Quadratic hyponormality and 2-hyponormality for Toeplitz operators

Sang Hoon Lee and Woo Young Lee

Abstract. In this note we prove the conjecture given in [CLL]: Let $0 < \alpha < 1$ and let ψ be the conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. If $\varphi = \psi + \lambda\bar{\psi}$ then T_φ is quadratically hyponormal if and only if T_φ is 2–hyponormal.

Mathematics Subject Classification (2000). Primary 47B20, 47B35; Secondary 47B37.

Keywords. Toeplitz operators, subnormal, 2-hyponormal, quadratically hyponormal.

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if T has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Evidently, normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal. Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \cdots\}$. An element $f \in L^2(\mathbb{T})$ is said to be analytic if $f \in H^2(\mathbb{T})$, and co-analytic if $f \in L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. If P denotes the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$ the operators T_φ on $H^2(\mathbb{T})$ defined by

$$T_\varphi g := P(\varphi g) \qquad (g \in H^2(\mathbb{T}))$$

is called the *Toeplitz operator with symbol* φ.

The Bram–Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

Supported by a grant (R14-2003-006-01000-0) from the Korea Research Foundation.

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([Bra],[Con, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{0.1}$$

Condition (0.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (0.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (0.1) for all k. If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B, and if we define T to be *k–hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k$$

is positive, or equivalently, the $(k+1) \times (k+1)$ operator matrix in (0.1) is positive (via the operator version of Choleski's Algorithm), then the Bram–Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k–hyponormal for every $k \geq 1$ ([CMX]).

Recall ([Ath],[CMX],[CoS]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators. If $k = 2$ then T is called *quadratically hyponormal*, and if $k = 3$ then T is said to be *cubically hyponormal*. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general. The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality (cf. [Cu1], [Cu2], [CuF], [CuL1], [CuL2], [CMX], [DPY], [McCP]).

On the other hand, P.R. Halmos ([Hal]) suggested the following problem (Halmos's Problem 5):

Is every subnormal Toeplitz operator either normal or analytic ?

As we know, this problem was answered in the negative by C. Cowen and J. Long [CoL]. They constructed a symbol φ for which T_φ is unitarily equivalent to a weighted shift.

Theorem 1. ([CoL],[Cow2]) *Let $0 < \alpha < 1$ and let ψ be a conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. If $\varphi = (1-\alpha^2)^{-1}(\psi + \alpha\bar{\psi})$, then T_φ is subnormal but neither normal nor analytic.*

Directly connected with the Halmos's Problem 5 is the following problem:

Which Toeplitz operators are subnormal ?

As a first inquiry we posed the following question in [CuL1], [CuL3]:

$$\text{Is every 2-hyponormal Toeplitz operator subnormal?} \tag{1.1}$$

In [CuL1] it was shown that every 2-hyponormal Toeplitz operator with a trigonometric polynomial symbol is subnormal. However the question (1.1) was answered in the negative in [CLL]: there is a gap between 2-hyponormality and subnormality for Toeplitz operators. This answer also uses the symbol constructed in [CoL].

Theorem 2. ([CLL, Theorem 6]) *Let $0 < \alpha < 1$ and let ψ be the conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. Let $\varphi = \psi + \lambda\bar{\psi}$ and let T_φ be the corresponding Toeplitz operator on H^2. Then*

(i) *T_φ is hyponormal if and only if λ is in the closed unit disk $|\lambda| \le 1$.*

(ii) *T_φ is subnormal if and only if $\lambda = \alpha$ or λ is in the circle $\left|\lambda - \frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}\right| = \frac{\alpha^k(1-\alpha^2)}{1-\alpha^{2k+2}}$ for $k = 0, 1, 2\cdots$.*

(iii) *T_φ is 2–hyponormal if and only if λ is in the unit circle $|\lambda| = 1$ or in the closed disk $\left|\lambda - \frac{\alpha}{1+\alpha^2}\right| \le \frac{\alpha}{1+\alpha^2}$.*

We were tempted to consider the gap between quadratic hyponormality and 2-hyponormality for Toeplitz operators. So in [CLL], we proposed the following:

Conjecture. *In Theorem 2, we have that T_φ is quadratically hyponormal if and only of T_φ is 2-hyponormal.*

In the sequel we prove the above conjecture. We begin with:

Lemma 3. *Let T be a weighted shift. Then $T + \lambda T^*$ is (weakly) k-hyponormal if and only if $T + |\lambda| T^*$ is (weakly) k-hyponormal.*

Proof. This follows from the observation that $T + \lambda T^*$ is unitarily equivalent to $e^{\frac{i\theta}{2}}(T + |\lambda|\, T^*)$ with $|\lambda| = \lambda e^{-i\theta}$ (cf. [Cow1, Lemma 2.1]). □

We now have:

Theorem 4. *For $0 < \alpha < 1$, let $T \equiv W_\beta$ be the weighted shift with weight sequence $\beta = \{\beta_n\}_{n=0}^\infty$, where*

$$\beta_n := \left(\sum_{j=0}^{n} \alpha^{2j}\right)^{\frac{1}{2}}. \tag{4.1}$$

If $S_\lambda := T + \lambda\, T^*$ $(\lambda \in \mathbb{C})$, then

(i) S_λ *is hyponormal if and only if* $|\lambda| \le 1$.
(ii) S_λ *is subnormal if and only if* $\lambda = 0$ *or* $|\lambda| = \alpha^k$ *for some* $k = 0, 1, 2, \cdots$.
(iii) S_λ *is* 2*–hyponormal if and only if* $|\lambda| = 1$ *or* $|\lambda| \le \alpha$.
(iv) S_λ *is quadratically hyponormal if and only if* $|\lambda| = 1$ *or* $|\lambda| \le \alpha$.

Proof. The statements (i) – (iii) are known from [Cow1,Theorem 2.3] and [CLL, Theorem 5]. Thus it suffices to focus on the assertion (iv). Let D be the diagonal operator, $D := \text{diag}(\alpha^n)$ Then we have

$$[T^*, T] = D^2 \quad \text{and} \quad [S_\lambda^*, S_\lambda] = (1 - |\lambda|^2)[T^*, T] = (1 - |\lambda|^2)D^2.$$

Define

$$A_l := \alpha^l T + \frac{\lambda}{\alpha^l} T^* \quad (l = 0, \pm 1, \pm 2, \cdots).$$

Then we have

$$DA_l = A_{l+1} D \quad \text{and} \quad A_l^* D = D A_{l+1}^* \quad (l = 0, \pm 1, \pm 2, \cdots). \tag{4.2}$$

Towards statement(iv), observe that if $|\lambda| = 1$ or $|\lambda| \le \alpha$ then by (iii) S_λ is quadratically hyponormal.

For the converse, we may assume $\lambda \ge 0$, in view of Lemma 3. We suppose that S_λ is quadratically hyponormal and $\lambda \ne 1$. We must show that $\lambda \le \alpha$. Evidently, $[S_\lambda^{*2}, S_\lambda^2] \ge 0$. Write

$$\begin{aligned} C &:= \frac{1}{1 - \lambda^2}[S_\lambda^{*2}, S_\lambda^2] \\ V &:= (1 + \alpha^2)[T^*, T]^{\frac{1}{2}} \left(\frac{\lambda}{\alpha^2} T + T^*\right). \end{aligned}$$

Note that

$$V = \frac{1 + \alpha^2}{\alpha} D A_1^*.$$

Then a straightforward calculation shows that (cf. [CLL, Proof of Theorem 5])

$$C - V^* V = \frac{(1 + \alpha^2)(\alpha^2 - \lambda^2)}{\alpha^2}[T^*, T]^2.$$

Thus we have that by (4.2)

$$\begin{aligned} [S_\lambda^{*2}, S_\lambda^2] &= (1 - \lambda^2) C \\ &= (1 - \lambda^2)\left(V^* V + \frac{(1 + \alpha^2)(\alpha^2 - \lambda^2)}{\alpha^2}[T^*, T]^2\right) \\ &= (1 - \lambda^2)\left(\frac{(1 + \alpha^2)^2}{\alpha^2} A_1 D^2 A_1^* + \frac{(1 + \alpha^2)(\alpha^2 - \lambda^2)}{\alpha^2} D^4\right) \\ &= \frac{(1 - \lambda^2)(1 + \alpha^2)^2}{\alpha^2} D \left(S_\lambda S_\lambda^* + \frac{\alpha^2 - \lambda^2}{1 + \alpha^2} D^2\right) D. \end{aligned}$$

From the observation that if D is positive and injective then $DTD \geq 0$ if and only if $T \geq 0$, we can see that

$$[S_\lambda^{*2}, S_\lambda^2] \geq 0 \iff S_\lambda S_\lambda^* + \frac{\alpha^2 - \lambda^2}{1+\alpha^2} D^2 \geq 0$$
$$\iff \langle (S_\lambda S_\lambda^* + \frac{\alpha^2 - \lambda^2}{1+\alpha^2} D^2)x,\ x\rangle \geq 0 \quad \text{for all } x \in \ell_2.$$

Note that Ker S_λ^* is nontrivial: more precisely,

$$\operatorname{Ker} S_\lambda^* = \bigvee \left\{ (1, 0, -\lambda\frac{\beta_0}{\beta_1}, 0, \lambda^2 \frac{\beta_0\beta_2}{\beta_1\beta_3}, 0, -\lambda^3 \frac{\beta_0\beta_2\beta_4}{\beta_1\beta_3\beta_5}, \cdots) \right\}.$$

So if we take $x(\neq 0) \in \operatorname{Ker} S_\lambda^*$, then

$$\langle (S_\lambda S_\lambda^* + \frac{\alpha^2 - \lambda^2}{1+\alpha^2} D^2)x,\ x\rangle = \frac{\alpha^2 - \lambda^2}{1+\alpha^2} ||Dx||^2.$$

Thus if $[S_\lambda^{*2}, S_\lambda^2] \geq 0$ then we have that $\frac{\alpha^2-\lambda^2}{1+\alpha^2}||Dx||^2 \geq 0$, and hence $\lambda \leq \alpha$, which proves the result. □

We therefore have:

Corollary 5. *Let $0 < \alpha < 1$ and let ψ be the conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. If $\varphi = \psi + \lambda\bar{\psi}$ then T_φ is quadratically hyponormal if and only if T_φ is 2–hyponormal.*

Proof. It was shown in [CoL] that $T_{\psi+\alpha\bar{\psi}}$ is unitarily equivalent to $(1-\alpha^2)^{\frac{3}{2}}T$, where T is the weighted shift in Theorem 4. Thus T_ψ is unitarily equivalent to $(1-\alpha^2)^{\frac{1}{2}}(T - \alpha T^*)$, so T_φ is unitarily equivalent to

$$(1-\alpha^2)^{\frac{1}{2}}(1-\lambda\,\alpha)(T + \frac{\lambda - \alpha}{1-\lambda\,\alpha}T^*) \quad \text{(cf. [Cow1, Theorem 2.4])}.$$

Therefore the result follows at once from Theorem 4. □

We conclude with:

Problem 6. *Find the values of λ for which S_λ in Theorem 4 is a cubically hyponormal operator. More generally, determine the set*

$$\mathfrak{H}_k \equiv \{\lambda \in \mathbb{C} :\ S_\lambda \text{ is weakly } k\text{-hyponormal}\}.$$

In[CuP] it was shown that there exists a non-subnormal polynomially hyponormal operator. Also in [McCP] it was shown that there exists a non-subnormal polynomially hyponormal operator if and only if there exists one which is a weighted shift although no concrete weighted shift has yet been found. We would be tempted to consider this gap for Toeplitz operators. At present we guess that, in Theorem 4,

$$S_\lambda \text{ is polynomially hyponormal} \iff S_\lambda \text{ is 2-hyponormal.}$$

If indeed this were true then we would get a concrete example of Toeplitz operator which is polynomially hyponormal but not subnormal. In fact, we were unable to decide whether or not there exists a non-subnormal polynomially hyponormal Toeplitz operator.

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Sang Hoon Lee
Department of Mathematics
Seoul National University
Seoul 151-742, Korea
e-mail: `shlee@math.skku.ac.kr`

Woo Young Lee
Department of Mathematics
Seoul National University
Seoul 151-742, Korea
e-mail: `wylee@math.snu.ac.kr`