COWEN SETS FOR TOEPLITZ OPERATORS
WITH FINITE RANK SELFCOMMUTATORS

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Abstract. Cowen’s theorem states that if \( \varphi \in L^{\infty}(\mathbb{T}) \) then the Toeplitz operator \( T_\varphi \) is hyponormal if and only if the following ‘Cowen’ set \( \mathcal{E}(\varphi) \) is nonempty:

\[
\mathcal{E}(\varphi) = \{ k \in H^{\infty}(\mathbb{T}) : ||k||_{\infty} \leq 1 \text{ and } \varphi - k \varphi \in H^{\infty}(\mathbb{T}) \}.
\]

In this paper, we give a complete description on the Cowen set \( \mathcal{E}(\varphi) \) if the selfcommutator \([T_\varphi^*, T_\varphi] \) is of finite rank. In particular, it is shown that the solution for the cases where \( \varphi \) is of bounded type has a connection with a \( H^{\infty} \) optimization problem.

1. Introduction

A bounded linear operator \( A \) on a Hilbert space \( \mathcal{H} \) is said to be hyponormal if its selfcommutator \([A^*, A] = A^*A - AA^* \) is positive semidefinite. Recall that given \( \varphi \in L^{\infty}(\mathbb{T}) \), the Toeplitz operator with symbol \( \varphi \) is the operator \( T_\varphi \) on the Hardy space \( H^2(\mathbb{T}) \) of the unit circle \( \mathbb{T} = \partial \mathbb{D} \) in the complex plane \( \mathbb{C} \) defined by

\[
T_\varphi f = P(\varphi \cdot f),
\]

where \( f \in H^2(\mathbb{T}) \) and \( P \) denotes the orthogonal projection that maps \( L^2(\mathbb{T}) \) onto \( H^2(\mathbb{T}) \). Relationships between hyponormal operators and Toeplitz-like operators were discovered in papers [NF] and [Cla]. More recently, the problem of determining which symbols induce hyponormal Toeplitz operators was completely solved by C. Cowen [Co] in 1988. Here we shall employ an equivalent variant of Cowen’s theorem that was proposed by T. Nakazi and K. Takahashi in [NT].

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Cowen’s theorem. [Co], [NT] Suppose that \( \varphi \in L^\infty(\mathbb{T}) \) is arbitrary and put
\[
\mathcal{E}(\varphi) := \{ k \in H^\infty(\mathbb{T}) : ||k||_{\infty} \leq 1 \text{ and } \varphi - k\varphi \in H^\infty(\mathbb{T}) \}.
\]
Then \( T_\varphi \) is hyponormal if and only if the set \( \mathcal{E}(\varphi) \) is nonempty.

Cowen’s method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution of a certain functional equation involving its symbol. This approach has been put to use in the works [CL1], [CL2], [FL], [GS], [HKL], [HL], [NT], [Zhu] to study Toeplitz operators on the Hardy space of the unit circle.

Now the set \( \mathcal{E}(\varphi) \) will be called the Cowen set for the function \( \varphi \in L^\infty(\mathbb{T}) \). The question about the Cowen set \( \mathcal{E}(\varphi) \) is of great interest. Indeed, \( \mathcal{E}(\varphi) \) has been studied intensively in recent literature because when \( \varphi \) is of bounded type (i.e., quotient of two bounded analytic functions), it has a connection with the following \( H^\infty \) optimization problem which naturally arise in robust control theory (cf. [FF]):

**\( H^\infty \) optimization problem.** Let \( k_0 \in L^\infty(\mathbb{T}) \) and \( \theta \) a fixed inner function in \( H^\infty(\mathbb{T}) \). Find \( \mu \) where
\[
\mu = \text{dist} (k_0, \theta H^\infty) \equiv \inf_{h \in H^\infty} ||k_0 - \theta h||_{\infty}.
\]

In this paper it is shown that via Nehari’s theorem and Adamyan-Arov-Krein theorem, a solution of a \( H^\infty \) optimization problem provides information on \( \mathcal{E}(\varphi) \) when \( \varphi \) is of bounded type and \( T_\varphi \) has finite rank selfcommutator.

## 2. Main Results

We begin with the connection between Hankel and Toeplitz operators. For \( \varphi \) in \( L^\infty(\mathbb{T}) \), the Hankel operator \( H_\varphi : H^2 \to H^2 \) is defined by
\[
H_\varphi f = J(I - P)(\varphi f),
\]
where \( J : (H^2)^\perp \to H^2 \) is given by \( Jz^{-n} = z^{n-1} \) for \( n \geq 1 \). For \( \zeta \in L^\infty(\mathbb{T}) \), we define
\[
\tilde{\zeta} = \overline{\zeta(\overline{z})}.
\]
The following is a basic connection between Hankel and Toeplitz operators:
\[
T_\varphi \psi - T_\varphi T_\psi = H_\varphi^* H_\psi \quad (\varphi, \psi \in L^\infty) \quad \text{and} \quad H_\varphi T_h = H_{\varphi h} = T_h^* H_\varphi \quad (h \in H^\infty).
\]
From this we can see that if $k \in \mathcal{E}(\varphi)$ then
\[
[T^* \varphi, T \varphi] = H^*_\varphi H^*_\varphi - H^*_\varphi H \varphi = H^*_\varphi H^*_\varphi - H^*_k \varphi H \varphi = H^*_\varphi (1 - T_k T^*_k) H \varphi.
\]
For an inner function $\theta$, we write
\[
\mathcal{H}(\theta) \equiv H^2 \ominus \theta H^2.
\]
If $\varphi \in L^\infty$, write
\[
\varphi_+ \equiv P(\varphi) \in H^2 \quad \text{and} \quad \varphi_- \equiv (I - P)(\varphi) \in zH^2.
\]
Thus we can write $\varphi = \varphi_+ + \varphi_-$. Assume that $\varphi$ is of bounded type, i.e., there are functions $\psi_1, \psi_2$ in $H^\infty(\mathbb{D})$ such that
\[
\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}
\]
for almost all $z \in \mathbb{T}$. Since $T_z H \varphi = H \varphi T_z$ it follows from Beurling’s theorem that $\ker H^2_\varphi = \theta H^2$ and $\ker H^2_{\varphi^*} = \theta_+ H^2$ for some inner functions $\theta, \theta_+$. If $T \varphi$ is hyponormal then by (1), $||H_{\varphi^* +} f|| \geq ||H_{\varphi^*} f||$ for all $f \in H^2$, so that
\[
\theta_+ H^2 = \ker H_{\varphi^* +} \subseteq \ker H_{\varphi^*} = \theta H^2,
\]
which implies that $\theta$ divides $\theta_+$, i.e., $\theta_+ = \theta_0 \theta$ for some inner function $\theta_0$. Thus if $\varphi = \varphi_+ + \varphi_-$ is of bounded type and $T \varphi$ is hyponormal then we can write (cf. [GS])
\[
\varphi_+ = \theta_0 \theta \bar{a} \quad \text{and} \quad \varphi_- = \theta \bar{b},
\]
where $a \in \mathcal{H}(\theta_0 \theta)$ and $b \in \mathcal{H}(\theta)$. If $k_0 \in H^\infty$ is a solution of equation
\[
(2) \quad b - k_0 a = \theta h \quad \text{for some} \ h \in H^2
\]
then $\mathcal{E}(\varphi)$ can be written as
\[
\mathcal{E}(\varphi) = \{\theta_0 (k_0 + \theta f) : f \in H^\infty \text{ and } ||k_0 + \theta f||_\infty \leq 1\}.
\]
By Nehari’s Theorem [Ne], we have
\[
(3) \quad \text{dist} (k_0, \theta H^\infty) = \inf_{f \in H^\infty} ||\bar{\theta} k_0 + f||_\infty = ||H_{\bar{\theta} k_0}||.
\]
Thus we have (see [GS, Theorem 8])
\[
(4) \quad T \varphi \text{ is hyponormal} \iff ||H_{\bar{\theta} k_0}|| \leq 1.
\]

The following theorem is our main result, which gives a description on the Cowen set $\mathcal{E}(\varphi)$ when the selfcommutator $[T^* \varphi, T \varphi]$ is of finite rank. In fact we can prove more:
Theorem 1. If $\varphi$ is of bounded type then we have that:

(a) If $\ker H_\varphi \not\subseteq \ker [T_\varphi^*, T_\varphi]$ then $E(\varphi)$ is empty;
(b) If $\ker H_\varphi = \ker [T_\varphi^*, T_\varphi]$ and $\text{rank} [T_\varphi^*, T_\varphi] < \infty$ then $E(\varphi)$ contains infinitely many inner functions;
(c) If $\ker H_\varphi \subsetneq \ker [T_\varphi^*, T_\varphi]$ then $E(\varphi)$ contains a unique function which is inner.

If instead $\varphi$ is not of bounded type such that $T_\varphi$ is hyponormal then $E(\varphi)$ contains a unique function.

To prove Theorem 1 we need auxiliary lemmas.

The following lemma is another version of Cowen’s theorem.

Lemma 2. [CL1], [CL2, Lemma 1] If $\varphi \equiv \varphi_1 + \varphi_2 \in L^\infty$, then $E(\varphi) \neq \emptyset$ if and only if the equation $H_\varphi k = \bar{z}\tilde{\varphi}$ admits a solution $k$ satisfying $||k||_\infty \leq 1$.

T. Nakazi and K. Takahashi [NT] noticed that if $T_\varphi$ is a hyponormal operator such that its selfcommutator is of finite rank then $E(\varphi)$ contains a finite Blaschke product.

Lemma 3. (Nakazi-Takahashi Theorem) [NT] A Toeplitz operator $T_\varphi$ is hyponormal and the rank of the selfcommutator $[T_\varphi^*, T_\varphi]$ is finite if and only if there exists a finite Blaschke product $k$ in $E(\varphi)$ of the form

$$k(z) = e^{i\theta} \prod_{j=1}^n \frac{z - \beta_j}{1 - \beta_j z}$$

such that $\text{deg}(k) = \text{rank} [T_\varphi^*, T_\varphi]$, where $\text{deg}(k)$ denotes the degree of $k$ – meaning the number of zeros of $k$ in the open unit disk $\mathbb{D}$.

The following lemma is a solution of a $H^\infty$ optimization problem.

Lemma 4. If $b$ and $q$ are finite Blaschke products then

$$\text{deg}(b) \geq \text{deg}(q) \iff \text{dist}(b, qH^\infty) < 1.$$
where \( \text{wind}(\cdot) \) denotes the winding number with respect to the origin: indeed, this follows from the fact that (see [Ni, Appendix 4, Theorem 41])

\[
\text{dist} \left( u, H^\infty \right) < 1 \iff T_u \text{ is left invertible} \iff \text{wind} \left( u \right) \geq 0,
\]

where the second implication comes from the observation that \( T_u \) is Fredholm and hence, by Coburn’s theorem \( T_u \) is left or right invertible and the Fredholm index of \( T_u \) is equal to \( -\text{wind} \left( u \right) \). Applying (6) to \( u = \frac{b}{q} \) gives that

\[
\text{dist} \left( b, q H^\infty \right) < 1 \iff \text{wind} \left( \frac{b}{q} \right) \geq 0 \iff \deg \left( b \right) \geq \deg \left( q \right).
\]

\[\Box\]

We are ready for:

**Proof of Theorem 1.** From (1) we can see that if \( T_\varphi \) is hyponormal then

\[
\ker H_\varphi \subseteq \ker [T_\varphi^*, T_\varphi],
\]

which proves statement (a).

Towards statement (b), suppose \( \varphi \) is of bounded type. So we can write \( \varphi = \theta_0 \theta a + \bar{\theta} b \) for \( a \in \mathcal{H}(\theta_0 \theta) \) and \( b \in \mathcal{H}(\theta) \). Now suppose \( \ker H_\varphi = \ker [T_\varphi^*, T_\varphi] \) and \( \text{rank} [T_\varphi^*, T_\varphi] < \infty \). Since \( \ker H_\varphi = \theta_0 \theta H^2 \) it follows that

\[
\text{ran} [T_\varphi^*, T_\varphi] = (\ker [T_\varphi^*, T_\varphi])^\perp = (\ker H_\varphi)^\perp = H^2 \ominus \theta_0 \theta H^2,
\]

which implies that \( \theta_0 \theta \) is a finite Blaschke product since \( \text{ran} [T_\varphi^*, T_\varphi] \) is finite dimensional. Also, by Lemma 3 there exists a finite Blaschke product \( \theta_0 k_0 \) in \( \mathcal{E}(\varphi) \) such that \( \deg(\theta_0 k_0) = \text{rank} [T_\varphi^*, T_\varphi] \). Thus \( k_0 \) is a finite Blaschke product such that \( \deg(\theta_0 k_0) = \text{rank} H_\varphi = \deg(\theta_0 \theta) \), and hence \( \deg(k_0) = \deg(\theta) \). So by Lemma 4, we have that \( \text{dist} \left( k_0, \theta H^\infty \right) < 1 \), and hence by (3), \( ||H_{\theta k_0}|| < 1 \). Remembering Adamyan-Arov-Krein theorem which states that if \( f \in L^\infty \) and \( \text{dist} \left( f, H^\infty \right) < 1 \) then \( f + H^\infty \) contains a unimodular function, we can see that if \( ||H_{\theta k_0}|| < 1 \), then \( k_0 + \theta H^\infty \) contains an inner function. Thus \( \theta_0 k_0 + \theta_0 \theta H^\infty \) contains an inner function, and in turn, \( \mathcal{E}(\varphi) \) contains an inner function. Since

\[
1 > \text{dist} \left( \theta k_0, H^\infty \right) = \text{dist} \left( \bar{z} \theta k_0, \bar{z} H^\infty \right) = \text{dist} \left( \bar{z} \theta k_0 + \bar{z} c, H^\infty \right) \text{ for a suitable } c = ||H_{\bar{z} \theta (k_0 + \theta c)}||,
\]
we can choose different constants \( \alpha_n \) (\( n \in \mathbb{Z}_+ \)) such that \( ||H_{\tilde{z}\theta(k_0 + \theta \alpha_n)}|| < 1 \). Applying again Adamyan-Arov-Krein theorem to \( H_{\tilde{z}\theta(k_0 + \theta \alpha_n)} \), there exists \( q_n \in H^\infty \) such that \( k_0 + \theta \alpha_n + z\theta q_n \) are inner functions. Evidently, \( \theta_0 k_0 + \theta\theta_0(\alpha_n + z\theta q_n) \in \mathcal{E}(\varphi) \) and are different. This proves statement (b).

Towards statement (c), suppose \( \ker H_{\varphi} \not\subseteq \ker [T^*_\varphi, T_\varphi] \). If \( \mathcal{E}(\varphi) \) contains a function \( k \) which is not inner then \( \ker (1 - T_k T^*_k) = \{0\} \); indeed if \( g = T_k T^*_k g \) then \( ||g||^2 = ||T^*_k g||^2 \), and hence

\[
\int |g|^2 d\mu = ||g||^2 = ||T^*_k g||^2 \leq ||\tilde{k}||^2 ||g||^2 d\mu,
\]

which implies that \( g = 0 \) a.e. if \( \tilde{k} \) is not inner. Thus by (1) we have that \( \ker [T^*_\varphi, T_\varphi] \subseteq \ker H_{\varphi} \), which forces that \( \ker H_{\varphi} = \ker [T^*_\varphi, T_\varphi] \), a contradiction. If instead \( \mathcal{E}(\varphi) \) contains two different inner functions then \( \mathcal{E}(\varphi) \) has a function which is not inner: for if \( k_1 \) and \( k_2 \) (\( k_1 \neq k_2 \)) are inner functions in \( \mathcal{E}(\varphi) \) then we can easily see that \( k_1 + k_2 \) and \( k_2 \) is not an inner function since every inner function is an extreme point of the unit ball of \( H^\infty \). Thus \( \mathcal{E}(\varphi) \) contains a unique inner function. This proves statement (c).

For the second assertion write \( \varphi = \varphi_+ + \varphi_- \). If \( \varphi \) is not of bounded type then by an argument of Abrahamse [Ab, Lemma 3], we have that \( \ker H_{\varphi_+} = \ker H_{\varphi} = \{0\} \). Thus the solution \( k \) of the equation \( H_{\varphi_+} k = \tilde{z}\varphi_- \) should be unique. Thus the second assertion follows at once from Lemma 2. \( \square \)

We would like to remark that if \( H_{\tilde{\varphi} k_0} \) attains its norm (e.g., it is of finite rank) then \( \text{dist} (k_0, \theta H^\infty) = 1 \) implies that \( \mathcal{E}(\varphi) \) contains a unique inner function. To see this, recall (cf. [Ni, p.202]) that if \( f \in \tilde{L}^\infty \) and \( H_f \) attains its norm then \( f + H^\infty \) contains a unique element of least norm which is of the form \( \lambda \frac{h}{h'} \), where \( \lambda \in \mathbb{C} \), \( h \) is an outer function and \( \nu \) is an inner function. So if \( ||H_{\tilde{\varphi} k_0}|| = 1 \) and \( H_{\tilde{\varphi} k_0} \) attains its norm then by (3), \( \tilde{\varphi} k_0 + H^\infty \) contains a unique unimodular function. Thus \( \mathcal{E}(\varphi) \) contains a unique inner function.

We now turn our attention to the cases of Toeplitz operators with symbols that are trigonometric polynomials. If \( \varphi \) is a trigonometric polynomial of the form \( \varphi(z) = \sum_{n=-m}^N a_n z^n \), where \( a_{-m} \) and \( a_N \) are nonzero, then the rank of the selfcommutator \( [T^*_\varphi, T_\varphi] \) is finite. Thus if \( T_\varphi \) is hyponormal then by Lemma 3, \( \mathcal{E}(\varphi) \) contains a finite Blaschke product.

We now have:
Corollary 5. Let $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ be such that $T_\varphi$ is a hyponormal operator.

(a) If $\text{rank} [T_\varphi^*, T_\varphi] < N$ then $\mathcal{E}(\varphi)$ contains a unique finite Blaschke product;
(b) If $\text{rank} [T_\varphi^*, T_\varphi] = N$ then $\mathcal{E}(\varphi)$ contains infinitely many inner functions. Furthermore if $b \in \mathcal{E}(\varphi)$ is a finite Blaschke product then $\deg(b) \geq N$.

Proof. Since $\ker H_{\overline{\varphi}} = z^N H^2$, Part (a) corresponds to the case where $\ker H_{\overline{\varphi}} \subset \ker [T_\varphi^*, T_\varphi]$ and Part (b) corresponds to the case where $\ker H_{\overline{\varphi}} = \ker [T_\varphi^*, T_\varphi]$. Thus the statement (a) and the first assertion of statement (b) follow at once from Theorem 1 together with Lemma 3.

For the second assertion of statement (b), assume to the contrary that $b \in \mathcal{E}(\varphi)$ is a finite Blaschke product of degree less than $N$. By Lemma 3, there exists a finite Blaschke product $k \in \mathcal{E}(\varphi)$ of degree $N$. Then we have

$$\hat{k}(j) = \hat{b}(j) \quad \text{for } j = 1, \ldots, N - 1,$$

where $\hat{f}(j)$ means the $j$-th Fourier coefficients of $f \in H^\infty$. Thus by the uniqueness argument of [HL, Lemma 1] we should have that $b = k$, a contradiction. \qed

Corollary 5(a) is an extended result of [HKL, Corollary 4]. The following is an immediate result from Corollary 5.

Corollary 6. Suppose that $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ and that $k$ is a finite Blaschke product in $\mathcal{E}(\varphi)$.

(a) If $\deg(k) < N$ then $\text{rank} [T_\varphi^*, T_\varphi] = \deg(k)$;
(b) If $\deg(k) \geq N$ then $\text{rank} [T_\varphi^*, T_\varphi] = N$.

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References


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