\mathcal{H}_i = \mathcal{H}$, and define an operator T on $\mathcal{K}$ by

$$T = \oplus_{n=1}^{\infty}\frac{1}{x_n}T_{x_n}.$$

Then by the preceding consideration, T is ∞-hyponormal, but $\widetilde{T}$ is not $(1+\varepsilon)$-hyponormal for any $\varepsilon > 0$. This completes the proof. □

3. Aluthge Iteration

For $T \in \mathcal{L}(\mathcal{H})$, we define $\widetilde{T}^{(1)} := \widetilde{T}$ and $\widetilde{T}^{(n+1)} := \widetilde{(\widetilde{T}^{(n)})}$ for every $n \in \mathbb{N}$. In [6, Conjecture 1.11], the following was conjectured: For every $T \in \mathcal{L}(\mathcal{H})$ the sequence $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges in the norm topology to a limit $\widetilde{T}_L$ which is a quasinormal operator. We show that $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ need not converge in the weak operator topology below.

Let $T \equiv W_\alpha$ be a unilateral weighted shift on $\ell_2(\mathbb{Z}_+)$ with weight sequence $\alpha \equiv \{\alpha_n\}_{n\in\mathbb{Z}_+}$, where $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, that is, $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis for $\ell_2(\mathbb{Z}_+)$. If U is the unilateral unweighted shift and $D = \text{diag}\{\alpha_n\}$ then $T = UD$ is the polar decomposition of T. A straightforward calculation shows that $\widetilde{T} = D^{\frac{1}{2}}UD^{\frac{1}{2}}$ is a unilateral weighted shift with weight sequence $\{\sqrt{\alpha_n\alpha_{n+1}}\}_{n=0}^{\infty}$. By induction we can see that

$$\widetilde{T}^{(n)}e_k = \left(\prod_{j=0}^{n}\alpha_{j+k}^{\binom{n}{j}}\right)^{\frac{1}{2^n}} e_{k+1}, \qquad \text{for all } n \geq 1, \tag{3.1}$$

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.

Lemma 3.1. *Suppose that a and b are any distinct positive real numbers. Let $T := W_\alpha$ be a unilateral weighted shift whose weights are either a or b. Suppose* *that only finitely many weights of T are equal to a. Then the sequence of the first weights of $\widetilde{T}^{(n)}$ converges to b.*

Proof. Recall that the first weight of $\widetilde{T}^{(n)}$ is

$$\widetilde{\alpha}_0^{(n)} := \left(\prod_{j=0}^{n} \alpha_j^{\binom{n}{j}} \right)^{\frac{1}{2^n}}. \tag{3.2}$$

Let p be the largest number satisfying $\alpha_p = a$. Then for $n > p$, we have

$$\begin{aligned} \log \widetilde{\alpha}_0^{(n)} &= \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} \log \alpha_j \\ &= \frac{1}{2^n} \left[\sum_{j=0}^{p} \binom{n}{j} \log \alpha_j + \sum_{j=p+1}^{n} \binom{n}{j} \log b \right] \\ &= \frac{1}{2^n} \sum_{j=0}^{p} \frac{n!}{j!(n-j)!} \log \alpha_j + \frac{1}{2^n} \sum_{j=p+1}^{n} \binom{n}{j} \log b. \end{aligned}$$

Moreover, for $j = 1, ..., p$, we have

$$\frac{n!}{2^n j!(n-j)!} = \frac{n \cdot (n-1) \cdots (n-j+1)}{j! 2^n} \leq \frac{n^p}{2^n}, \tag{3.3}$$

which converges to 0 as $n \to \infty$. Hence since $\frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} = 1$ for any $n \in \mathbb{N}$, by (3.3) we have

$$\begin{aligned} \lim_{n\to\infty} \log \widetilde{\alpha}_0^{(n)} &= \lim_{n\to\infty} \frac{1}{2^n} \sum_{j=p+1}^{n} \binom{n}{j} \log b \\ &= \lim_{n\to\infty} \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} \log b \\ &= \log b. \end{aligned}$$

Hence the sequence of the first weights of $\widetilde{T}^{(n)}$ converges to b. □

Proposition 3.2. *Suppose a and b are any distinct positive real numbers. Then there is a unilateral weighted shift $T := W_\alpha$ with weight sequence α such that the sequence of the first weights of $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ have two subsequences converging to a and b, respectively.*

Proof. We first take $\alpha_0 = a$. Consider a weight sequence $\beta^{(1)} : a, b, b, ...$ Then by Lemma 3.1 with $\beta^{(1)}$, there is m_1 such that $|b - \widetilde{\beta}_{m_1}^{(1)}| < \frac{1}{2}$, where $\{\widetilde{\beta}_n^{(1)}\}_{n=1}^{\infty}$ is the sequence of the first weights of $\{\widetilde{W}_{\beta^{(1)}}^{(n)}\}_{n=1}^{\infty}$ induced by (3.1). Let

$$p_1 := \max\{ \binom{m_1}{j} : 0 \leq j \leq m_1 \}$$

and put $\alpha_1 = \cdots = \alpha_{p_1} = b$. And consider a weight sequence $\gamma^{(1)} := \{\gamma_n^{(1)}\}_{n=1}^{\infty}$ with

$$\gamma_n^{(1)} = \begin{cases} a & n = 0, \\ b & 1 \leq n \leq p_1, \\ a & p_1 < n. \end{cases}$$

By Lemma 3.1 with $\gamma^{(1)}$, there is n_1 such that $|a-\widetilde{\gamma}_{n_1}^{(1)}|<\frac{1}{2}$, where $\{\widetilde{\gamma}_n^{(1)}\}_{n=1}^{\infty}$ is the sequence of the first weights of $\{\widetilde{W}_{\gamma^{(1)}}^{(n)}\}_{n=1}^{\infty}$. Let

$$q_1 := \max\{\binom{n_1}{j}, p_1+1 : 0\le j\le n_1\}$$

and put $\alpha_{p_1+1}=...=\alpha_{q_1}=a$. Consider a weight sequence $\beta^{(2)}:=\{\beta_n^{(2)}\}_{n=1}^{\infty}$ with

$$\beta_n^{(2)}=\begin{cases} a & n=0,\\ b & 1\le n\le p_1,\\ a & p_1<n\le q_1,\\ b & q_1<n.\end{cases}$$

Applying Lemma 3.1 with $\beta^{(2)}$, we obtain m_2 such that $|b-\widetilde{\beta}_{m_2}^{(2)}|<\frac{1}{2^2}$, where $\{\widetilde{\beta}_n^{(2)}\}_{n=1}^{\infty}$ is the sequence of the first weights of $\{\widetilde{W}_{\beta^{(2)}}^{(n)}\}_{n=1}^{\infty}$. Let

$$p_2 := \max\{\binom{m_2}{j}, q_1+1 : 0\le j\le m_2\}.$$

Put $\alpha_{q_1+1}=...=\alpha_{p_2}=b$. Then similarly we may obtain a sequence $\gamma^{(2)}$ and n_2 such that $|a-\widetilde{\gamma}_{n_2}^{(2)}|<\frac{1}{2^2}$. Repeating this process alternately with n_k and m_k, we have $\beta^{(k)}$, $\gamma^{(k)}$, p_k, and q_k with

$$q_k := \max\{\binom{n_k}{j}, p_k+1 : 0\le j\le n_k\},\qquad k\in\mathbb{N},$$

and

$$p_{k+1} := \max\{\binom{m_{k+1}}{j}, q_k+1 : 0\le j\le m_{k+1}\},\qquad k\in\mathbb{N},$$

such that

$$|a-\widetilde{\gamma}_{n_k}^{(k)}|<\frac{1}{2^k}\qquad\text{and}\qquad |b-\widetilde{\beta}_{m_k}^{(k)}|<\frac{1}{2^k},\qquad k\in\mathbb{N}. \tag{3.4}$$

According to the above construction, we obtain a sequence α with

$$\alpha : \overbrace{\underbrace{\overbrace{a,b,...,b}^{(p_1)},a,...,a}_{(q_1)},b,...,b}^{(p_2)},a,a,...,a}_{(q_2)}\text{ (with the outer underbrace }(q_2)\text{ over } a,b,\dots,a),\ .....,$$

satisfying $\widetilde{\alpha}_0^{(m_k)}=\widetilde{\beta}_{m_k}^{(k)}$ and $\widetilde{\alpha}_0^{(n_k)}=\widetilde{\gamma}_{n_k}^{(k)}$, where $\{\widetilde{\alpha}_0^{(n)}\}_{n=1}^{\infty}$ is the sequence of the first weights of $\{\widetilde{W}_{\alpha}^{(n)}\}_{n=1}^{\infty}$. Hence by (3.4) we have

$$|a-\widetilde{\alpha}_0^{(n_k)}|<\frac{1}{2^k}\qquad\text{and}\qquad |b-\widetilde{\alpha}_0^{(m_k)}|<\frac{1}{2^k},\qquad k\in\mathbb{N}.$$

Thus the proof is complete. □

The following comes at once from Proposition 3.2.

Corollary 3.3. *There exists an operator T such that the sequence $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ does not converge in the weak operator topology.*

The operator T in the proof in Proposition 3.2 is not hyponormal. We were unable to decide whether $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges in the strong operator topology (or the weak operator topology) when T is hyponormal.

Theorem 3.4. *Let $T \equiv W_\alpha$ be a hyponormal bilateral weighted shift on $\ell_2(\mathbb{Z})$ with a weight sequence $\alpha \equiv \{\alpha_n\}_{n\in\mathbb{Z}}$. Let $a := \inf\{\alpha_n\}_{n\in\mathbb{Z}}$ and $b := \sup\{\alpha_n\}_{n\in\mathbb{Z}}$. Then $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges to a quasinormal operator in the norm topology if and only if $a = b$.*

Proof. Since the necessity is obvious, we only consider the sufficiency. Assume that $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges to a quasinormal operator $\widetilde{T}_L$ in the norm topology and suppose $a < b$. If U is the bilateral unweighted shift and $D = \operatorname{diag}\{\alpha_n\}$ then $T = UD$ is the polar decomposition of T. Thus $\widetilde{T} = D^{\frac{1}{2}}UD^{\frac{1}{2}}$ is also a bilateral weighted shift, and hence if the sequence $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges in the norm topology to an operator $\widetilde{T}_L$ then $\widetilde{T}_L$ must be a bilateral weighted shift. Note that T is hyponormal and the spectrum, $\sigma(T)$, of T is the annulus $\{\lambda : a \le |\lambda| \le b\}$. Thus for every n, $\widetilde{T}^{(n)}$ is hyponormal and $\sigma(\widetilde{T}^{(n)}) = \sigma(T)$ (cf. [6, Theorem 1.3]). Note that since $\widetilde{T}_L$ is a fixed point of the mapping $\widetilde{\ }: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, it follows that $\widetilde{T}_L$ is quasinormal (cf. [6, Proposition 1.10]). On the other hand, since σ, a set-valued function, is a continuous function when restricted to the set of hyponormal operators (cf. [5]), it follows that

$$\sigma(\widetilde{T}_L) = \sigma(T) = \{\lambda : a \le |\lambda| \le b\}.$$

Observe that a bilateral weighted shift W_β with weight sequence $\beta \equiv \{\beta_n\}$ is quasinormal if and only if either W_β is a scalar multiple of the bilateral unweighted shift or there exists an $n_0 \in \mathbb{Z}$ such that $\beta_n = 0$ for all $n < n_0$ and $\beta_n = \beta_{n_0}$ for all $n \ge n_0$. Thus the spectrum of a bilateral quasinormal weighted shift is a (possibly degenerated) circle or a disk with center 0. Thus $\widetilde{T}_L$ is not quasinormal, which is a contradiction. □

The following example shows the existence of an operator T such that $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converge in the strong operator topology but not the norm topology.

Example 3.5. Let $T \equiv W_\alpha$ be a hyponormal bilateral weighted shift on $\ell_2(\mathbb{Z})$ with weight sequence $\alpha \equiv \{\alpha_n\}_{n\in\mathbb{Z}}$, where α_n is given by

$$\alpha_n := \begin{cases} \frac{1}{2} & (n < 0), \\ 1 & (n \ge 0). \end{cases}$$

By Theorem 3.4, $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ does not converge to a quasinormal operator in the norm topology. In fact SOT-$\lim_{n\to\infty} \widetilde{T}^{(n)} = B$ (where B is the bilateral unweighted shift). Indeed, we first observe that

$$\widetilde{T}^{(n)} e_k = \left(\prod_{j=0}^{n} \alpha_{j+k}^{\binom{n}{j}} \right)^{\frac{1}{2^n}} e_{k+1}, \qquad \text{for all } n \in \mathbb{Z}.$$

So the weight sequence of $\widetilde{T}^{(n)}$ is composed of

$$\widetilde{\alpha}_k^{(n)} := \left(\prod_{j=0}^{n} \alpha_{j+k}^{\binom{n}{j}} \right)^{\frac{1}{2^n}}, \qquad k \in \mathbb{Z}.$$

For $n > k$, we have

$$|\log \widetilde{\alpha}_{-k}^{(n)}| = |\frac{1}{2^n} \cdot \sum_{j=0}^{n} \binom{n}{j} \log \alpha_{j-k}| = |\log 2 \cdot \frac{1}{2^n} \sum_{j=0}^{k-1} \frac{n!}{j!(n-j)!}|.$$

By (3.3) obviously

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{j=0}^{k-1} \frac{n!}{j!(n-j)!} = 0,$$

for a fixed $k \in \mathbb{N}$. Hence $\lim_{n\to\infty} \widetilde{\alpha}_{-k}^{(n)} = 1$ for each $k \in \mathbb{N}$. Since $\lim_{n\to\infty} \widetilde{\alpha}_{k}^{(n)} = 1$ for $k \geq 0$ obviously, SOT-$\lim_{n\to\infty} \widetilde{T}^{(n)} = B$.

If T is quasinormal, obviously $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges to T. If T is a hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^{\infty}$, which converges to α, then by the previous argument, $\widetilde{T}^{(n)}$ is a weighted shift with weight sequence

$$\left\{ \left(\prod_{j=0}^{n} \alpha_{j+k}^{\binom{n}{j}} \right)^{\frac{1}{2^n}} \right\}_{k=0}^{\infty} \qquad \text{for each } n \in \mathbb{Z}_+,$$

whose k-th weight, by a straightforward calculation, converges to α for each $k = 0, 1, \ldots$. Consequently, $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges to αU (where U is the unilateral unweighted shift) in the norm topology. Note that αU is quasinormal. Thus we may revise Conjecture 1.11 in [6] as following.

Conjecture 3.6 ([6])**.** If $T \in \mathcal{L}(\mathcal{H})$ is a p-hyponormal operator with $0 < p \leq \infty$, then $\{\widetilde{T}^{(n)}\}_{n=1}^{\infty}$ converges in the strong operator topology.

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