HYPONORMAL TOEPLITZ OPERATORS WITH RATIONAL SYMBOLS

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Mathematics Subject Classification (2000): 47B20, 47B35

Abstract. In this paper we consider the self-commutators of Toeplitz operators T_φ with rational symbols φ using the classical Hermite-Fejér interpolation problem. Our main theorem is as follows. Let $\varphi = \overline{g} + f \in L^\infty$ and let $f = \theta\overline{a}$ and $g = \theta\overline{b}$, where θ is a finite Blaschke product of degree d and $a, b \in \mathcal{H}(\theta) := H^2 \ominus \theta H^2$. Then $\mathcal{H}(\theta)$ is a reducing subspace of $[T_\varphi^*, T_\varphi]$, and $[T_\varphi^*, T_\varphi]$ has the following representation relative to the direct sum $\mathcal{H}(\theta) \oplus \mathcal{H}(\theta)^\perp$:

$$[T_\varphi^*, T_\varphi] = A(a)^* W M(\varphi) W^* A(a) \bigoplus 0_\infty,$$

where $A(a) := P_{\mathcal{H}(\theta)} M_a \mid_{\mathcal{H}(\theta)}$ (M_a is the multiplication operator with symbol a), W is the unitary operator from $\mathbb{C}^d$ onto $\mathcal{H}(\theta)$ defined by $W := (\phi_1, \cdots, \phi_d)$ ($\{\phi_j\}$ is an orthonormal basis for $\mathcal{H}(\theta)$), and $M(\varphi)$ is a matrix associated with the classical Hermite-Fejér interpolation problem. Hence, in particular, T_φ is hyponormal if and only if $M(\varphi)$ is positive. Moreover the rank of the self-commutator $[T_\varphi^*, T_\varphi]$ is given by $\operatorname{rank}[T_\varphi^*, T_\varphi] = \operatorname{rank} M(\varphi)$.

1 Introduction

For φ in $L^\infty(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$, the *Toeplitz operator with symbol* φ is the operator T_φ on the Hardy space $H^2(\mathbb{T})$ of the unit circle given by

$$T_\varphi f := P(\varphi f) \quad (f \in H^2(\mathbb{T})),$$

where P denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. A bounded linear operator A is called hyponormal if its self-commutator $[A^*, A] := A^*A - AA^*$ is positive (semidefinite). Normal Toeplitz operators were characterized by a property of their symbols in the early 1960's by A. Brown and P. Halmos [BH] and 25 years passed before the exact nature of the relationship between the symbol $\varphi \in L^\infty$ and the positivity of the self-commutator $[T_\varphi^*, T_\varphi]$ was understood (via Cowen's theorem [Co]). We shall employ an equivalent variant of Cowen's theorem [Co], that was first proposed by Nakazi and Takahashi [NT].

This work was supported by a grant (R14-2003-006-01000-0) from the Korea Research Foundation.

Cowen's Theorem. *For $\varphi \in L^\infty$, write*

$$\mathcal{E}(\varphi) := \{k \in H^\infty : ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

Cowen's theorem is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution with specified properties to a certain functional equation involving the operator's symbol. This approach has been put to use in the works [CL], [FL], [Gu1], [Gu2], [GS], [HL], [NT], [Zhu] to study Toeplitz operators on $H^2(\mathbb{T})$. Particular attention has been paid to Toeplitz operators with polynomial symbols. In particular, K. Zhu [Zhu] has applied Cowen's criterion and Schur's algorithm [Sch] to the Schur function Φ_N to obtain an abstract characterization of those polynomial symbols that correspond to hyponormal Toeplitz operators.

On the other hand, a function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are functions ψ_1, ψ_2 in $H^\infty(\mathbb{D})$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$$

for almost all z in $\mathbb{T}$. Evidently, rational functions in L^∞ are of bounded type. In this paper we present an explicit description of the self-commutators of Toeplitz operators with bounded type symbols associated with a finite Blaschke product (or equivalently, rational symbols).

2 Preliminaries and auxiliary lemmas

Let J be the unitary operator on L^2 defined by

$$J(f)(z) = \overline{z}f(\overline{z}).$$

For $\varphi \in L^\infty$, the operator on H^2 defined by

$$H_\varphi f = J(I-P)(\varphi f)$$

is called the *Hankel operator* H_φ with symbol φ. If we define the function $\widetilde{v}$ by $\widetilde{v}(z) := \overline{v(\overline{z})}$, then H_φ can be viewed as the operator on H^2 defined by

$$(1.1) \qquad \langle zuv, \overline{\varphi}\rangle = \langle H_\varphi u, \widetilde{v}\rangle \quad \text{for all } v \in H^\infty.$$

The following is a basic connection between Hankel and Toeplitz operators ([Ni]):

$$T_{\varphi\psi} - T_\varphi T_\psi = H^*_{\overline{\varphi}} H_\psi \ \ (\varphi, \psi \in L^\infty) \quad \text{and} \quad H_\varphi T_h = H_{\varphi h} = T^*_{\widetilde{h}} H_\varphi \ \ (h \in H^\infty).$$

From this we can see that if $k \in \mathcal{E}(\varphi)$ then

$$[T^*_\varphi, T_\varphi] = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_\varphi H_\varphi = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_{k\,\overline{\varphi}} H_{k\,\overline{\varphi}} = H^*_{\overline{\varphi}}(1 - T_{\widetilde{k}} T^*_{\widetilde{k}}) H_{\overline{\varphi}}.$$

If θ is an inner function, the degree of θ, denoted by $\deg(\theta)$, is defined as n if θ is a finite Blaschke product of the form

$$\theta(z) = e^{i\xi} \prod_{j=1}^{n} \frac{z-\beta_j}{1-\overline{\beta_j}z} \qquad (|\beta_j| < 1 \text{ for } j = 1, \cdots, n),$$

otherwise the degree of θ is infinite. For an inner function θ, write

$$\mathcal{H}(\theta) := H^2 \ominus \theta H^2.$$

Note that $\ker H_{\overline{\theta}} = \theta H^2$ and $\operatorname{ran} H^*_{\overline{\theta}} = \mathcal{H}(\theta)$. It was shown [Ab, Lemma 6] that if T_φ is hyponormal and φ is not in H^∞ then

$$\varphi \text{ is of bounded type } \iff \overline{\varphi} \text{ is of bounded type.}$$

In [Ab], it was also shown that

$$\varphi \text{ is of bounded type } \iff \ker H_\varphi \neq \{0\} \iff \varphi = \overline{\theta} b,$$

where θ is an inner function and $b \in H^\infty$ satisfies that the inner parts of b and θ are coprime. So we have

$$\ker H_{\overline{\theta}b} = \theta H^2 \quad \text{and} \quad \operatorname{cl} \operatorname{ran} H_{\overline{\theta}b} = \mathcal{H}(\widetilde{\theta}). \tag{1.2}$$

On the other hand, when we study the hyponormality of Toeplitz operators T_φ with symbols φ, we may assume that $\varphi(0) = 0$ because the hyponormality of an operator is invariant under translation by scalars. Thus if $\varphi = \overline{g} + f \in L^\infty$ $(f, g \in H^2)$, then we will assume that $f(0) = g(0) = 0$ throughout the paper. Therefore we can see (cf. [GS], [Gu2]) that if $\varphi = \overline{g} + f \in L^\infty$ $(f, g \in H^2)$ is of bounded type and T_φ is hyponormal then we can write

$$f = \theta_1 \theta_2 \overline{a} \quad \text{and} \quad g = \theta_1 \overline{b} \tag{1.3}$$

for some inner functions θ_1 and θ_2, where $a \in \mathcal{H}(\theta_1\theta_2)$ and $b \in \mathcal{H}(\theta_1)$.

To prove the main result we need several auxiliary lemmas. The first lemma gives a way to compute the rank of a product of two Hankel operators.

Lemma 2.1 (Axler-Chang-Sarason Theorem [ACS]). *For $\varphi, \psi \in L^\infty$,*

$$\operatorname{rank}(H^*_\varphi H_\psi) = \min\{\operatorname{rank}(H_\varphi),\ \operatorname{rank}(H_\psi)\}.$$

The next result is a characterization of hyponormal Toeplitz operators whose self-commutator is of finite rank.

Lemma 2.2 (Nakazi-Takahashi Theorem [NT]). *A Toeplitz operator T_φ is hyponormal and $[T^*_\varphi, T_\varphi]$ is a finite rank operator if and only if there exits a finite Blaschke product k in $\mathcal{E}(\varphi)$. In this case, we can choose k such that* $\deg(k) = \operatorname{rank}[T^*_\varphi, T_\varphi]$.

For a subspace $\mathcal{M}$ of H^2, let $P_{\mathcal{M}}$ be the orthogonal projection onto $\mathcal{M}$. Then we have:

Lemma 2.3. *If $f=\theta_1\theta_2\overline{a}$ for $a\in\mathcal{H}(\theta_1\theta_2)$ then*

$$\overline{\theta_2}P_{\theta_2H^2}(f)=P(\theta_1\overline{a})=\theta_1\overline{P_{\mathcal{H}(\theta_1)}(a)}+c \quad \textit{for some constant } c.$$

Proof. Let $g\in H^2$ be arbitrary. Then

$$\langle\overline{\theta_2}P_{\theta_2H^2}(f),\ g\rangle=\langle P_{\theta_2H^2}(\theta_1\theta_2\overline{a}),\ \theta_2 g\rangle=\langle\theta_1\theta_2\overline{a},\ \theta_2 g\rangle=\langle P(\theta_1\overline{a}),\ g\rangle.$$

Therefore we have that $P(\theta_1\overline{a})=\overline{\theta_2}P_{\theta_2H^2}(f)$. For the second equality, let $a_1:=P_{\mathcal{H}(\theta_1)}(a)$ and $a_2:=a-a_1$. Then we have

$$P(\theta_1\overline{a})=P(\theta_1\overline{a_1})+P(\theta_1\overline{a_2})=\theta_1\overline{a_1}+P(\theta_1\overline{a_2}).$$

But since $\mathcal{H}(\theta_1\theta_2)=\mathcal{H}(\theta_1)\oplus\theta_1\mathcal{H}(\theta_2)$ for inner functions θ_1 and θ_2, it follows that $a_2\in\theta_1\mathcal{H}(\theta_2)$. Therefore we can conclude that $P(\theta_1\overline{a_2})\in P(\overline{\mathcal{H}(\theta_2)})\in\mathbb{C}$. This completes the proof. □

Lemma 2.4. *Let $\varphi=\overline{g}+f\in L^\infty$. If $f=\theta_1\theta_2\overline{a}$ and $g=\theta_1\overline{b}$ for $a\in\mathcal{H}(\theta_1\theta_2)$ and $b\in\mathcal{H}(\theta_1)$, then $\theta_1\,\mathcal{H}(\theta_2)\subseteq$ ran $[T_\varphi^*,T_\varphi]\subseteq\mathcal{H}(\theta_1\theta_2)$.*

Proof. Observe that

$$[T_\varphi^*,T_\varphi]=H_{\overline{f}}^*H_{\overline{f}}-H_{\overline{g}}^*H_{\overline{g}}=H_{\overline{\theta_1\theta_2 a}}^*H_{\overline{\theta_1\theta_2 a}}-H_{\overline{\theta_1 b}}^*H_{\overline{\theta_1 b}}. \tag{2.1}$$

Since $\mathrm{cl\,ran}\,(H_{\overline{\theta_1\theta_2 a}}^*H_{\overline{\theta_1\theta_2 a}})=\mathrm{cl\,ran}\,H_{\overline{\theta_1\theta_2 a}}^*=\mathcal{H}(\theta_1\theta_2)$ and $\mathrm{cl\,ran}\,(H_{\overline{\theta_1 b}}^*H_{\overline{\theta_1 b}})=\mathcal{H}(\theta_1)$, we can see that $\theta_1\mathcal{H}(\theta_2)\subseteq\mathrm{ran}\,[T_\varphi^*,T_\varphi]\subseteq\mathcal{H}(\theta_1\theta_2)$. □

Lemma 2.5. *Let $\varphi=\overline{g}+f\in L^\infty$, where f and g are in H^2. If φ is of bounded type and T_φ is hyponormal then*

$$\mathrm{rank}\,[T_\varphi^*,T_\varphi]=\min\{\deg(k):k \textit{ is an inner function in } \mathcal{E}(\varphi)\}.$$

Proof. If φ is of bounded type such that T_φ is hyponormal then $\mathcal{E}(\varphi)$ contains at least an inner function (see [Le]). If $\mathcal{E}(\varphi)$ has no finite Blaschke product then by Lemma 2.2 we have that for all k in $\mathcal{E}(\varphi)$, $\mathrm{rank}\,[T_\varphi^*,T_\varphi]=\infty=\deg(k)$. If instead $\mathcal{E}(\varphi)$ has a finite Blaschke product then it suffices to show that there exists an inner function k in $\mathcal{E}(\varphi)$ such that $\mathrm{rank}\,(H_{\overline{k}})\le\mathrm{rank}(H_{\overline{f}})$. We assume to the contrary that $\mathrm{rank}\,(H_{\overline{f}})<\mathrm{rank}\,(H_{\overline{k}})$ for all inner functions k in $\mathcal{E}(\varphi)$. Since k is an inner function we have that

$$[T_\varphi^*,T_\varphi]=H_{\overline{f}}^*H_{\overline{f}}-H_{\overline{g}}^*H_{\overline{g}}=H_{\overline{f}}^*H_{\overline{f}}-H_{k\overline{f}}^*H_{k\overline{f}}=H_{\overline{f}}^*H_{\overline{k}}H_{\overline{k}}^*H_{\overline{f}}.$$

By Lemma 2.1 we see that

$$\mathrm{rank}\,[T_\varphi^*,T_\varphi]=\mathrm{rank}\,(H_{\overline{f}}^*H_{\overline{k}})=\min\{\mathrm{rank}\,(H_{\overline{f}}),\mathrm{rank}\,(H_{\overline{k}})\}.$$

But since $\mathrm{rank}\,(H_{\overline{f}})<\deg(k)$, it follows that $\mathrm{rank}\,[T_\varphi^*,T_\varphi]<\deg(k)$, which contradicts Lemma 2.2. This completes the proof. □

The following lemma is a slight extension of [Gu2, Corollary 3.5], in which the rank of the self-commutator is finite.

Lemma 2.6. *Let $\varphi = \overline{g} + f \in L^\infty$, where f and g are in H^2. Assume that*

$$f = \theta_1\theta_2\overline{a} \quad \textit{and} \quad g = \theta_1\overline{b} \tag{2.2}$$

for $a \in \mathcal{H}(\theta_1\theta_2)$ and $b \in \mathcal{H}(\theta_1)$. Let $\psi := \theta_1\overline{P_{\mathcal{H}(\theta_1)}(a)} + \overline{g}$. Then T_φ is hyponormal if and only if T_ψ is. Moreover, in the cases where T_φ is hyponormal,

$$\text{rank}\,[T_\varphi^*, T_\varphi] = \deg(\theta_2) + \text{rank}\,[T_\psi^*, T_\psi].$$

Proof. The first assertion follows at once from [Gu2, Corollary 3.5] together with Lemma 2.3.

For the rank formula, note that $\mathcal{E}(\varphi) = \{k_1\theta_2 : k_1 \in \mathcal{E}(\psi)\}$. Therefore by Lemma 2.5 we have that rank $[T_\varphi^*, T_\varphi] = \deg(\theta_2) + \text{rank}\,[T_\psi^*, T_\psi]$. □

3 Main results

In view of Lemma 2.6, when we study the hyponormality of Toeplitz operators with bounded type symbols φ, we may assume that the symbol $\varphi = \overline{g} + f \in L^\infty$ is of the form

$$f = \theta\overline{a} \quad \text{and} \quad g = \theta\overline{b}, \tag{3.1}$$

where θ is an inner function and $a, b \in \mathcal{H}(\theta)$ such that the inner parts of a, b and θ are coprime.

On the other hand, if $\varphi = \overline{g} + f \in L^\infty$, where f and g are rational functions then we can show that φ can be written in the form (3.1) with a finite Blaschke product θ. C. Gu [Gu1] showed that if $\varphi = \overline{g} + f \in L^\infty$, where f and g are rational functions then the problem determining the hyponormality of T_φ is exactly the tangential Hermite-Fejér interpolation problem. By comparison, using the classical Hermite-Fejér interpolation problem, we will give an explicit description of the self-commutator $[T_\varphi^*, T_\varphi]$.

To begin with, let θ be a finite Blaschke product of degree d. We can write

$$\theta = e^{i\xi}\prod_{k=1}^{n}\left(\widetilde{B_k}\right)^{m_k} \qquad (\text{where } \widetilde{B_k} := \frac{z-\alpha_k}{1-\overline{\alpha_k}z}). \tag{3.2}$$

So $d = \sum_{k=1}^{n} m_k$. For our purpose, rewrite θ as in the form $\theta = e^{i\xi}\prod_{j=1}^{d} B_j$, where

$$B_j := \widetilde{B_k} \quad \text{if } \sum_{l=0}^{k-1} m_l < j \le \sum_{l=0}^{k} m_l$$

and, for notational convenience, $m_0 := 0$. For example, the first Balschke product $\widetilde{B_1}$ is repeated m_1 times and so on. Let

$$\phi_j := \frac{q_j}{1-\overline{\alpha_j}z}\,B_{j-1}B_{j-2}\cdots B_1 \quad (1 \le j \le d), \tag{3.3}$$

where $\phi_1 := q_1(1-\overline{\alpha_1}z)^{-1}$ and $q_j := (1-|\alpha_j|^2)^{\frac{1}{2}}$. It is well known that $\{\phi_j\}_1^d$ is an orthonormal basis for $\mathcal{H}(\theta)$ (cf. [FF,Theorem X.1.5]).

Let $\varphi = \overline{g} + f \in L^\infty$, where $g = \theta\overline{b}$ and $f = \theta\overline{a}$ for $a, b \in \mathcal{H}(\theta)$ and write

$$\mathcal{C}(\varphi) := \{k \in H^\infty : \varphi - k\overline{\varphi} \in H^\infty\}.$$

Then k is in $\mathcal{C}(\varphi)$ if and only if $\overline{\theta}b - k\overline{\theta}a \in H^2$, or equivalently,

$$b - ka \in \theta H^2. \tag{3.4}$$

Note that $\theta^{(n)}(\alpha_i) = 0$ for all $0 \le n < m_i$. Thus the condition (3.4) is equivalent to the following equation: for all $1 \le i \le n$,

$$\begin{pmatrix} k_{i,0} \\ k_{i,1} \\ k_{i,2} \\ \vdots \\ k_{i,m_i-2} \\ k_{i,m_i-1} \end{pmatrix} = \begin{pmatrix} a_{i,0} & 0 & 0 & 0 & \cdots & 0 \\ a_{i,1} & a_{i,0} & 0 & 0 & \cdots & 0 \\ a_{i,2} & a_{i,1} & a_{i,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{i,m_i-2} & a_{i,m_i-3} & \ddots & \ddots & a_{i,0} & 0 \\ a_{i,m_i-1} & a_{i,m_i-2} & \cdots & a_{i,2} & a_{i,1} & a_{i,0} \end{pmatrix}^{-1} \begin{pmatrix} b_{i,0} \\ b_{i,1} \\ b_{i,2} \\ \vdots \\ b_{i,m_i-2} \\ b_{i,m_i-1} \end{pmatrix}, \tag{3.5}$$

where

$$k_{i,j} := \frac{k^{(j)}(\alpha_i)}{j!}, \quad a_{i,j} := \frac{a^{(j)}(\alpha_i)}{j!} \quad \text{and} \quad b_{i,j} := \frac{b^{(j)}(\alpha_i)}{j!}.$$

Thus k is in $\mathcal{C}(\varphi)$ if and only if k is a function in H^∞ for which

$$\frac{k^{(j)}(\alpha_i)}{j!} = k_{i,j} \qquad (1 \le i \le n,\ 0 \le j < m_i), \tag{3.6}$$

where the $k_{i,j}$ are determined by the equation (3.5). If in addition $||k||_\infty \le 1$ is required then this is exactly the classical Hermite-Fejér interpolation problem.

To construct a polynomial $k(z) = p(z)$ satisfying (3.6), let $p_i(z)$ be the polynomial of order $d - m_i$ defined by

$$p_i(z) := \prod_{\substack{k=1 \\ k \ne i}}^{n} \Big(\frac{z - \alpha_k}{\alpha_i - \alpha_k}\Big)^{m_k}.$$

Also consider the polynomial $p(z)$ of degree $d - 1$ defined by

$$p(z) := \sum_{i=1}^{n} \Big(k'_{i,0} + k'_{i,1}(z - \alpha_i) + k'_{i,2}(z - \alpha_i)^2 + \cdots + k'_{i,m_i-1}(z - \alpha_i)^{m_i-1}\Big)\, p_i(z), \tag{3.7}$$

where the $k'_{i,j}$ are obtained by the following equations:

$$k'_{i,j} = k_{ij} - \sum_{k=0}^{j-1} \frac{k'_{i,k}\, p_i^{(j-k)}(\alpha_i)}{(j-k)!} \quad (1 \le i \le n;\ 0 \le j < m_i) \quad \text{and} \quad k'_{i,0} = k_{i,0} \quad (1 \le i \le n).$$

Then $p(z)$ satisfies (3.6) (See [FF]). But $p(z)$ may not be contractive.

On the other hand, if ψ is a function in H^∞, let $A(\psi)$ be the operator on $\mathcal{H}(\theta)$ defined dy

$$A(\psi) := P_{\mathcal{H}(\theta)} M_\psi \,|_{\mathcal{H}(\theta)}, \tag{3.8}$$

where M_ψ is the multiplication operator with symbol ψ. Now let W be the unitary operator from $\mathbb{C}^d$ onto $\mathcal{H}(\theta)$ defined by

$$W := (\phi_1, \phi_2, \cdots, \phi_d),$$

where the ϕ_j are the functions in (3.3).

We then have:

Lemma 3.1. ([FF, Theorems X.1.5 and X.5.6]) *Let θ be the Blaschke product in (3.2) and let $\{\phi_j\}_1^d$ be the orthonormal basis for $\mathcal{H}(\theta)$ in (3.3). Then $A(z) = P_{\mathcal{H}(\theta)}M_z\,|_{\mathcal{H}(\theta)}$ is unitarily equivalent to the lower triangular matrix M on $\mathbb{C}^d$ defined by*

$$\begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ q_1q_2 & \alpha_2 & 0 & 0 & 0 & \cdots & 0 \\ -q_1\overline{\alpha_1}q_3 & q_2q_3 & \alpha_3 & 0 & 0 & \cdots & 0 \\ q_1\overline{\alpha_2\alpha_3}q_4 & -q_2\overline{\alpha_3}q_4 & q_3q_4 & \alpha_4 & 0 & \cdots & 0 \\ -q_1\overline{\alpha_2\alpha_3\alpha_4}q_5 & q_2\overline{\alpha_3\alpha_4}q_5 & -q_3\overline{\alpha_4}q_5 & q_4q_5 & \alpha_5 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ (-1)^d q_1\left(\prod_{j=2}^{d-1}\overline{\alpha_j}\right)q_d & (-1)^{d-1}q_2\left(\prod_{j=3}^{d-1}\overline{\alpha_j}\right)q_d & \cdots & \cdots & -q_{d-2}\overline{\alpha_{d-1}}q_d & q_{d-1}q_d & \alpha_d \end{pmatrix}.$$

Moreover, if p is a polynomial defined in (3.7) then $A(p)W = Wp(M)$.

Our main theorem now follows:

Theorem 3.2. *Let $\varphi = \overline{g} + f \in L^\infty$ and let $f = \theta\overline{a}$ and $g = \theta\overline{b}$, where θ is a finite Blaschke product and $a, b \in \mathcal{H}(\theta)$. Then $\mathcal{H}(\theta)$ is a reducing subspace of $[T_\varphi^*, T_\varphi]$, and $[T_\varphi^*, T_\varphi]$ has the following representation relative to the direct sum $\mathcal{H}(\theta) \oplus \mathcal{H}(\theta)^\perp$:*

$$[T_\varphi^*, T_\varphi] = A(a)^* W M(\varphi) W^* A(a) \bigoplus 0_\infty, \tag{3.9}$$

where $A(a)$ is invertible and $M(\varphi) := I_{\mathcal{H}(\theta)} - p(M)^*p(M)$. Hence, in particular, T_φ is hyponormal if and only if $M(\varphi)$ is positive. Moreover the rank of the self-commutator $[T_\varphi^*, T_\varphi]$ is given by

$$\operatorname{rank}\,[T_\varphi^*, T_\varphi] = \operatorname{rank} M(\varphi).$$

Proof. From the proof of Lemma 2.4 we can see that $\operatorname{ran}[T_\varphi^*.T_\varphi] \subseteq \mathcal{H}(\theta)$. Therefore $\mathcal{H}(\theta)$ is a reducing subspace of $[T_\varphi^*, T_\varphi]$.

Towards the equality (3.9), let u and v be in $\mathcal{H}(\theta)$. Suppose $k = p$ is a polynomial in (3.7). Since $\ker H_{\overline{\theta}} = \theta H^2$, we have that $H_{\overline{\theta}k}u = H_{\overline{\theta}}(P_{\mathcal{H}(\theta)}(ku))$. Note that $H_{\overline{\theta}}^* H_{\overline{\theta}}$ is the projection onto $\mathcal{H}(\theta)$. Thus we have that

$$\begin{aligned} \langle H_{\widetilde{\overline{\theta}k}} H_{\overline{\theta}k}u, v\rangle &= \langle H_{\overline{\theta}k}u, H_{\overline{\theta}k}v\rangle \\ &= \langle P_{\mathcal{H}(\theta)}ku, P_{\mathcal{H}(\theta)}kv\rangle \\ &= \langle A(k)u, A(k)v\rangle. \end{aligned} \tag{3.10}$$

Thus by Lemma 3.1 we have that

$$H_{\widetilde{\overline{\theta}k}} H_{\overline{\theta}k}|_{\mathcal{H}(\theta)} = A(k)^*A(k) = Wk(M)^*k(M)W^*. \tag{3.11}$$

Hence by (3.11) we get

$$\big(H_{\overline{\theta}}^* H_{\overline{\theta}} - H_{\widetilde{\overline{\theta}k}} H_{\overline{\theta}k}\big)|_{\mathcal{H}(\theta)} = W\big(I_{\mathcal{H}(\theta)} - k(M)^*k(M)\big)W^*.$$

Since k satisfies the equality (3.5) and hence $\varphi - k\overline{\varphi} \in H^\infty$, it follows that

$$\begin{aligned}
[T_\varphi^*, T_\varphi]|_{\mathcal{H}(\theta)} &= \big(H_{\overline{f}}^* H_{\overline{f}} - H_{\overline{g}}^* H_{\overline{g}}\big)|_{\mathcal{H}(\theta)} \\
&= \big(H_{\overline{f}}^* H_{\overline{f}} - H_{k\overline{f}}^* H_{k\overline{f}}\big)|_{\mathcal{H}(\theta)} \\
&= \big(H_{\overline{\theta}a}^* H_{\overline{\theta}a} - H_{k\overline{\theta}a}^* H_{k\overline{\theta}a}\big)|_{\mathcal{H}(\theta)} \\
&= T_a^* (H_{\overline{\theta}}^* H_{\overline{\theta}} - H_{\overline{\theta}k}^* H_{\overline{\theta}k}) T_a|_{\mathcal{H}(\theta)} \\
&= A(a)^* W \big(I_{\mathcal{H}(\theta)} - p(M)^* p(M)\big) W^* A(a) \\
&= A(a)^* W M(\varphi) W^* A(a),
\end{aligned}$$

which gives (3.9).

For the invertibility of $A(a)$, suppose $A(a)^* f = 0$ for some $f \in \mathcal{H}(\theta)$. Then $P_{\mathcal{H}(\theta)}(\overline{a}f) = 0$ and hence

$$\overline{a}f = \theta g \quad \text{for some } g \in H^2,$$

or equivalently, $\overline{a}\overline{\theta}f = g$. Note that $\overline{\theta}f \in {H^2}^\perp$ and hence $\overline{a}\overline{\theta}f \in {H^2}^\perp \cap H^2 = \{0\}$. Thus we have $f = 0$, which implies that $A(a)^*$ is 1-1. Since $A(a)$ is a finite dimensional operator, $A(a)$ is invertible. This completes the proof. $\square$

Example 3.3. C. Gu [Gu1] showed that if $\varphi = f + \overline{g} \in L^\infty$, where f and g are rational functions then the problem determining the hyponormality of T_φ is exactly the tangential Hermite-Fejér interpolation problem. In fact we can show that this problem is equivalent to our problem. However our solution has an advantage which gives an explicit description of the self-commutator $[T_\varphi^*, T_\varphi]$ even though this method is not simpler than the method of [Gu1]. To see this consider the function $\varphi = \overline{g} + f$, where

$$f(z) := 3\frac{z-\frac{1}{2}}{1-\frac{1}{2}z} + 2\frac{z-\frac{1}{3}}{1-\frac{1}{3}z} + \frac{13}{6} \quad \text{and} \quad g(z) := \frac{z-\frac{1}{2}}{1-\frac{1}{2}z} + \frac{z-\frac{1}{3}}{1-\frac{1}{3}z} + \frac{5}{6}.$$

Thus if $\theta := \frac{z-\frac{1}{2}}{1-\frac{1}{2}z} \cdot \frac{z-\frac{1}{3}}{1-\frac{1}{3}z}$ then

$$a := 3\frac{z-\frac{1}{3}}{1-\frac{1}{3}z} + 2\frac{z-\frac{1}{2}}{1-\frac{1}{2}z} + \frac{13}{6}\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\cdot\frac{z-\frac{1}{3}}{1-\frac{1}{3}z} \quad \text{and} \quad b := \frac{z-\frac{1}{3}}{1-\frac{1}{3}z} + \frac{z-\frac{1}{2}}{1-\frac{1}{2}z} + \frac{5}{6}\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\cdot\frac{z-\frac{1}{3}}{1-\frac{1}{3}z}$$

are in $\mathcal{H}(\theta)$, and $f = \theta\overline{a}$ and $g = \theta\overline{b}$. So a straightforward calculation shows that $p(z)$ satisfying (3.8) is given by $p(z) = -z + \frac{5}{6}$ and $M = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{6}}{3} & \frac{1}{3} \end{pmatrix}$. Thus we have that

$$M(\varphi) := I - p(M)^* p(M) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{7}{9} & -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{2}{9} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{3}{4} \end{pmatrix}.$$

Since $\phi_1 = \frac{\sqrt{3}}{2}\frac{1}{1-\frac{1}{2}z}$ and $\phi_2 = \frac{2\sqrt{2}}{3}\frac{1}{1-\frac{1}{3}z}\cdot\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}$ form a basis for $\mathcal{H}(\theta)$, we have that

$$\begin{aligned}
[T_\varphi^*, T_\varphi] &= A(a)^* W M_\varphi W^* A(a) \bigoplus 0_\infty \\
&= \begin{pmatrix} \frac{3}{5} & 2\sqrt{6} \\ 0 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} \frac{2}{9} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 \\ 2\sqrt{6} & -\frac{2}{5} \end{pmatrix} \bigoplus 0_\infty \\
&= \begin{pmatrix} \frac{512}{25} & -\frac{16\sqrt{6}}{25} \\ -\frac{16\sqrt{6}}{25} & \frac{3}{25} \end{pmatrix} \bigoplus 0_\infty.
\end{aligned}$$

By comparison, the tangential Hermite-Fejér matrix induced by φ is given by (using the notations in [Gu1])

$$A^*\Gamma A - B^*\Gamma B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 24 & 24 \\ 0 & 24 & 24 \end{pmatrix}.$$

Corollary 3.4. *Let $\varphi = \overline{g} + f \in L^\infty$ and let $f = \theta\overline{a}$ and $g = \theta\overline{b}$, where θ is a finite Blaschke product and $a, b \in \mathcal{H}(\theta)$. If T_φ is hyponormal and* $\operatorname{rank}[T_\varphi^*, T_\varphi] < \deg(\theta)$ *then $\mathcal{E}(\varphi)$ has exactly one element.*

Proof. Suppose $\operatorname{rank}([T_\varphi^*, T_\varphi]) < \deg(\theta)$. By Theorem 3.2 we have that

$$\operatorname{rank}\big(I_{\mathcal{H}(\theta)} - p(M)^* p(M)\big) < \deg(\theta).$$

Therefore the norm of $p(M)$ should be one. By an argument of [FF, p.302] - there exists a unique solution k to (3.7) such that $||k||_\infty \le 1$ if and only if $||p(M)|| = 1$, $\mathcal{E}(\varphi)$ has exactly one element. □

Theorem 3.5. *Let $\varphi = \overline{g} + f \in L^\infty$ and let $f = \theta\overline{a}$ and $g = \theta\overline{b}$, where θ is a finite Blaschke product and $a, b \in \mathcal{H}(\theta)$. Let θ_1 be a factor of θ and let*

$$\varphi_{\theta_1} := \overline{\theta_1} P_{\mathcal{H}(\theta_1)}(b) + \theta_1 \overline{P_{\mathcal{H}(\theta_1)}(a)}.$$

If T_φ is hyponormal then $T_{\varphi_{\theta_1}}$ is. Moreover, in the cases where T_φ is hyponormal, the rank of $[T^*_{\varphi_{\theta_1}}, T_{\varphi_{\theta_1}}]$ is given by

$$\operatorname{rank}[T^*_{\varphi_{\theta_1}}, T_{\varphi_{\theta_1}}] = \begin{cases} \operatorname{rank}[T_\varphi^*, T_\varphi] & \text{if } \operatorname{rank}[T_\varphi^*, T_\varphi] < \deg(\theta_1) \\ \deg(\theta_1) & \text{if } \operatorname{rank}[T_\varphi^*, T_\varphi] \ge \deg(\theta_1). \end{cases}$$

Proof. Let $a_1 := P_{\mathcal{H}(\theta_1)}(a)$, $b_1 := P_{\mathcal{H}(\theta_1)}(b)$, $a_2 := a - a_1$ and $b_2 := b - b_1$. If T_φ is hyponormal then by Cowen's theorem there exists a function $k \in H^\infty$ with $||k||_\infty \le 1$ for which

$$\overline{\theta}b - k\overline{\theta}a = h \qquad \text{for some } h \in H^2,$$

or equivalently,

$$\begin{aligned} \overline{\theta}\big(b_1 + b_2 - k(a_1 + a_2)\big) = h &\Longleftrightarrow \overline{\theta}(b_1 - ka_1) - \overline{\theta}(b_2 - ka_2) = h \\ &\Longleftrightarrow \overline{\theta_1}(b_1 - ka_1) - \overline{\theta_1}(b_2 - ka_2) = \theta_2 h, \end{aligned}$$

where $\theta := \theta_1\theta_2$. Since b_1 and b_2 are orthogonal and $b_1 \in \mathcal{H}(\theta_1)$, it follows that $b_2 \in \theta_1 H^2$. Thus $\overline{\theta_1} b_2 \in H^2$. Similarly, we have that $\overline{\theta_1} a_2 \in H^2$. Therefore we have that

$$\overline{\theta_1}(b_1 - ka_1) = \overline{\theta_1}(b_2 - ka_2) + \theta_2 h \in H^2,$$

or

$$\overline{\theta_1} P_{\mathcal{H}(\theta_1)}(b) - k\overline{\theta_1} P_{\mathcal{H}(\theta_1)}(a) \in H^2.$$

Therefore by Cowen's theorem $T_{\varphi_{\theta_1}}$ is hyponormal.

For the rank formula, suppose that $\text{rank}\,[T^*_\varphi, T_\varphi] < \deg(\theta_1)$. By the Nakazi-Takahashi theorem, there exists a finite Blaschke product $k \in H^\infty$ such that $\deg(k) = \text{rank}\,[T^*_\varphi, T_\varphi] < \deg(\theta_1)$. Since $\mathcal{E}(\varphi) \subseteq \mathcal{E}(\varphi_{\theta_1})$ it follows $k \in \mathcal{E}(\varphi_{\theta_1})$. By Lemma 2.5 and Corollary 3.4 we have that

$$\text{rank}\,[T^*_\varphi, T_\varphi] = \deg(k) = \text{rank}\,[T^*_{\varphi_{\theta_1}}, T_{\varphi_{\theta_1}}].$$

For the other case we will show that if $\text{rank}\,[T^*_{\varphi_{\theta_1}}, T_{\varphi_{\theta_1}}] < \deg(\theta_1)$ then $\text{rank}\,[T^*_\varphi, T_\varphi] < \deg(\theta_1)$. To prove this suppose $\text{rank}\,[T^*_{\varphi_{\theta_1}}, T_{\varphi_{\theta_1}}] < \deg(\theta_1)$. By Corollary 3.4, $\mathcal{E}(\varphi_{\theta_1})$ has exactly one element. Since $\mathcal{E}(\varphi) \subseteq \mathcal{E}(\varphi_{\theta_1})$, $\mathcal{E}(\varphi)$ also consists of one element and hence by Lemma 2.5 we have that

$$\text{rank}\,[T^*_\varphi, T_\varphi] = \text{rank}\,[T^*_{\varphi_{\theta_1}}, T_{\varphi_{\theta_1}}] < \deg(\theta_1).$$

This completes the proof. □

Corollary 3.6. *Suppose that φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-N}^{N} a_n z^n$, where a_{-N} and a_N are nonzero. Let $\varphi_j := T_{\overline{z}^j}\varphi + \overline{T_{\overline{z}^j}\overline{\varphi}}$. If T_φ is hyponormal then T_{φ_j} is hyponormal for each $j = 0, 1, 2, \cdots, N$. In the cases where T_φ is hyponormal we have*

$$\text{rank}\,[T^*_{\varphi_j}, T_{\varphi_j}] = \begin{cases} N - j & \text{if } \ \text{rank}\,[T^*_\varphi, T_\varphi] \geq N - j \\ \text{rank}\,[T^*_\varphi, T_\varphi] & \text{if } \ \text{rank}\,[T^*_\varphi, T_\varphi] < N - j. \end{cases}$$

Proof. This follows at once from Theorem 3.5. □

Acknowledgement. The authors are grateful to the referee for several helpful suggestions.

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