HYPONORMALITY OF TOEPLITZ OPERATORS WITH RATIONAL SYMBOLS

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Abstract. In this article we introduce a notion of 'division' for rational functions and then give a criterion for hyponormality of $T_{\overline{g}+f}$ (f, g are rational functions) in the cases where g divides f. Furthermore we show that we may assume, without loss of generality, that g divides f when we consider the hyponormality of $T_{\overline{g}+f}$.

1 Introduction

A bounded linear operator A on a Hilbert space $\mathfrak{H}$ is said to be hyponormal if its selfcommutator $[A^*, A] = A^*A - AA^*$ is positive semidefinite. Recall that given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ is the operator T_φ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$ in the complex plane $\mathbb{C}$ defined by

$$T_\varphi f = P(\varphi \cdot f),$$

where $f \in H^2(\mathbb{T})$ and P denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Normal Toeplitz operators were characterized by a property of their symbols in the early 1960's by A. Brown and P. Halmos [BH]. The problem of determining which symbols induce hyponormal Toeplitz operators was completely solved by C. Cowen [Co] in 1988. Here we shall employ an equivalent variant of Cowen's theorem that was proposed by T. Nakazi and K. Takahashi in [NT].

Cowen's theorem. ([Co], [NT]) *Suppose that* $\varphi \in L^\infty(\mathbb{T})$ *is arbitrary and put*

$$\mathcal{E}(\varphi) := \{k \in H^\infty(\mathbb{T}) : ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\}.$$

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Then T_φ is hyponormal if and only if the set $\mathcal{E}(\varphi)$ is nonempty.

Cowen's method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution of a certain functional equation involving its symbol.

A function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are functions ψ_1, ψ_2 in $H^\infty(\mathbb{D})$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$$

for almost all z in $\mathbb{T}$. Evidently, rational functions in L^∞ are of bounded type.

For an inner function θ, write

$$\mathcal{H}(\theta) := H^2 \ominus \theta H^2.$$

Note that $\ker H_{\overline{\theta}} = \theta H^2$ and $\operatorname{ran} H^*_{\overline{\theta}} = \mathcal{H}(\theta)$. It was shown [Ab, Lemma 6] that if T_φ is hyponormal and φ is not in H^∞ then

$$\varphi \text{ is of bounded type } \Longleftrightarrow \overline{\varphi} \text{ is of bounded type.}$$

In [Ab], it was also shown that

$$\varphi \text{ is of bounded type } \Longleftrightarrow \ker H_\varphi \neq \{0\} \Longleftrightarrow \varphi = \overline{\theta} b,$$

where $b \in H^\infty$ and θ is an inner function such that the inner parts of b and θ are relatively prime. Therefore we can see (cf. [GS], [Gu2]) that if $\varphi = \overline{g} + f$ $(f, g \in H^2)$ is of bounded type and T_φ is hyponormal then we can write

$$f = \theta_1 \theta_2 \overline{a} \quad \text{and} \quad g = \theta_1 \overline{b}$$

for some inner functions θ_1 and θ_2, where $a \in \mathcal{H}(\theta_1\theta_2)$ and $b \in \mathcal{H}(\theta_1)$. Here we assume that the inner parts of a and $\theta_1\theta_2$ are relatively prime and also the inner parts of b and θ_1 are relatively prime.

Let θ be a finite Blaschke product of degree d. We can write

$$\theta = e^{i\xi} \prod_{i=1}^{n} B_i^{n_i}, \tag{1}$$

where $B_i(z) := \frac{z-\alpha_i}{1-\overline{\alpha_i}z}$, $(|\alpha_i| < 1)$, $n_i \geq 1$ and $\sum_{i=1}^n n_i = d$. Let $\theta = e^{i\xi} \prod_{j=1}^d B_j$ and each zero of θ be repeated according to its multiplicity. Note that this Blaschke product is precisely the same Blaschke product in (1). Let

$$\phi_j := \frac{d_j}{1-\overline{\alpha_j}z} B_{j-1}B_{j-2}\cdots B_1 \quad (1 \leq j \leq d), \tag{2}$$

where $\phi_1 := d_1(1-\overline{\alpha_1}z)^{-1}$ and $d_j := (1-|\alpha_j|^2)^{\frac{1}{2}}$. It is well known that $\{\phi_j\}_{j=1}^d$ forms an orthonormal basis for $\mathcal{H}(\theta)$ (cf. [FF,Theorem X.1.5]).

Let $f \in H^\infty$ be a rational function such that $f(0) = 0$. Then we may write

$$f = p_m(z) + \sum_{i=1}^{n} \sum_{j=0}^{l_i - 1} \frac{a_{ij}}{(1 - \overline{\alpha_i} z)^{l_i - j}} \qquad (0 < |\alpha_i| < 1),$$

where $p_m(z)$ denotes a polynomial of degree m. Let

$$\theta = z^m \prod_{i=1}^{n} B_i^{l_i},$$

where $B_i(z) := \frac{z - \alpha_i}{1 - \overline{\alpha_i} z}$. Observe that

$$\frac{a_{ij}}{1 - \overline{\alpha_i} z} = \frac{\overline{\alpha_i} a_{ij}}{1 - |\alpha_i|^2} \Big(\frac{z - \alpha_i}{1 - \overline{\alpha_i} z} + \frac{1}{\overline{\alpha_i}} \Big).$$

Letting $a := \theta \overline{f}$, we can see that $a \in \mathcal{H}(\theta)$ and $f = \theta \overline{a}$. Thus if $\varphi = \overline{g} + f \in L^\infty$, where f and g are rational functions with $f(0) = g(0) = 0$ and if T_φ is hyponormal, then we can write

$$f = \theta_1 \theta_2 \overline{a}, \quad g = \theta_1 \overline{b}$$

for some finite Blaschke products θ_1, θ_2 and $a \in \mathcal{H}(\theta_1 \theta_2)$ and $b \in \mathcal{H}(\theta_1)$, where the inner parts of a and $\theta_1 \theta_2$ are relatively prime and the inner parts of b and θ_1 are relatively prime.

It was shown in [Zhu] that the hyponormality of T_φ with polynomial symbols φ can be reduced to a Carathéodory-Schur interpolation problem (also see [HL] for another criterion). By comparison, it was observed in [Gu1] that the hyponormality of T_φ with rational symbols φ can be reduced to a tangential Hermite-Fejér interpolation problem. In this article we define the division $\frac{f}{g}$ for rational functions f, g and present a criterion for hyponormality of $T_{\overline{g}+f}$ when g divides f, where f, g are rational functions. Furthermore we show that the condition "g divides f" can be assumed without loss of generality when we study the hyponormality of $T_{\overline{g}+f}$.

2 Main Results

We need several auxiliary lemmas to understand the main results.

Lemma 1. *If θ_1 is a Blaschke product and θ_2 is an inner function then*

$$\mathcal{H}(\theta_1 \theta_2) \subset \mathcal{H}(\theta_1) \cdot \mathcal{H}(z \theta_2). \tag{3}$$

In particular, if θ_1 and θ_2 are finite Blaschke products then

$$\mathcal{H}(\theta_1 \theta_2) = \mathcal{H}(\theta_1) \cdot \mathcal{H}(z \theta_2). \tag{4}$$

Proof. We first observe that for any inner functions θ_1 and θ_2,

$$\mathcal{H}(\theta_1 \theta_2) = \theta_2 \mathcal{H}(\theta_1) + \mathcal{H}(\theta_2),$$

and hence

$$\mathcal{H}(\theta_1)\cdot\mathcal{H}(z\theta_2)=\mathcal{H}(\theta_1)\cdot\Big[\theta_2\mathcal{H}(z)+\mathcal{H}(\theta_2)\Big]=\theta_2\mathcal{H}(\theta_1)+\mathcal{H}(\theta_1)\cdot\mathcal{H}(\theta_2).$$

We now claim that if θ_1 is a Blaschke product then

$$\mathcal{H}(\theta_2)\subset\mathcal{H}(\theta_1)\cdot\mathcal{H}(z\theta_2). \tag{5}$$

Towards (5), let θ_1 be a Blaschke product of degree N (possibly, infinite). Then we can write

$$\theta_1=e^{i\xi_1}\prod_{i=1}^{N}B_i,$$

where $B_i(z)=\frac{z-\alpha_i}{1-\overline{\alpha_i}z}$ ($|\alpha_i|<1$). We write $H_0^2:=\{zf:\ f\in H^2\}$ and $\overline{\mathcal{L}}:=\{\overline{f}:\ f\in\mathcal{L}\}$ for $\mathcal{L}\subset L^2(\mathbb{T})$. Then for any inner function θ,

$$\mathcal{H}(\theta)=\{f\in H^2:\overline{\theta}f\in\overline{H_0^2}\}. \tag{6}$$

Suppose $f\in\mathcal{H}(\theta_2)$. Then we have that

$$\begin{aligned}(1-\overline{\alpha_1}z)f\in\mathcal{H}(z\theta_2)&\Longleftrightarrow\overline{z}\overline{\theta_2}(1-\overline{\alpha_1}z)f\in\overline{H_0^2}\\&\Longleftrightarrow\overline{z}\overline{\theta_2}f-\overline{\theta_2}\overline{\alpha_1}f\in\overline{H_0^2},\end{aligned}$$

which implies that $f\in\mathcal{H}(\theta_1)\cdot\mathcal{H}(z\theta_2)$. This proves (5). Hence from (5), we have that

$$\mathcal{H}(\theta_2)\subset\mathcal{H}(\theta_1)\cdot\mathcal{H}(z\theta_2)=\theta_2\mathcal{H}(\theta_1)+\mathcal{H}(\theta_1)\cdot\mathcal{H}(\theta_2),$$

which implies that $\mathcal{H}(\theta_1\theta_2)\subset\mathcal{H}(\theta_1)\cdot\mathcal{H}(z\theta_2)$. This proves (3).

Further if θ_1 and θ_2 are finite Blaschke products then by (6), $\mathcal{H}(\theta_1)\cdot\mathcal{H}(z\theta_2)\subset\mathcal{H}(\theta_1\theta_2)$, which together with (3) proves (4). □

The inclusion (3) of Lemma 1 need not hold if θ_1 is a singular inner function even though θ_2 is a finite Blaschke product. For example, if $\theta_1=e^{\frac{z+1}{z-1}}$ and $\theta_2=z$, then evidently, $1\in\mathcal{H}(\theta_1\theta_2)$, whereas $1\notin\mathcal{H}(\theta_1)\cdot\mathcal{H}(z\theta_2)$. Indeed,

$$\begin{aligned}1\in\mathcal{H}(\theta_1)\cdot\mathcal{H}(z\theta_2)&\Longrightarrow 1\in\mathcal{H}(\theta_1)\cdot(a+bz)\quad\text{for some } a,b\in\mathbb{C}\\&\Longrightarrow\frac{1}{a+bz}\in\mathcal{H}(\theta_1)\\&\Longrightarrow\frac{1}{1-\overline{c}z}\in\mathcal{H}(\theta_1)\quad\text{for some } |c|<1\\&\Longrightarrow\Big\langle e^{\frac{z+1}{z-1}},\ \frac{1}{1-\overline{c}z}\Big\rangle=0\\&\Longrightarrow e^{\frac{c+1}{c-1}}=0,\end{aligned}$$

a contradiction. Hence $1\notin\mathcal{H}(\theta_1)\cdot\mathcal{H}(z\theta_2)$.

Lemma 2. *If θ is an inner function then*

$$\mathcal{H}(z\theta) = \{\theta\overline{c} : c \in \mathcal{H}(z\theta)\} = H^2 \cap \{\theta\overline{c} : c \in H^2\}.$$

Proof. Evidently, $\mathcal{H}(z\theta) \supseteq \{\theta\overline{c} : c \in \mathcal{H}(z\theta)\}$. Note that

$$f \in \mathcal{H}(z\theta) \Longrightarrow \overline{z\theta} f \in \overline{H_0^2} \Longrightarrow f \in z\theta\overline{H_0^2} \Longrightarrow f \in \theta\overline{H^2}.$$

Therefore $f = \theta\overline{c}$ for some $c \in H^2$ and hence $c = \theta\overline{f}$. Observe that $\overline{z\theta}c = \overline{z\theta}\theta\overline{f} = \overline{zf} \in \overline{H_0^2}$. Thus $c \in \mathcal{H}(z\theta)$, which proves the first equality. For the second equality, it suffices to prove that if $\theta\overline{c} \in H^2$ then $c \in \mathcal{H}(z\theta)$. This follows at once from the observation:

$$\theta\overline{c} \in H^2 \Longrightarrow \overline{z}\overline{\theta}c \in \overline{H_0^2} \Longrightarrow c \in \mathcal{H}(z\theta),$$

which completes the proof. □

Lemma 3. *Let $f = \theta_1\theta_2\overline{a}$ and $g = \theta_1\overline{b}$ for $a \in \mathcal{H}(\theta_1\theta_2)$ and $b \in \mathcal{H}(\theta_1)$. Then*

$$\frac{f}{g} \in \mathcal{H}(z\theta_2) \Longleftrightarrow \frac{a}{b} \in \mathcal{H}(z\theta_2).$$

Proof. Write $\phi := \frac{f}{g}$ and $\psi := \frac{a}{b}$. Observe that $\phi = \frac{\theta_1\theta_2\overline{a}}{\theta_1\overline{b}} = \theta_2\overline{\psi}$, which implies that $\psi = \theta_2\overline{\phi}$. Note that

$$\phi \in H^2 \Longleftrightarrow \theta_2\overline{\psi} \in H^2 \Longleftrightarrow \overline{z\theta_2}\psi \in \overline{H_0^2}.$$

Therefore

$$\begin{aligned}\phi \in \mathcal{H}(z\theta_2) &\Longleftrightarrow \phi \in H^2 \text{ and } \overline{z\theta_2}\phi \in \overline{H_0^2}\\ &\Longleftrightarrow \overline{z\theta_2}\psi \in \overline{H_0^2} \text{ and } \psi \in H^2\\ &\Longleftrightarrow \psi \in \mathcal{H}(z\theta_2).\end{aligned}$$

□

The following lemma is used in proving the main theorem.

Lemma 4. ([Gu2, Corollary 3.5]) *Let $\varphi = \overline{g} + f \in L^\infty$, where $f = \theta_1\theta_2\overline{a}$ and $g = \theta_1\overline{b}$ for $a \in \mathcal{H}(\theta_1\theta_2)$ and $b \in \mathcal{H}(\theta_1)$. Let*

$$\varphi' := \overline{\theta_1\overline{b}} + \theta_1\overline{P_{\mathcal{H}(\theta_1)}(a)}.$$

Then T_φ is hyponormal if and only if $T_{\varphi'}$ is. Moreover, $\mathcal{E}(\varphi) = \{k\theta_2 : \ k \in \mathcal{E}(\varphi')\}$.

Lemma 4 says that when we study the hyponormality of Toeplitz operators T_φ with bounded type symbols φ, we may assume that the symbol $\varphi = \overline{g} + f \in L^\infty$ is of the form

$$f = \theta\overline{a} \quad \text{and} \quad g = \theta\overline{b},$$

where θ is an inner function and $a, b \in \mathcal{H}(\theta)$.

In view of Lemmas 1 and 3 we can introduce a notion of "division" for rational functions.

Definition 5. Let $f = \theta_1\theta_2\overline{a}$ and $g = \theta_1\overline{b}$ for $a \in \mathcal{H}(\theta_1\theta_2)$ and $b \in \mathcal{H}(\theta_1)$, where the θ_i are finite Blaschke products for $i = 1, 2$. We shall say that g *divides* f if $\frac{f}{g} \in \mathcal{H}(z\theta_2)$, or equivalently, $\frac{a}{b} \in \mathcal{H}(z\theta_2)$.

We examine Definition 5 for the cases of polynomials. For example if $f = \sum_{j=1}^N a_j z^j$ and $g = \sum_{j=1}^n b_j z^j$, put

$$\theta_1 := z^n, \quad \theta_2 := z^{N-n}, \quad a := \sum_{j=0}^{N-1} \overline{a_{N-j}} z^j, \quad \text{and} \quad b := \sum_{j=0}^{n-1} \overline{b_{n-j}} z^j.$$

Then

$$f = \theta_1\theta_2\overline{a}, \quad g = \theta_1\overline{b}, \quad a \in \mathcal{H}(\theta_1\theta_2), \quad \text{and} \quad b \in \mathcal{H}(\theta_1).$$

Thus g divides f if and only if $\frac{a}{b} \in \mathcal{H}(z\theta_2) = \mathcal{H}(z^{N-n+1})$, i.e., $\frac{a}{b} = \sum_{j=0}^{N-n} c_j z^j \in \mathcal{H}(z^{N-n+1})$ for some c_j $(0 \le j \le N-n)$. This exactly coincides with the usual concept of division for polynomials.

We then have:

Theorem 6. *Let $\varphi = \overline{g} + f \in L^\infty$, where $f = \theta_1\theta_2\overline{a}$ and $g = \theta_1\overline{b}$ for $a \in \mathcal{H}(\theta_1\theta_2)$ and $b \in \mathcal{H}(\theta_1)$ with finite Blaschke products θ_1 and θ_2. If g divides f and $\psi := \frac{a}{b} \in \mathcal{H}(z\theta_2)$ then the following are equivalent:*

(i) *T_φ is hyponormal;*

(ii) *There exists a function $k \in H^\infty$ with $||k||_\infty \le 1$ such that $k\psi \in 1 + \theta_1 H^2$.*

(iii) *T_ζ is hyponormal, where $\zeta = \overline{\theta}_1 + \theta_1 \overline{P_{\mathcal{H}(z\theta_1)}(\psi)}$.*

Moreover if T_φ is hyponormal then $|\psi(\alpha)| \ge 1$ for each zero α of θ_1. In particular, if $\theta_1 = \theta_2$ then

$$T_\varphi \text{ is hyponormal} \iff T_{\overline{\theta}_1 + \theta_1\overline{\psi}} \text{ is hyponormal.} \tag{7}$$

Proof. (i) $\Leftrightarrow$ (ii): Let $\varphi' := \theta_1 \overline{P_{\mathcal{H}(\theta_1)}(a)} + \overline{g}$. Then by Lemma 4 we have that $\mathcal{E}(\varphi) = \{k\theta_2 : k \in \mathcal{E}(\varphi')\}$. Therefore

$$\begin{aligned} T_\varphi \text{ is hyponormal} \iff \exists k' \in \mathcal{E}(\varphi) &\iff \overline{\theta_1} b - k'\overline{\theta_1\theta_2} a \in H^2 \text{ and } ||k'||_\infty \le 1 \\ &\iff \overline{\theta_1} b - k\overline{\theta_1} a \in H^2 \text{ and } ||k||_\infty \le 1 \quad (k' = k\theta_2) \\ &\iff b(1 - k\psi) \in \theta_1 H^2 \text{ and } ||k||_\infty \le 1 \end{aligned}$$

But since the inner parts of b and θ_1 are relatively prime and by assumption, $\psi \in H^2$, it follows that

$$\begin{aligned} k' \in \mathcal{E}(\varphi) &\iff 1 - k\psi \in \theta_1 H^2 \text{ and } ||k||_\infty \le 1 \\ &\iff k\psi \in 1 + \theta_1 H^2 \text{ and } ||k||_\infty \le 1. \end{aligned}$$

(ii) ⇔ (iii): Observe that

$$\begin{aligned} k\psi \in 1+\theta_1 H^2 &\Longleftrightarrow \overline{\theta_1} - k\overline{\theta_1}\psi \in H^2 \\ &\Longleftrightarrow \overline{\theta_1} - k\overline{\theta_1\overline{\psi}} \in H^2 \\ &\Longleftrightarrow \overline{\theta_1} - k\overline{P(\theta_1\overline{\psi})} \in H^2. \end{aligned}$$

But since $P(\theta_1\overline{\psi}) = \theta_1\overline{P_{\mathcal{H}(z\theta_1)}(\psi)}$, it follows that T_φ is hyponormal if and only if $T_{\overline{\theta_1}+\theta_1\overline{P_{\mathcal{H}(z\theta_1)}(\psi)}}$ is hyponormal.

On the other hand, if T_φ is a hyponormal operator and if $\theta_1(\alpha)=0$ then by (ii)

$$k(\alpha)\psi(\alpha) = 1 \Longrightarrow k(\alpha) = \frac{1}{\psi(\alpha)},$$

which implies that $|\psi(\alpha)| \geq 1$ since $||k||_\infty \leq 1$. The last assertion (7) follows at once from the observation that if $\theta_1 = \theta_2$ then $P_{\mathcal{H}(z\theta_1)}(\psi) = \psi$. □

Example 7. *Let*

$$\varphi = \overline{z}\prod_{j=1}^{8}(\overline{B_j} - \frac{1}{2}) + \frac{8}{7}z\prod_{j=1}^{9}(B_j - \frac{1}{2}),$$

where $B_j(z) = \frac{z+\frac{1}{2}}{1+\frac{1}{2}z}$ $(1 \leq j \leq 9)$. Then T_φ is not hyponormal.

Proof. Observe that

$$g = z\prod_{j=1}^{8}B_j \cdot \prod_{j=1}^{8}(1-\frac{1}{2}\overline{B_j}) \quad \text{and} \quad f = \frac{8}{7}z\prod_{j=1}^{9}B_j \cdot \prod_{j=1}^{9}(1-\frac{1}{2}\overline{B_j}).$$

Then

$$\psi(z) = \frac{8}{7}\left(1 - \frac{1}{2}B_9(z)\right).$$

Thus $\psi(0) = \frac{6}{7} < 1$. Therefore by Theorem 6, T_φ is not hyponormal. □

Example 8. *Let*

$$\varphi = \overline{z}\prod_{j=1}^{8}(\overline{B_j} - \frac{1}{2}) + 4z\prod_{j=1}^{9}(B_j - \frac{1}{2}),$$

where $B_j(z) = \frac{z+\frac{1}{2}}{1+\frac{1}{2}z}$ $(1 \leq j \leq 9)$. Then T_φ is hyponormal.

Proof. Observe that

$$g = z\prod_{j=1}^{8}B_j \cdot \prod_{j=1}^{8}(1-\frac{1}{2}\overline{B_j}) \quad \text{and} \quad f = z\prod_{j=1}^{9}B_j \cdot 4\prod_{j=1}^{9}(1-\frac{1}{2}\overline{B_j}).$$

Then

$$\psi(z) = 4\left(1 - \frac{1}{2}B_9(z)\right).$$

Put

$$k(z) := \frac{1 + z\prod_{j=1}^{8}B_j(z)}{4(1-\frac{1}{2}B_9(z))}.$$

Then $||k||_\infty \leq 1$ and $k(z)\cdot\psi(z) = 1 + z\prod_{j=1}^{8}B_j(z)$. Therefore by Theorem 6 (ii), T_φ is hyponormal. □

Corollary 9. *Let $\varphi = \overline{g} + f \in L^\infty$, where $f = \theta_1\theta_2\overline{a}$ and $g = \theta_1\overline{b}$ for $a \in \mathcal{H}(\theta_1\theta_2)$ and $b \in \mathcal{H}(\theta_1)$ with finite Blaschke products θ_1 and θ_2. If $\psi := \frac{f}{g} \in \mathcal{H}(z\theta_2)$ and $|a(\alpha)| = |b(\alpha)|$ for some zero α of θ_1 then*

$$T_\varphi \text{ is hyponormal} \Longleftrightarrow \mathcal{E}(\varphi) = \left\{\frac{b(\alpha)}{a(\alpha)}\theta_2\right\}$$

Proof. By Theorem 6 we have that

$$T_\varphi \text{ hyponormal} \Longleftrightarrow \exists k \in H^\infty \text{ with } ||k||_\infty \le 1 \text{ such that } k\psi \in 1 + \theta_1 H^2.$$

Thus if T_φ is hyponormal then $k(\alpha)\psi(\alpha) = 1$ for each zero α of θ_1, so that $k(\alpha) = \frac{b(\alpha)}{a(\alpha)}$. Therefore by the maximum modulus principle, $k(z) = \frac{b(\alpha)}{a(\alpha)}$, and hence $\mathcal{E}(\varphi) = \left\{\frac{b(\alpha)}{a(\alpha)}\theta_2\right\}$. □

In Theorem 6, the conditions "g divides f" and "$\theta_1 = \theta_2$" seem to be too rigid. However the following theorem shows that we may assume, without loss of generality, that g divides f and moreover $\theta_1 = \theta_2$ when we consider the hyponormality of T_φ.

Theorem 10. *Let $\varphi = \overline{g} + f \in L^\infty$, where $f = \theta_1\theta_2\overline{a}$ and $g = \theta_1\overline{b}$ for $a \in \mathcal{H}(\theta_1\theta_2)$ and $b \in \mathcal{H}(\theta_1)$ with a finite Blaschke product θ_1. If we let*

$$f_c = \theta_1^2\overline{P_{\mathcal{H}(\theta_1)}(a)} + \theta_1\overline{c} \quad \textit{for } c \in \mathcal{H}(z\theta_1)$$

and put $\varphi_c := \overline{g} + f_c$, then we have:

(i) *T_φ is hyponormal if and only if T_{φ_c} is;*

(ii) *g divides f_c for some $c \in \mathcal{H}(z\theta_1)$.*

Proof. Write

$$\theta_1 = e^{i\xi}\prod_{i=1}^{n} B_i^{n_i},$$

where

$$B_i(z) := \frac{z - \alpha_i}{1 - \overline{\alpha_i}z}, \quad (|\alpha_i| < 1), \quad n_i \ge 1, \quad \text{and} \quad \sum_{i=1}^{n} n_i = d.$$

Put $a_0 = P_{\mathcal{H}(\theta_1)}(a)$. Then for each $c \in \mathcal{H}(z\theta_1)$, $f_c = \theta_1^2(\overline{a_0 + \theta_1 c})$ and $P_{\mathcal{H}(\theta_1)}(a_0+\theta_1 c) = a_0$. Thus by Lemma 4, T_φ is hyponormal if and only if T_{φ_c} is. To prove (ii) observe that by Lemma 2,

$$\begin{aligned}
\phi_c := \frac{f_c}{g} \in \mathcal{H}(z\theta_1) &\Longleftrightarrow \theta_1^2\overline{a_0} + \theta_1\overline{c} \in (\theta_1\overline{b}) \cdot \mathcal{H}(z\theta_1) \text{ for some } c \in \mathcal{H}(z\theta_1) \\
&\Longleftrightarrow \theta_1^2\overline{a_0} + \theta_1\overline{c} \in (\theta_1\overline{b}) \cdot \mathcal{H}(z\theta_1) \text{ for some } c \in H^2 \\
&\Longleftrightarrow \theta_1^2(\overline{a_0 + \theta_1 c}) = \theta_1\overline{b}\theta_1\overline{k} \ \text{ for some } \theta_1\overline{k} \in H^2 \\
&\Longleftrightarrow a_0 + \theta_1 c = bk \text{ for some } k \in \mathcal{H}(z\theta_1) \\
&\Longleftrightarrow a_0 - bk \in \theta_1 H^2 \text{ for some } k \in \mathcal{H}(z\theta_1). \qquad (8)
\end{aligned}$$

Note that $\theta_1^{(n)}(\alpha_i) = 0$ for all $0 \le n < n_i$. Thus the condition (8) is equivalent to the following equation: for all $1 \le i \le n$,

$$\begin{bmatrix} k_{i,0} \\ k_{i,1} \\ k_{i,2} \\ \vdots \\ k_{i,n_i-2} \\ k_{i,n_i-1} \end{bmatrix} = \begin{bmatrix} b_{i,0} & 0 & 0 & 0 & \cdots & 0 \\ b_{i,1} & b_{i,0} & 0 & 0 & \cdots & 0 \\ b_{i,2} & b_{i,1} & b_{i,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ b_{i,n_i-2} & b_{i,n_i-3} & \ddots & \ddots & b_{i,0} & 0 \\ b_{i,n_i-1} & b_{i,n_i-2} & \ldots & b_{i,2} & b_{i,1} & a_{i,0} \end{bmatrix}^{-1} \begin{bmatrix} a_{i,0} \\ a_{i,1} \\ a_{i,2} \\ \vdots \\ a_{i,n_i-2} \\ a_{i,n_i-1} \end{bmatrix}, \tag{9}$$

where

$$k_{i,j} := \frac{k^{(j)}(\alpha_i)}{j!}, \quad a_{i,j} := \frac{a_0^{(j)}(\alpha_i)}{j!} \quad \text{and} \quad b_{i,j} := \frac{b^{(j)}(\alpha_i)}{j!}.$$

Thus there exists a function $k \in H^\infty$ satisfying (8) if and only if there exists a function $k \in H^\infty$ for which

$$\frac{k^{(j)}(\alpha_i)}{j!} = k_{i,j} \qquad (1 \le i \le n,\ 0 \le j < n_i), \tag{10}$$

where the $k_{i,j}$ are determined by the equation (9). If in addition $||k||_\infty \le 1$ is required then this is exactly the classical Hermite-Fejér interpolation problem. But it is well known that there always exists a polynomial k satisfying (10) (cf. [FF]). We now find such a funtion k in $\mathcal{H}(z\theta_1)$. Observe that

$$\begin{aligned} a_0 - bk \in \theta_1 H^2 &\Longleftrightarrow P_{\mathcal{H}(\theta_1)}(a_0 - bk) = 0 \\ &\Longleftrightarrow a_0 - P_{\mathcal{H}(\theta_1)}(bk) = 0 \\ &\Longleftrightarrow a_0 - P_{\mathcal{H}(\theta_1)}(bP_{\mathcal{H}(\theta_1)}k) = 0 \\ &\Longleftrightarrow P_{\mathcal{H}(\theta_1)}(a_0 - bP_{\mathcal{H}(\theta_1)}k) = 0 \\ &\Longleftrightarrow a_0 - bP_{\mathcal{H}(\theta_1)}k \in \theta_1 H^2. \end{aligned}$$

If we put $k_1 := P_{\mathcal{H}(\theta_1)}k$ then k_1 satisfies (8) and $k_1 \in \mathcal{H}(z\theta_1)$. This completes the proof. □

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