Joint hyponormality of rational Toeplitz pairs

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Abstract. We characterize hyponormal "rational" Toeplitz pairs which are pairs of Toeplitz operators whose symbols are rational functions in L^∞. The main result of this article is as follows. If $\mathbf{T} = (T_\phi, T_\psi)$ is a hyponormal rational Toeplitz pair then $\phi - \beta\psi \in H^2$ for some constant β; in other words, their co-analytic parts necessarily coincide up to a constant multiple. As a corollary we get a complete characterization of hyponormal rational Toeplitz pairs.

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1. Introduction

In the monograph [CL], R. Curto and the second named author characterized the joint hyponormality of Toeplitz pairs which are pairs of Toeplitz operators with trigonometric polynomial symbols. We here obtain a complete characterization of hyponormal *rational* Toeplitz pairs which are pairs of Toeplitz operators whose symbols are rational functions in L^∞.

The Bram-Halmos criterion on subnormality ([Br]) states that an operator T on a Hilbert space $\mathcal{H}$ is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$ for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$. It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1). \tag{1.1}$$

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In 1988, the notion of "joint hyponormality" (for the general case of n-tuples of operators) was first formally introduced by A. Athavale [At]. He conceived joint hyponormality as a notion at least as strong as requiring that the linear span of the operator coordinates consist of hyponormal operators, the latter notion being called weak joint hyponormality.

Subnormality, joint hyponormality, and weak joint hyponormality have been studied by A. Athavale [At], J. Conway and W. Szymanski [CS], R. Curto [Cu], R. Curto, P. Muhly, and J. Xia [CMX], R. Douglas, V. Paulsen, and K. Yan [DPY], R. Douglas and K. Yan [DY], D. Farenick and R. McEachin [FM], C. Gu [Gu], S. McCullough and V. Paulsen [McCP], D. Xia [Xi], R. Curto and the second named author [CL], and others. Joint hyponormality originated from questions about commuting normal extensions of commuting operators, and it has also been considered with an aim at understanding the gap between hyponormality and subnormality for single operators. To date, much of the research on joint hyponormality has dealt with commuting tuples of hyponormal operators.

The study of jointly hyponormal Toeplitz operators started from D. Farenick and R. McEachin [FM]. They studied operators that form hyponormal pairs in the presence of the unilateral shift. Since the unilateral shift is a Toeplitz operator on the Hardy space of the unit circle, one can ask whether the results in [FM] extend to *Toeplitz pairs*, that is, pairs whose coordinates are Toeplitz operators on the Hardy space of the unit circle. R. Curto and the second named author [CL] gave a complete characterization of hyponormal *trigonometric* Toeplitz pairs which are pairs of Toeplitz operators with trigonometric polynomial symbols. C. Gu [Gu] studied the joint hyponormality of Toeplitz pairs whose coordinates have the same co-analytic parts. The purpose of this article is to provide a complete characterization of hyponormal rational Toeplitz pairs.

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. For $A, B \in \mathcal{B}(\mathcal{H})$, we let $[A, B] := AB - BA$; $[A, B]$ is the *commutator* of A and B. Given an n-tuple $\mathbf{T} = (T_1, \dots, T_n)$ of operators on $\mathcal{H}$, we let $[\mathbf{T}^*, \mathbf{T}] \in \mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$ denote the *self-commutator* of $\mathbf{T}$, defined by

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] & \dots & [T_n^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] & \dots & [T_n^*, T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*, T_n] & [T_2^*, T_n] & \dots & [T_n^*, T_n] \end{pmatrix}.$$

(This definition of self-commutator for n-tuples of operators on a Hilbert space was introduced by A. Athavale in [At].) By analogy with the case $n = 1$, we shall say ([At], [CMX]) that $\mathbf{T}$ is *jointly hyponormal* (or simply, *hyponormal*) if $[\mathbf{T}^*, \mathbf{T}]$ is a positive operator on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Clearly, the hyponormality of an n-tuple requires as a necessary condition that every coordinate in the tuple be hyponormal.

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the unit circle. The Hilbert space $L^2 \equiv L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and the Hardy space $H^2 \equiv H^2(\mathbb{T})$ is the closed linear span of

$\{e_n : n = 0, 1, \ldots\}$. An element $f \in L^2$ is said to be analytic if $f \in H^2$, and co-analytic if $f \in L^2 \ominus H^2$. If P denotes the orthogonal projection of L^2 onto H^2, then for every $\phi \in L^\infty \equiv L^\infty(\mathbb{T})$, the operator T_ϕ on H^2 defined by

$$T_\phi g := P(\phi g) \qquad (g \in H^2)$$

is called the *Toeplitz operator* with symbol ϕ. If ϕ is a trigonometric polynomial of the form $\phi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero, then the nonnegative integers m and N denote the co-analytic and analytic degrees of ϕ, respectively. If $\phi \in L^\infty$, write

$$\phi_+ \equiv P(\phi) \in H^2 \quad \text{and} \quad \phi_- \equiv \overline{(I-P)(\phi)} \in zH^2.$$

Thus we can write $\phi = \overline{\phi_-} + \phi_+$. D. Farenick and R. MaEachin [FM] showed that if U is the unilateral shift on H^2 then the hyponormality of (U, T) implies that T is necessarily a Toeplitz operator. Furthermore they proved that if $\phi = \overline{\phi_-} + \phi_+ \in L^\infty$ and if $\psi = \overline{\phi_-} + U^*\phi_+ \in L^\infty$ then (U, T_ϕ) is hyponormal if and only if the single Toeplitz operator T_ψ is hyponormal. R. Curto and the second-named author [CL] have studied the hyponormality of $\mathbf{T} = (T_\phi, T_\psi)$ when both symbols ϕ and ψ are trigonometric polynomials. In [CL], a complete characterization of hyponormal Toeplitz pairs in this case was given. The core of the main result in [CL] is that the hyponormality of $\mathbf{T} = (T_\phi, T_\psi)$ (where ϕ and ψ are trigonometric polynomials) forces that *the co-analytic parts of ϕ and ψ necessarily coincide up to a constant multiple*, that is,

$$\phi_- = \beta\psi_- \quad \text{for some } \beta \in \mathbb{C}. \tag{1.2}$$

C. Gu [Gu] gave a characterization of hyponormal Toeplitz pairs $\mathbf{T} = (T_\phi, T_\psi)$ under the constraint (1.2) on the symbol - the assumption of equal co-analytic parts. In this article we show that if ϕ and ψ are rational functions in L^∞, then the condition "symbols have equal co-analytic parts" is indeed necessary for the hyponormality of the pair $\mathbf{T} = (T_\phi, T_\psi)$: this follows a spirit of the main result in [CL]. Thus we get a characterization of hyponormal "rational" Toeplitz pairs. A key step for the proof of the main result is accomplished by a direct and careful analysis on the self-commutator of the pair.

The organization of the article is as follows. In Section 2 we introduce basic facts about Toeplitz operators and Hankel operators. In Section 3 we provide auxiliary lemmas to be used in proving the main results. Section 4 is devoted to prove the main results.

Observe that if (T_1, T_2) is hyponormal, then so is $(T_1 - \lambda_1, T_2 - \lambda_2)$ for every $\lambda_1, \lambda_2 \in \mathbb{C}$. Thus if $\phi, \psi \in L^\infty$ have Fourier coefficients $\hat{\phi}(n), \hat{\psi}(n)$ for every $n \in \mathbb{Z}$, respectively, then the hyponormality of (T_ψ, T_ϕ) is independent of the particular values of $\hat{\phi}(0)$ and $\hat{\psi}(0)$. Therefore, throughout the article, we will assume that the 0-th coefficient $\hat{\phi}(0)$ of the given symbol ϕ of a Toeplitz operator is zero.

2. Preliminaries

A bounded linear operator A is called hyponormal if its self-commutator $[A^*, A] := A^*A - AA^*$ is positive (semidefinite). Normal Toeplitz operators were characterized by a property of their symbols in the early 1960's by A. Brown and P. Halmos [BH] and 25 years passed before the exact nature of the relationship between the symbol $\phi \in L^\infty$ and the positivity of the self-commutator $[T_\phi^*, T_\phi]$ was understood (via Cowen's theorem [Co]). We shall employ an equivalent variant of Cowen's theorem [Co], that was first proposed by Nakazi and Takahashi [NT].

Cowen's Theorem. *For $\phi \in L^\infty$, write*

$$\mathcal{E}(\phi) := \{k \in H^\infty : ||k||_\infty \le 1 \text{ and } \phi - k\overline{\phi} \in H^\infty\}.$$

Then T_ϕ is hyponormal if and only if $\mathcal{E}(\phi)$ is nonempty.

Cowen's method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution with specified properties to a certain functional equation involving the operator's symbol. This approach has been put to use in many works to study hyponormal Toeplitz operators on the Hardy space of the unit circle.

Let J be the unitary operator on L^2 defined by $J(f)(z) = \overline{z}f(\overline{z})$. For $\phi \in L^\infty$, the operator on H^2 defined by

$$H_\phi f := J(I-P)(\phi f) \quad (f \in H^2)$$

is called the *Hankel operator* H_ϕ with symbol ϕ. If we define the function $\widetilde{v}$ by $\widetilde{v}(z) := \overline{v(\overline{z})}$, then H_ϕ can be viewed as the operator on H^2 defined by $\langle zuv, \overline{\phi}\rangle = \langle H_\phi u, \widetilde{v}\rangle$ for all $v \in H^\infty$. We write $H_0^2 := \{zf : f \in H^2\}$ and $\overline{\mathcal{L}} = \{\overline{f} : f \in \mathcal{L}\}$ for $\mathcal{L} \subset L^2(\mathbb{T})$. We write, for an inner function θ,

$$\mathcal{H}(\theta) := H^2 \ominus \theta H^2.$$

The following is a basic connection between Hankel and Toeplitz operators ([Ni]):

(i) $T_{\phi\psi} - T_\phi T_\psi = H^*_{\overline{\phi}} H_\psi$ $(\phi, \psi \in L^\infty)$;
(ii) $H_\phi T_h = H_{\phi h} = T^*_{\widetilde{h}} H_\phi$ $(h \in H^\infty, \phi \in L^\infty)$.

From this we can see that if $k \in \mathcal{E}(\phi)$ then

$$[T_\phi^*, T_\phi] = H^*_{\overline{\phi}}H_{\overline{\phi}} - H^*_\phi H_\phi = H^*_{\overline{\phi}}H_{\overline{\phi}} - H^*_{k\overline{\phi}}H_{k\overline{\phi}} = H^*_{\overline{\phi}}(1 - T_{\widetilde{k}}T^*_{\widetilde{k}})H_{\overline{\phi}}. \tag{2.1}$$

We here observe that if $\mathbf{T} = (T_\phi, T_\psi)$ then the self-commutator of $\mathbf{T}$ can be expressed as:

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T_\phi^*, T_\phi] & [T_\psi^*, T_\phi] \\ [T_\phi^*, T_\psi] & [T_\psi^*, T_\psi] \end{pmatrix} = \begin{pmatrix} H^*_{\overline{\phi_+}}H_{\overline{\phi_+}} - H^*_{\overline{\phi_-}}H_{\overline{\phi_-}} & H^*_{\overline{\phi_+}}H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}}H_{\overline{\phi_-}} \\ H^*_{\overline{\psi_+}}H_{\overline{\phi_+}} - H^*_{\overline{\phi_-}}H_{\overline{\psi_-}} & H^*_{\overline{\psi_+}}H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}}H_{\overline{\psi_-}} \end{pmatrix}.$$

A function $\phi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are functions $\psi_1, \psi_2 \in H^\infty(\mathbb{D})$ such that

$$\phi(z) = \frac{\psi_1(z)}{\psi_2(z)}$$

for almost all z in $\mathbb{T}$. Evidently, rational functions in L^∞ are of bounded type. We recall ([Ab, Lemma 6]) that if T_ϕ is hyponormal and ϕ is not analytic then ϕ is of bounded type if and only if $\overline{\phi}$ is of bounded type. In [Ab, Lemma 3], it was also shown that

$$\phi \text{ is of bounded type} \iff \ker H_\phi \neq \{0\} \iff \phi = \overline{\theta} b,$$

where θ is an inner function, $b \in H^\infty$, and θ and b are relatively prime, i.e., there does not exist a nonconstant inner function ω such that $\theta = \omega\theta_0$ and $b = \omega b_0$ for some $\theta_0.b_0 \in H^\infty$. Thus if $f \in H^2$ is such that $\overline{f}$ is of bounded type and $f(0) = 0$ then we can write

$$f = \theta\overline{b},$$

where θ is an inner function and $b \in \mathcal{H}(\theta)$ satisfies that b and θ are coprime. In particular, we can easily show that

$$f = \theta\overline{b} \text{ is a rational function} \iff \theta \text{ is a finite Blaschke product.} \tag{2.2}$$

Since $T_{\overline{z}} H_\phi = H_\phi T_z$ it follows from Beurling's theorem that $\ker H_{\overline{\phi_-}} = \theta_0 H^2$ and $\ker H_{\overline{\phi_+}} = \theta_+ H^2$ for some inner functions θ_0, θ_+. If T_ϕ is hyponormal and ϕ is not analytic then by (2.1), $||H_{\overline{\phi_+}} f|| \geq ||H_{\overline{\phi_-}} f||$ for all $f \in H^2$, so that

$$\theta_+ H^2 = \ker H_{\overline{\phi_+}} \subset \ker H_{\overline{\phi_-}} = \theta_0 H^2, \tag{2.3}$$

which implies that θ_0 divides θ_+, i.e., $\theta_+ = \theta_0\theta_1$ for some inner function θ_1. Thus if $\phi = \overline{\phi_-} + \phi_+$ is of bounded type, $\phi_+(0) = \phi_-(0) = 0$, and T_ϕ is hyponormal then we can write

$$\phi_+ = \theta_0\theta_1\overline{a} \quad \text{and} \quad \phi_- = \theta_0\overline{b}, \quad \text{where } a \in \mathcal{H}(\theta_0\theta_1) \text{ and } b \in \mathcal{H}(\theta_0). \tag{2.4}$$

On the other hand, as in (2.3), the hyponormality of Toeplitz pairs is also related to the kernels of Hankel operators involved with the analytic and co-analytic parts of the symbol. Indeed it was shown ([Gu, Lemma 6.2]) that if neither ϕ nor ψ is analytic and if $(T_\phi, T_\psi)$is hyponormal, then

$$\ker(H_{\overline{\phi_+}}) \subset \ker(H_{\overline{\psi_-}}) \quad \text{and} \quad \ker(H_{\overline{\psi_+}}) \subset \ker(H_{\overline{\phi_-}}). \tag{2.5}$$

Throughout the article, whenever writing $f = \theta\overline{a} \in H^2$ (where θ is an inner function and $a \in \mathcal{H}(\theta)$), we will assume that θ and a are relatively prime.

3. Auxiliary Lemmas

To prove the main result we need several auxiliary lemmas. The first two lemmas are technical lemmas about Hankel operators. If $\mathcal{M}$ is a closed subspace of L^2, we write $P_{\mathcal{M}}$ for the orthogonal projection of L^2 onto $\mathcal{M}$.

Lemma 1. *If θ_0 and θ_1 are inner functions, then*

$$H_{\overline{\theta_0\theta_1}} = T_{\widetilde{\theta}_1} H_{\overline{\theta_0}} + H_{\overline{\theta_0\theta_1}} P_{\theta_0 H^2}.$$

Proof. We first claim that $H_{\overline{\theta_0\theta_1}}|_{\mathcal{H}(\theta_0)} = T_{\widetilde{\theta}_1} H_{\overline{\theta_0}}|_{\mathcal{H}(\theta_0)}$. Indeed, if $f \in \mathcal{H}(\theta_0)$ then $\overline{\theta_0} f \in \overline{H_0^2}$, and hence $\overline{z}\theta_0\overline{f} \in H^2$. Thus

$$H_{\overline{\theta_0\theta_1}} f = J(I-P)(\overline{\theta_0\theta_1} f) = J(\overline{\theta_0\theta_1} f) = \overline{z}\widetilde{\theta}_0\widetilde{\theta}_1\overline{\widetilde{f}} = T_{\widetilde{\theta}_1}(\overline{z}\widetilde{\theta}_0\overline{\widetilde{f}}) = T_{\widetilde{\theta}_1} H_{\overline{\theta_0}} f.$$

But since $\ker T_{\widetilde{\theta}_1} H_{\overline{\theta_0}} = \ker H_{\overline{\theta_0}} = \theta_0 H^2$, it follows that

$$H_{\overline{\theta_0\theta_1}} = H_{\overline{\theta_0\theta_1}} P_{\mathcal{H}(\theta_0)} + H_{\overline{\theta_0\theta_1}} P_{\theta_0 H^2} = T_{\widetilde{\theta}_1} H_{\overline{\theta_0}} + H_{\overline{\theta_0\theta_1}} P_{\theta_0 H^2}.$$

□

Lemma 2. *If θ_0 and θ_1 are inner functions, then*

$$H_{\overline{\theta_0\theta_1}} P_{\theta_0 H^2} = P_{\mathcal{H}(\widetilde{\theta}_1)} H_{\overline{\theta_0\theta_1}}.$$

Proof. For $f \in H^2$, write $f = f_1 + f_2 + f_3$, where $f_1 \in \mathcal{H}(\theta_0)$, $f_2 \in \theta_0\theta_1 H^2$ and $f_3 \in \mathcal{H}(\theta_0\theta_1) \ominus \mathcal{H}(\theta_0)$. Thus $f_3 = \theta_0 h$ for some $h \in H^2$. Observe that

$$f_3 = \theta_0 h \in \mathcal{H}(\theta_0\theta_1) \Longleftrightarrow \overline{\theta_0\theta_1}\theta_0 h \in \overline{H_0^2} \Longleftrightarrow h \in \mathcal{H}(\theta_1).$$

Therefore we have that

$$H_{\overline{\theta_0\theta_1}} P_{\theta_0 H^2} f = H_{\overline{\theta_0\theta_1}} \theta_0 h = J(I-P)(\overline{\theta_0\theta_1}\theta_0 h) = J(\overline{\theta_1} h) = \overline{z}\widetilde{\theta}_1\overline{\widetilde{h}}.$$

On the other hand, we have that

$$\begin{aligned} P_{\mathcal{H}(\widetilde{\theta}_1)} H_{\overline{\theta_0\theta_1}} f &= P_{\mathcal{H}(\widetilde{\theta}_1)} H_{\overline{\theta_0\theta_1}} (f_1 + f_3) \\ &= P_{\mathcal{H}(\widetilde{\theta}_1)} J(I-P)(\overline{\theta_0\theta_1} f_1 + \overline{\theta_1} h) \\ &= P_{\mathcal{H}(\widetilde{\theta}_1)} J(\overline{\theta_0\theta_1} f_1 + \overline{\theta_1} h) \\ &= P_{\mathcal{H}(\widetilde{\theta}_1)} (\overline{z}\widetilde{\theta}_0\widetilde{\theta}_1\overline{\widetilde{f}_1}) + P_{\mathcal{H}(\widetilde{\theta}_1)} (\overline{z}\widetilde{\theta}_1\overline{\widetilde{h}}) \\ &= \overline{z}\widetilde{\theta}_1\overline{\widetilde{h}}, \end{aligned}$$

where the last equality comes from the following observation:

$$f_1 \in \mathcal{H}(\theta_0) \Longrightarrow \overline{z}\widetilde{\theta}_0\overline{\widetilde{f}_1} \in H^2 \Longrightarrow \overline{z}\widetilde{\theta}_0\widetilde{\theta}_1\overline{\widetilde{f}_1} \in \widetilde{\theta}_1 H^2 \Longrightarrow P_{\mathcal{H}(\widetilde{\theta}_1)}(\overline{z}\widetilde{\theta}_0\widetilde{\theta}_1\overline{\widetilde{f}_1}) = 0.$$

This completes the proof. □

For a notational convenience we adopt the following notation: If $\phi = \overline{\phi_-} + \phi_+ \in L^\infty$ and if θ_1 and θ_2 are inner functions, we write

$$\phi_{\theta_1,\theta_2} := \overline{P_{H_0^2}(\overline{\theta_1}\phi_-)} + P_{H_0^2}(\overline{\theta_2}\phi_+).$$

Thus we have that $(\phi_{\theta_1,\theta_2})_+ = P_{H_0^2}(\overline{\theta_2}\phi_+)$ and $(\phi_{\theta_1,\theta_2})_- = P_{H_0^2}(\overline{\theta_1}\phi_-)$. We also abbreviate

$$\phi_\theta \equiv \phi_{\theta,\theta}.$$

Lemma 3. *Let $\phi = \overline{\phi_-} + \phi_+ \in L^\infty$ be of the form*

$$\phi_+ = \theta_0\theta_1\overline{a} \quad \text{and} \quad \phi_- = \theta_0\overline{b},$$

where θ_1 and θ_2 are inner functions, and $a \in \mathcal{H}(\theta_0\theta_1)$ and $b \in \mathcal{H}(\theta_0)$. Then we have:

(i) *Suppose θ is a factor of θ_1 and ω is a factor of θ_0. If T_ϕ is hyponormal then $T_{\phi_{\omega,\theta}}$ is hyponormal. Further if $\omega = 1$ then the converse is also true.*

(ii) *If θ is an arbitrary inner function and T_ϕ is hyponormal then T_{ϕ_θ} is hyponormal.*

Proof. (i) Write $\theta_1 = \theta\Delta_1$ and $\theta_0 = \omega\Delta_0$. If T_ϕ is hyponormal then by Cowen's theorem there exist a function $k \in H^\infty$ with $||k||_\infty \le 1$ and a function $h \in H^2$ for which $\overline{\theta_0}b - k\overline{\theta_0\theta_1}a = h$, that is, $ka = \theta_1(b - \theta_0 h)$. Since a and θ_1 are relatively prime, it follows that $k = \theta_1\zeta$ for some $\zeta \in H^\infty$. If we put $k_1 := \Delta_1\zeta\omega$ then $||k_1||_\infty \le 1$ and $\overline{\Delta_0}b - k_1\overline{\theta_0\Delta_1}a \in \omega H^2$, so that $\overline{P_{H_0^2}(\Delta_0\overline{b})} - k_1\,\overline{P_{H_0^2}(\theta_0\Delta_1\overline{a})} \in H^2$, which implies $k_1 \in \mathcal{E}(\phi_{\omega,\theta})$, that is, $T_{\omega,\theta}$ is hyponormal. For the second assertion, observe that if $k \in \mathcal{E}(\phi_{1,\theta})$ then $k\theta \in \mathcal{E}(\phi)$.

(ii) Note that if $h \in H^\infty$ and $\phi \in L^\infty$ then $H_{\overline{P_{H_0^2}(\overline{h}\phi)}} = H_{\overline{P(\overline{h}\phi)}} = H_{\overline{\phi}h} = H_{\overline{\phi}}T_h$. Thus,

$$\begin{aligned}
[T^*_{\phi_\theta}, T_{\phi_\theta}] &= H^*_{\overline{P_{H_0^2}(\overline{\theta}\phi_+)}}H_{\overline{P_{H_0^2}(\overline{\theta}\phi_+)}} - H^*_{\overline{P_{H_0^2}(\overline{\theta}\phi_-)}}H_{\overline{P_{H_0^2}(\overline{\theta}\phi_-)}} \\
&= H^*_{\theta\overline{\phi_+}}H_{\theta\overline{\phi_+}} - H^*_{\theta\overline{\phi_-}}H_{\theta\overline{\phi_-}} \\
&= T^*_\theta(H^*_{\overline{\phi_+}}H_{\overline{\phi_+}} - H^*_{\overline{\phi_-}}H_{\overline{\phi_-}})T_\theta \\
&= T^*_\theta[T^*_\phi, T_\phi]T_\theta,
\end{aligned}$$

which implies that if T_ϕ is hyponormal then so is T_{ϕ_θ}. □

The converse of Lemma 3(ii) is not true in general: indeed, a straightforward calculation shows that if $\phi(z) = z^{-2} + 3z^{-1} + 2z + 2z^2$ then T_ϕ is not hyponormal, whereas T_{ϕ_z} is hyponormal.

Lemma 4. *Let $\phi = \overline{\phi_-} + \phi_+ \in L^\infty$ and $\psi = \overline{\psi_-} + \psi_+ \in L^\infty$ be of the form*

$$\phi_+ = \theta_0\theta_1\overline{a},\ \ \phi_- = \theta_0\overline{b},\ \ \psi_+ = \theta_2\theta_3\overline{c} \ \ \text{and} \ \ \psi_- = \theta_2\overline{d},$$

where $a \in \mathcal{H}(\theta_0\theta_1)$, $b \in \mathcal{H}(\theta_0)$, $c \in \mathcal{H}(\theta_2\theta_3)$, and $d \in \mathcal{H}(\theta_2)$. Suppose θ is an inner function. If $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal then $\mathbf{T}_\theta = (T_{\phi_\theta}, T_{\psi_\theta})$ is hyponormal.

Proof. From the same argument as in the proof of Lemma 3(ii), we have that

$$\begin{aligned}
[\mathbf{T}_\theta^*, \mathbf{T}_\theta] &= \begin{pmatrix} [T^*_{\phi_\theta}, T_{\phi_\theta}] & [T^*_{\psi_\theta}, T_{\phi_\theta}] \\ [T^*_{\phi_\theta}, T_{\psi_\theta}] & [T^*_{\psi_\theta}, T_{\psi_\theta}] \end{pmatrix} \\
&= \begin{pmatrix} T_\theta^* H^*_{\overline{\phi_+}} H_{\overline{\phi_+}} T_\theta - T_\theta^* H^*_{\overline{\phi_-}} H_{\overline{\phi_-}} T_\theta & T_\theta^* H^*_{\overline{\phi_+}} H_{\overline{\psi_+}} T_\theta - T_\theta^* H^*_{\overline{\psi_-}} H_{\overline{\phi_-}} T_\theta \\ T_\theta^* H^*_{\overline{\psi_+}} H_{\overline{\phi_+}} T_\theta - T_\theta^* H^*_{\overline{\phi_-}} H_{\overline{\psi_-}} T_\theta & T_\theta^* H^*_{\overline{\psi_+}} H_{\overline{\psi_+}} T_\theta - T_\theta^* H^*_{\overline{\psi_-}} H_{\overline{\psi_-}} T_\theta \end{pmatrix} \\
&= \begin{pmatrix} T_\theta^* & 0 \\ 0 & T_\theta^* \end{pmatrix} [\mathbf{T}^*, \mathbf{T}] \begin{pmatrix} T_\theta & 0 \\ 0 & T_\theta \end{pmatrix},
\end{aligned}$$

which implies that if $\mathbf{T}$ is hyponormal then so is $\mathbf{T}_\theta$. □

If one coordinate of the Toeplitz pair has an analytic symbol then the hyponormality of the pair can be determined by the hyponormality of a single Toeplitz operator (cf. [CL, Theorem 1.10]; [Gu, Theorem 4.1]).

Lemma 5. *If $\phi \in H^\infty$ is such that $\phi = \theta\overline{a}$ for $a \in \mathcal{H}(\theta)$ and $\psi = \overline{\psi_-} + \psi_+ \in L^\infty$ is arbitrary then $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal if and only if $T_{\psi_{1,\theta}}$ is hyponormal.*

Proof. See [Gu, Theorem 4.1]. □

4. Main Results

Our main theorem is as follows:

Theorem 1. *Let ϕ and ψ be rational functions in L^∞. If $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal then $\phi - \beta\psi \in H^2$ for some constant β.*

Proof. In view of (2.2) and (2.4), we can write

$$\phi_+ = \theta_0\theta_1\overline{a},\ \phi_- = \theta_0\overline{b},\ \psi_+ = \theta_2\theta_3\overline{c} \text{ and } \psi_- = \theta_2\overline{d},$$

where the θ_j are finite Blaschke products, $a \in \mathcal{H}(\theta_0\theta_1)$, $b \in \mathcal{H}(\theta_0)$, $c \in \mathcal{H}(\theta_2\theta_3)$ and $d \in \mathcal{H}(\theta_2)$.

We split the proof into four steps.

STEP 1: *If θ is the greatest common inner factor of θ_1 and θ_3, then*

$$\text{(4.1)} \qquad \mathbf{T}_{1\theta} = (T_{\phi_{1,\theta}}, T_{\psi_{1,\theta}}) \text{ is hyponormal} \implies \mathbf{T} = (T_\phi, T_\psi) \text{ is hyponormal.}$$

Proof. We first claim that

$$\text{(4.2)} \qquad [T_\psi^*, T_\phi] = [T^*_{\psi_{1,\theta}}, T_{\phi_{1,\theta}}] + H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}}.$$

Write $\theta_1 = \theta\delta_1$ and $\theta_3 = \theta\delta_3$ for some inner functions δ_1, δ_3. Then we have that

(4.3)
$$\begin{aligned}
H^*_{\overline{\phi_+}} H_{\overline{\psi_+}} &= H^*_{\overline{\theta_0\theta_1 a}} H_{\overline{\theta_2\theta_3 c}} \\
&= \left(H_{\overline{\theta_0\theta_1}} T_a\right)^* \left(H_{\overline{\theta_2\theta_3}} T_c\right) \\
&= \left(T_{\widetilde{\theta}} H_{\overline{\theta_0\delta_1}} T_a + H_{\overline{\theta_0\theta_1}} P_{\theta_0\delta_1 H^2} T_a\right)^* \left(T_{\widetilde{\theta}} H_{\overline{\theta_2\delta_3}} T_c + H_{\overline{\theta_2\theta_3}} P_{\theta_2\delta_3 H^2} T_c\right) \\
&= T_a^* H^*_{\overline{\theta_0\delta_1}} T_{\overline{\widetilde{\theta}}} T_{\widetilde{\theta}} H_{\overline{\theta_2\delta_3}} T_c + T_a^* H^*_{\overline{\theta_0\delta_1}} T_{\overline{\widetilde{\theta}}} H_{\overline{\theta_2\theta_3}} P_{\theta_2\delta_3 H^2} T_c \\
&\qquad + T_a^* P_{\theta_0\delta_1 H^2} H^*_{\overline{\theta_0\theta_1}} T_{\widetilde{\theta}} H_{\overline{\theta_2\delta_3}} T_c + T_a^* P_{\theta_0\delta_1 H^2} H^*_{\overline{\theta_0\theta_1}} H_{\overline{\theta_2\theta_3}} P_{\theta_2\delta_3 H^2} T_c,
\end{aligned}$$

where the third equality follows from Lemma 1. Observe that

(4.4)
$$T_a^* H^*_{\overline{\theta_0\delta_1}} T_{\overline{\widetilde{\theta}}} T_{\widetilde{\theta}} H_{\overline{\theta_2\delta_3}} T_c = T_a^* H^*_{\overline{\theta_0\delta_1}} H_{\overline{\theta_2\delta_3}} T_c = H^*_{\overline{\theta_0\delta_1 a}} H_{\overline{\theta_2\delta_3 c}} = H^*_{\overline{(\phi_{1,\theta})_+}} H_{\overline{(\psi_{1,\theta})_+}}.$$

For each $h \in H^2$, if we write $h_1 := P_{\mathcal{H}(\theta)} h$ then

$$T_{\overline{\widetilde{\theta}}} H_{\overline{\theta_2\theta_3}}(\theta_2\delta_3 h) = T_{\overline{\widetilde{\theta}}} J(I-P)(\overline{\theta_2\theta_3}\theta_2\delta_3 h) = T_{\overline{\widetilde{\theta}}} J(\overline{\theta} h_1) = T_{\overline{\widetilde{\theta}}}\left(\overline{z}\widetilde{\theta}\widetilde{h_1}\right) = 0,$$

which implies $T_{\overline{\widetilde{\theta}}} H_{\overline{\theta_2\theta_3}} P_{\theta_2\delta_3 H^2} = 0$, and hence

$$T_a^* H^*_{\overline{\theta_0\delta_1}} T_{\overline{\widetilde{\theta}}} H_{\overline{\theta_2\theta_3}} P_{\theta_2\delta_3 H^2} T_c = 0. \tag{4.5}$$

Similarly, we have that $T_{\overline{\widetilde{\theta}}} H_{\overline{\theta_0\theta_1}} P_{\theta_0\delta_1 H^2} = 0$, and hence $P_{\theta_0\delta_1 H^2} H^*_{\overline{\theta_0\theta_1}} T_{\widetilde{\theta}} = 0$. Therefore we have that

$$T_a^* P_{\theta_0\delta_1 H^2} H^*_{\overline{\theta_0\theta_1}} T_{\widetilde{\theta}} H_{\overline{\theta_2\delta_3}} T_c = 0. \tag{4.6}$$

On the other hand, by Lemma 2, we have that

$$\begin{aligned}
T_a^* P_{\theta_0\delta_1 H^2} H^*_{\overline{\theta_0\theta_1}} H_{\overline{\theta_2\theta_3}} P_{\theta_2\delta_3 H^2} T_c &= T_a^* H^*_{\overline{\theta_0\theta_1}} P_{\mathcal{H}(\widetilde{\theta})} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\theta_2\theta_3}} T_c \\
&= H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}}.
\end{aligned} \tag{4.7}$$

Hence by (4.3)-(4.7), it follows that

$$\begin{aligned}
[T^*_\psi, T_\phi] &= H^*_{\overline{\phi_+}} H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\phi_-}} \\
&= H^*_{\overline{(\phi_{1,\theta})_+}} H_{\overline{(\psi_{1,\theta})_+}} + H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} - H^*_{\overline{(\psi_{1,\theta})_-}} H_{\overline{(\phi_{1,\theta})_-}} \\
&= [T^*_{\psi_{1,\theta}}, T_{\phi_{1,\theta}}] + H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}},
\end{aligned}$$

which proves (4.2). Applying (4.2) to $[T^*_\phi, T_\phi]$ and $[T^*_\psi, T_\psi]$ gives that

$$[\mathbf{T}^*, \mathbf{T}] = [\mathbf{T}^*_{1,\theta}, \mathbf{T}_{1,\theta}] + V,$$

where

$$V = \begin{pmatrix} H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} & H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} \\ H^*_{\overline{\psi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} & H^*_{\overline{\psi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} \end{pmatrix}.$$

But since

$$V = \begin{pmatrix} (P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}})^* & 0 \\ (P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}})^* & 0 \end{pmatrix} \begin{pmatrix} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} & P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} \\ 0 & 0 \end{pmatrix} \geq 0,$$

it follows that if $\mathbf{T}_{1\theta}$ is hyponormal then $\mathbf{T}$ is hyponormal. $\square$

STEP 2: *If θ is the greatest common inner factor of θ_1 and θ_3 and if ω is the greatest common inner factor of θ_0 and θ_2 then*

$$\mathbf{T} = (T_\phi, T_\psi) \text{ is hyponormal} \implies \big(\mathbf{T}_\omega\big)_{1,\theta} = (T_{(\phi_\omega)_{1,\theta}}, T_{(\psi_\omega)_{1,\theta}}) \text{ is hyponormal.}$$

Proof. By Lemma 4, $\mathbf{T}_\omega = (T_{\phi_\omega}, T_{\psi_\omega})$ is hyponormal. Thus without loss of generality we may assume that θ_0 and θ_2 are relatively prime. Also by (2.5) we have

$$\theta_0\theta_1\, H^2 \subset \theta_2\, H^2 \quad \text{and} \quad \theta_2\theta_3\, H^2 \subset \theta_0\, H^2.$$

Thus $\theta_1 = \theta_2\varpi_1$ and $\theta_3 = \theta_0\varpi_3$ for some inner functions ϖ_1, ϖ_3. But since θ_0 and θ_2 are relatively prime, it follows that

$$\delta_1 = \theta_2\Delta_1 \quad \text{and} \quad \delta_3 = \theta_0\Delta_3 \quad \text{form some inner functions } \Delta_1, \Delta_3. \tag{4.8}$$

Now let $a_1 := P_{\mathcal{H}(\theta_0\delta_1)}a$ and $c_1 := P_{\mathcal{H}(\theta_0\delta_3)}c$. Then we have that

$$\begin{aligned}
[\mathbf{T}_{1\theta}^*, \mathbf{T}_{1\theta}] &= \begin{pmatrix} [T^*_{\phi_{1,\theta}}, T_{\phi_{1,\theta}}] & [T^*_{\psi_{1,\theta}}, T_{\phi_{1,\theta}}] \\ [T^*_{\phi_{1,\theta}}, T_{\psi_{1,\theta}}] & [T^*_{\psi_{1,\theta}}, T_{\psi_{1,\theta}}] \end{pmatrix} \\
&= \begin{pmatrix} H^*_{\overline{\phi_+\theta}} H_{\overline{\phi_+\theta}} - H^*_{\overline{\phi_-}} H_{\overline{\phi_-}} & H^*_{\overline{\phi_+\theta}} H_{\overline{\psi_+\theta}} - H^*_{\overline{\psi_-}} H_{\overline{\phi_-}} \\ H^*_{\overline{\psi_+\theta}} H_{\overline{\phi_+\theta}} - H^*_{\overline{\phi_-}} H_{\overline{\psi_-}} & H^*_{\overline{\psi_+\theta}} H_{\overline{\psi_+\theta}} - H^*_{\overline{\psi_-}} H_{\overline{\psi_-}} \end{pmatrix} \\
&= \begin{pmatrix} H^*_{\overline{\theta_0\delta_1 a_1}} H_{\overline{\theta_0\delta_1 a_1}} - H^*_{\overline{\theta_0 b}} H_{\overline{\theta_0 b}} & H^*_{\overline{\theta_0\delta_1 a_1}} H_{\overline{\theta_2\delta_3 c_1}} - H^*_{\overline{\theta_2 d}} H_{\overline{\theta_0 b}} \\ H^*_{\overline{\theta_2\delta_3 c_1}} H_{\overline{\theta_0\delta_1 a_1}} - H^*_{\overline{\theta_0 b}} H_{\overline{\theta_2 d}} & H^*_{\overline{\theta_2\delta_3 c_1}} H_{\overline{\theta_2\delta_3 c_1}} - H^*_{\overline{\theta_2 d}} H_{\overline{\theta_2 d}} \end{pmatrix}.
\end{aligned}$$

Observe that $\ker [T^*_{\phi_{1,\theta}}, T_{\phi_{1,\theta}}] \supset \theta_0\delta_1 H^2$ and hence $\operatorname{ran} [T^*_{\phi_{1,\theta}}, T_{\phi_{1,\theta}}] \subset \mathcal{H}(\theta_0\delta_1)$. Thus $[T^*_{\phi_{1,\theta}}, T_{\phi_{1,\theta}}]$ has the following matrix representation:

$$[T^*_{\phi_{1,\theta}}, T_{\phi_{1,\theta}}] = \begin{pmatrix} H^*_{\overline{\theta_0\delta_1 a_1}} H_{\overline{\theta_0\delta_1 a_1}} - H^*_{\overline{\theta_0 b}} H_{\overline{\theta_0 b}} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \theta_0\delta_1 H^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \theta_0\delta_1 H^2 \end{pmatrix},$$

where the left-upper corner will be, without loss in simplifying the notation, understood as the restriction of $H^*_{\overline{\theta_0\delta_1 a_1}} H_{\overline{\theta_0\delta_1 a_1}} - H^*_{\overline{\theta_0 b}} H_{\overline{\theta_0 b}}$ to $\mathcal{H}(\theta_0\delta_1)$. By similar arguments for $[T^*_{\psi_{1,\theta}}, T_{\phi_{1,\theta}}]$, $[T^*_{\phi_{1,\theta}}, T_{\psi_{1,\theta}}]$ and $[T^*_{\psi_{1,\theta}}, T_{\psi_{1,\theta}}]$ using (4.8), we can see that the positivity of $[\mathbf{T}_{1\theta}, \mathbf{T}_{1\theta}]$ is equivalent to the positivity of the restriction, say E, of it to $\mathcal{H}(\theta_0\theta_1) \oplus \mathcal{H}(\theta_2\theta_3)$. Note that E can be written as:

(4.9)
$$E = \begin{pmatrix} H^*_{\overline{\theta_0\delta_1 a_1}} H_{\overline{\theta_0\delta_1 a_1}} - H^*_{\overline{\theta_0 b}} H_{\overline{\theta_0 b}} & 0 & H^*_{\overline{\theta_0\delta_1 a_1}} H_{\overline{\theta_2\delta_3 c_1}} - H^*_{\overline{\theta_2 d}} H_{\overline{\theta_0 b}} & 0 \\ 0 & 0 & 0 & 0 \\ H^*_{\overline{\theta_2\delta_3 c_1}} H_{\overline{\theta_0\delta_1 a_1}} - H^*_{\overline{\theta_0 b}} H_{\overline{\theta_2 d}} & 0 & H^*_{\overline{\theta_2\delta_3 c_1}} H_{\overline{\theta_2\delta_3 c_1}} - H^*_{\overline{\theta_2 d}} H_{\overline{\theta_2 d}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$: \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \mathcal{H}(\theta_0\theta_1) \cap \theta_0\delta_1 H^2 \\ \mathcal{H}(\theta_2\delta_3) \\ \mathcal{H}(\theta_2\theta_3) \cap \theta_2\delta_3 H^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \mathcal{H}(\theta_0\theta_1) \cap \theta_0\delta_1 H^2 \\ \mathcal{H}(\theta_2\delta_3) \\ \mathcal{H}(\theta_2\theta_3) \cap \theta_2\delta_3 H^2 \end{pmatrix},$$

where each entry of E should be understood as a restriction to a suitable subspace. Let

$$A_a := P_{\mathcal{H}(\theta_0\theta_1)} M_a|_{\mathcal{H}(\theta_0\theta_1)} \quad \text{and} \quad A_c := P_{\mathcal{H}(\theta_2\theta_3)} M_c|_{\mathcal{H}(\theta_2\theta_3)},$$

where M_x is a multiplication operator with symbol x. We now argue that A_a and A_c are invertible. To see this, suppose $A_a f = 0$ for some $f \in \mathcal{H}(\theta_0\theta_1)$. Then $P_{\mathcal{H}(\theta_0\theta_1)}(af) = 0$ and hence $af \in \theta_0\theta_1 H^2$. Since the inner part of a and $\theta_0\theta_1$ are relatively prime, it follows that $f \in \theta_0\theta_1 H^2$. So $f \in \theta_0\theta_1 H^2 \cap \mathcal{H}(\theta_0\theta_1) = \{0\}$, which implies that A_a is one-one. But since A_a is a finite dimensional operator (because $\theta_0\theta_1$ is a finite Blaschke product), it follows that A_a is invertible. Similarly, A_c is also invertible. Observe that

$$P_{\mathcal{H}(\theta_0\delta_1)} A_a|_{\mathcal{H}(\theta_0\theta_1)\cap\theta_0\delta_1 H^2} = P_{\mathcal{H}(\theta_0\delta_1)} M_a|_{\mathcal{H}(\theta_0\theta_1)\cap\theta_0\delta_1 H^2} = 0.$$

Thus A_a has the following matrix representation:

$$A_a = \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \mathcal{H}(\theta_0\theta_1) \cap \theta_0\delta_1 H^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \mathcal{H}(\theta_0\theta_1) \cap \theta_0\delta_1 H^2 \end{pmatrix}.$$

Since $a_1 = P_{\mathcal{H}(\theta_0\delta_1)} M_a|_{\mathcal{H}(\theta_0\delta_1)}$, we can see, by the same argument as for A_a, that a_1 is invertible. As a result, a_3 is also invertible. Thus the inverse of A_a is lower triangular; in fact, $A_a^{-1} = \begin{pmatrix} a_1^{-1} & 0 \\ -a_3^{-1} a_2 a_1^{-1} & a_3^{-1} \end{pmatrix}$. Similarly, the inverse of A_c is also lower triangular. Write

$$G := \begin{pmatrix} A_a^{-1} & 0 \\ 0 & A_c^{-1} \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\theta_0\theta_1) \\ \mathcal{H}(\theta_2\theta_3) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(\theta_0\theta_1) \\ \mathcal{H}(\theta_2\theta_3) \end{pmatrix}.$$

Since G is invertible, it follows from (4.9) that

$$[\mathbf{T}_{1\theta}^*, \mathbf{T}_{1\theta}] \text{ is positive} \iff D := G^* E G \geq 0.$$

Since A_a^{-1} and A_c^{-1} are both lower triangular, D should be of the form

(4.10)
$$D = \begin{pmatrix} d_1 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 0 \\ d_3 & 0 & d_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \mathcal{H}(\theta_0\theta_1) \cap \theta_0\delta_1 H^2 \\ \mathcal{H}(\theta_2\delta_3) \\ \mathcal{H}(\theta_2\theta_3) \cap \theta_2\delta_3 H^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \mathcal{H}(\theta_0\theta_1) \cap \theta_0\delta_1 H^2 \\ \mathcal{H}(\theta_2\delta_3) \\ \mathcal{H}(\theta_2\theta_3) \cap \theta_2\delta_3 H^2 \end{pmatrix}.$$

On the other hand, we know that

$$[\mathbf{T}_{1\theta}^*, \mathbf{T}_{1\theta}] = [\mathbf{T}^*, \mathbf{T}] - V,$$

where, for a notational convenience, each term will be understood as a restriction to $\mathcal{H}(\theta_0\theta_1) \oplus \mathcal{H}(\theta_2\theta_3)$ (Note that $\mathcal{H}(\theta_0\theta_1) \oplus \mathcal{H}(\theta_2\theta_3)$ reduces $[\mathbf{T}^*, \mathbf{T}]$, $[\mathbf{T}_{1\theta}^*, \mathbf{T}_{1\theta}]$ and V). Thus V can be viewed as:

$$V = \begin{pmatrix} H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} & H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} \\ H^*_{\overline{\psi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} & H^*_{\overline{\psi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\theta_0\theta_1) \\ \mathcal{H}(\theta_2\theta_3) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(\theta_0\theta_1) \\ \mathcal{H}(\theta_2\theta_3) \end{pmatrix}$$

Observe that

$$\begin{aligned} C &:= G^* \, V \, G \\ &= \begin{pmatrix} A_a^{-1} & 0 \\ 0 & A_c^{-1} \end{pmatrix}^* \begin{pmatrix} H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} & H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} \\ H^*_{\overline{\psi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} & H^*_{\overline{\psi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} \end{pmatrix} \begin{pmatrix} A_a^{-1} & 0 \\ 0 & A_c^{-1} \end{pmatrix} \\ &= \begin{pmatrix} {A_a^{-1}}^* H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} A_a^{-1} & {A_a^{-1}}^* H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} A_c^{-1} \\ {A_c^{-1}}^* H^*_{\overline{\psi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} A_a^{-1} & {A_c^{-1}}^* H^*_{\overline{\psi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} A_c^{-1} \end{pmatrix}. \end{aligned} \tag{4.11}$$

Since

$${A_a^{-1}}^* H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} A_a^{-1} = \Bigl(H_{\overline{\theta_0\theta_1 a}} A_a^{-1}\Bigr)^* P_{\mathcal{H}(\widetilde{\theta})} \Bigl(H_{\overline{\theta_0\theta_1 a}} A_a^{-1}\Bigr)$$

and

$$\begin{aligned} H_{\overline{\theta_0\theta_1 a}} A_a^{-1} &= H_{\overline{\theta_0\theta_1}} T_a A_a^{-1} \\ &= H_{\overline{\theta_0\theta_1}} P M_a \Bigl(P_{\mathcal{H}(\theta_0\theta_1)} M_a|_{\mathcal{H}(\theta_0\theta_1)}\Bigr)^{-1} \\ &= H_{\overline{\theta_0\theta_1}} P_{\mathcal{H}(\theta_0\theta_1)} M_a|_{\mathcal{H}(\theta_0\theta_1)} \Bigl(P_{\mathcal{H}(\theta_0\theta_1)} M_a|_{\mathcal{H}(\theta_0\theta_1)}\Bigr)^{-1} \\ &= H_{\overline{\theta_0\theta_1}}, \end{aligned}$$

it follows that

$$\begin{aligned} {A_a^{-1}}^* H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\phi_+}} A_a^{-1} &= H^*_{\overline{\theta_0\theta_1}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\theta_0\theta_1}} \\ &= H^*_{\overline{\theta_0\theta_1}} H_{\overline{\theta_0\theta_1}} P_{\theta_0\delta_1 H^2} \quad \text{(by Lemma 2)} \\ &= P_{\mathcal{H}(\theta_0\theta_1)} P_{\theta_0\delta_1 H^2} \quad (\text{since } H^*_{\overline{\zeta}} H_{\overline{\zeta}} = P_{\mathcal{H}(\zeta)} \text{ for an inner function } \zeta) \\ &= P_{\mathcal{H}(\theta_0\theta_1) \cap \theta_0\delta_1 H^2}. \end{aligned} \tag{4.12}$$

Similarly, we also have that

$${A_a^{-1}}^* H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} A_c^{-1} = {A_a^{-1}}^* H^*_{\overline{\theta_0\theta_1 a}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\theta_2\theta_3 c}} A_c^{-1} = H^*_{\overline{\theta_0\theta_1}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\theta_2\theta_3}}.$$

Thus, by Lemma 2, we get

$$\begin{aligned} P_{\mathcal{H}(\theta_0\delta_1)} {A_a^{-1}}^* H^*_{\overline{\phi_+}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} A_c^{-1} P_{\mathcal{H}(\theta_2\delta_3)} &= P_{\mathcal{H}(\theta_0\delta_1)} H^*_{\overline{\theta_0\theta_1}} P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\theta_2\theta_3}} P_{\mathcal{H}(\theta_2\delta_3)} \\ &= P_{\mathcal{H}(\theta_0\delta_1)} H^*_{\overline{\theta_0\theta_1}} H_{\overline{\theta_2\theta_3}} P_{\theta_2\delta_3 H^2} P_{\mathcal{H}(\theta_2\delta_3)} \\ &= 0. \end{aligned} \tag{4.13}$$

Similarly, we also have that

$$A_c^{-1^*} H_{\overline{\psi_+}}^* P_{\mathcal{H}(\widetilde{\theta})} H_{\overline{\psi_+}} A_c^{-1} = P_{\mathcal{H}(\theta_2\theta_3)\cap\theta_2\delta_3 H^2}. \tag{4.14}$$

Therefore by (4.11)-(4.14), C should be of the form

$$C = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 1 & * & * \\ 0 & * & 0 & 0 \\ * & * & 0 & 1 \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \mathcal{H}(\theta_0\theta_1)\cap\theta_0\delta_1 H^2 \\ \mathcal{H}(\theta_2\delta_3) \\ \mathcal{H}(\theta_2\theta_3)\cap\theta_2\delta_3 H^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \mathcal{H}(\theta_0\theta_1)\cap\theta_0\delta_1 H^2 \\ \mathcal{H}(\theta_2\delta_3) \\ \mathcal{H}(\theta_2\theta_3)\cap\theta_2\delta_3 H^2 \end{pmatrix}. \tag{4.15}$$

In particular, we have that

$$P_{\mathcal{H}(\theta_0\delta_1)\oplus\mathcal{H}(\theta_2\delta_3)}\, C\, P_{\mathcal{H}(\theta_0\delta_1)\oplus\mathcal{H}(\theta_2\delta_3)} = 0. \tag{4.16}$$

Observe, by (4.10), that $\mathbf{T}_{1\theta} = (T_{\phi_{1\theta}}, T_{\psi_{1\theta}})$ is hyponormal if and only if the following operator matrix $\widetilde{D}$ is positive:

$$\widetilde{D} := \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} : \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \mathcal{H}(\theta_2\delta_3) \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}(\theta_0\delta_1) \\ \mathcal{H}(\theta_2\delta_3) \end{pmatrix}. \tag{4.17}$$

But since by (4.10), (4.11), (4.16), and (4.17),

$$\begin{aligned} \widetilde{D} &= P_{\mathcal{H}(\theta_0\delta_1)\oplus\mathcal{H}(\theta_2\delta_3)} G^* \Big([\mathbf{T}^*, \mathbf{T}] - V\Big)\, G\, P_{\mathcal{H}(\theta_0\delta_1)\oplus\mathcal{H}(\theta_2\delta_3)} \\ &= P_{\mathcal{H}(\theta_0\delta_1)\oplus\mathcal{H}(\theta_2\delta_3)} \Big(G^*\, [\mathbf{T}^*, \mathbf{T}]\, G\, - C\Big) P_{\mathcal{H}(\theta_0\delta_1)\oplus\mathcal{H}(\theta_2\delta_3)} \\ &= P_{\mathcal{H}(\theta_0\delta_1)\oplus\mathcal{H}(\theta_2\delta_3)}\, G^*\, [\mathbf{T}^*, \mathbf{T}]\, G\, P_{\mathcal{H}(\theta_0\delta_1)\oplus\mathcal{H}(\theta_2\delta_3)} \\ &\geq 0, \end{aligned}$$

it follows that $\mathbf{T}_{1\theta}$ is hyponormal and this proves (4.1). □

STEP 3: *We claim that*

$$\theta_2 = \xi\theta_0 \quad \textit{for some nonzero } \xi \in \mathbb{C}. \tag{4.18}$$

Proof. In view of STEP 1, STEP 2, and Lemma 4, we may assume that θ_1 and θ_3 are relatively prime and θ_0 and θ_2 are relatively prime. Thus for (4.18) it suffices to show that θ_0 and θ_2 are constant. Since $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal it follows from again Lemma 4 that $\mathbf{T}_{\theta_0} = (T_{\phi_{\theta_0}}, T_{\psi_{\theta_0}})$ is hyponormal. Observe that

$$\begin{aligned} (\phi_{\theta_0})_+ &= P_{H_0^2}(\overline{\theta_0}\theta_0\theta_1\overline{a}) = P_{H_0^2}(\theta_1\overline{a}) = \theta_1\overline{P_{\mathcal{H}(\theta_1)}(a)}, \\ (\phi_{\theta_0})_- &= P_{H_0^2}(\overline{\theta_0}\theta_0\overline{b}) = 0, \\ (\psi_{\theta_0})_+ &= P_{H_0^2}(\overline{\theta_0}\theta_2\theta_3\overline{c}), \\ (\psi_{\theta_0})_- &= P_{H_0^2}(\overline{\theta_0}\theta_2\overline{d}). \end{aligned}$$

Thus ϕ_{θ_0} is analytic. Therefore by Lemma 5, T_ω is hyponormal, where

$$\omega = P_{H_0^2}\left(\overline{\theta_1}(\psi_{\theta_0})_+\right) + \overline{(\psi_{\theta_0})_-} = P_{H_0^2}\left(\overline{\theta_1} P_{H_0^2}(\overline{\theta_0}\theta_2\theta_3\overline{c})\right) + \overline{P_{H_0^2}(\overline{\theta_0}\theta_2\overline{d})}.$$

Since $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal we have, by (2.5), that $\theta_0\theta_1 H^2 \subset \theta_2 H^2$ and $\theta_2\theta_3 H^2 \subset \theta_0 H^2$. Since θ_0 and θ_2 are relatively prime we can see that $\theta_1 = \theta_2\Delta_1$ and $\theta_3 = \theta_0\Delta_3$ for some inner functions Δ_1 and Δ_3. Observe that

$$P_{H_0^2}(\overline{\theta_0}\theta_2\theta_3\overline{c}) = P_{H_0^2}(\theta_2\Delta_3\overline{c}) = \theta_2\Delta_3\overline{c_3},$$

where $c_3 := P_{\mathcal{H}(\theta_2\Delta_3)}(c)$. Since the inner part of $\Delta_1 c_3$ and Δ_3 are relatively prime (because c and θ_3 are relatively prime) we have that

$$\omega_+ = P_{H_0^2}(\overline{\theta_1}\theta_2\Delta_3\overline{c_3}) = P_{H_0^2}(\overline{\Delta_1}\Delta_3\overline{c_3}) = \Delta_3\overline{P_{\mathcal{H}(\Delta_3)}(\Delta_1 c_3)},$$

where Δ_3 and $P_{\mathcal{H}(\Delta_3)}(\Delta_1 c_3)$ are relatively prime. Since $\theta_0 d$ and θ_2 are relatively prime we also have that

$$\omega_- = P_{H_0^2}(\overline{\theta_0}\theta_2\overline{d}) = \theta_2\overline{P_{\mathcal{H}(\theta_2)}(\theta_0 d)},$$

where θ_2 and $P_{\mathcal{H}(\theta_2)}(\theta_0 d)$ are relatively prime. Therefore the hyponormality of T_ω forces that $\Delta_3 = \theta_2\zeta_3$ for some inner function ζ_3. Therefore $\theta_3 = \theta_0\Delta_3 = \theta_0\theta_2\zeta_3$ and hence θ_2 is a common inner factor of θ_1 and θ_3. But since θ_1 and θ_3 are relatively prime, we must have that θ_2 is a constant. Interchanging the roles of ϕ and ψ in the above argument gives that θ_0 is also a constant. This proves (4.18). □

We write $\mathcal{Z}(\theta)$ for the set of all zeros in $\mathbb{D}$ of the inner function θ.

STEP 4: *We conclude that*

$$\phi_- = \beta\psi_-, \quad \text{i.e.,} \quad \phi - \beta\psi \in H^2 \quad \textit{for some constant } \beta.$$

Proof. Suppose $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal. Then $T_{\phi-\beta\psi}$ is hyponormal for all $\beta \in \mathbb{C}$. In view of STEP 3, we may assume that $\theta_2 = \xi\theta_0$ for some $\xi \in \mathbb{C}$. Observe that

$$\phi - \alpha\psi = \theta_0\theta_1\theta_3\left(\overline{\theta_3 a} - \xi\alpha\,\overline{\theta_1 c}\right) + \overline{\theta_0}\left(b - \overline{\xi}\alpha d\right).$$

We want to show that $b = \beta_0 d$ for some $\beta_0 \in \mathbb{C}$. Assume to the contrary that $b \neq \beta d$ for any $\beta \in \mathbb{C}$. Then we can see that $(T_{\phi-\beta\psi}, T_\psi)$ is hyponormal (cf. [Gu, Lemma 5.1]). Let $\beta \in \mathcal{Z}(\theta_0)$, i.e., $\theta_0 = b_\beta\theta_\beta$ with $b_\beta := \frac{z-\beta}{1-\overline{\beta}z}$ and some finite Blaschke product θ_β. Since θ_0 and d are relatively prime, it follows that $d(\beta) \neq 0$. Write

$$\alpha := \xi\,\frac{b(\beta)}{d(\beta)}.$$

Then $\overline{b_\beta}(b - \overline{\xi}\alpha d) \in H^2$. Thus we have that

$$(\phi - \alpha\psi)_- = \theta_\beta\left(\overline{\overline{b_\beta}(b - \overline{\xi}\alpha d)}\right).$$

But since $(T_{\phi-\alpha\psi}, T_\psi)$ is hyponormal, applying STEP 3 with $(T_{\phi-\alpha\psi}, T_\psi)$ in place of (T_ϕ, T_ψ) gives that θ_0 is an inner factor of θ_β up to a unitary constant, a contradiction. Hence $b = \beta_0 d$ for some $\beta \in \mathbb{C}$, and hence $\phi_- = \beta\psi_-$ for some $\beta \in \mathbb{C}$. This completes the proof of Theorem 1. $\square$

The following corollary is a complete characterization of hyponormal rational Toeplitz pairs.

Corollary 2. Let $\phi = \overline{\phi_-} + \phi_+ \in L^\infty$ and $\psi = \overline{\psi_-} + \psi_+ \in L^\infty$ be of the form

$$\phi_+ = \theta_0\theta_1\overline{a},\ \phi_- = \theta_0\overline{b},\ \psi_+ = \theta_2\theta_3\overline{c} \text{ and } \psi_- = \theta_2\overline{d},$$

where the θ_j are finite Blaschke products, $a \in \mathcal{H}(\theta_0\theta_1)$, $b \in \mathcal{H}(\theta_0)$, $c \in \mathcal{H}(\theta_2\theta_3)$, and $d \in \mathcal{H}(\theta_2)$. Then $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal if and only if there exists a constant β such that

(i) $\phi_- = \beta\psi_-$;
(ii) $\theta_3 a - \overline{\beta}\theta_1 c \in \theta\theta_0 H^2$;
(iii) $T_{\psi'}$ is hyponormal, where $\psi' := P(\theta\theta_0\overline{\theta_1 c}) + \overline{\theta_2}d$.

Here θ is the greatest common inner factor of θ_1 and θ_3.

Proof. We first assume that θ_1 and θ_3 are relatively prime. Now suppose $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal. By Theorem 1, $\theta_0 = \theta_2$ (up to a unitary constant) and $\phi_- = \beta\psi_-$ for some $\beta \in \mathbb{C}$. Thus $\phi - \beta\psi = \theta_0\theta_1\theta_3(\overline{\theta_3 a} - \beta\overline{\theta_1 c}) \in z\, H^2$. We want to show that $\theta_0(\overline{\theta_3 a} - \beta\overline{\theta_1 c}) \in \overline{H^2}$. Assume $\theta_0(\overline{\theta_3 a} - \beta\overline{\theta_1 c}) \notin \overline{H^2}$. Then there exists a nonconstant factor ζ_0 of θ_0 such that $\phi - \beta\psi = \zeta_0\theta_1\theta_3\big[\theta_0\overline{\zeta_0}\,\big(\overline{\theta_3 a} - \beta\overline{\theta_1 c}\big)\big]$, where $\overline{\theta_0}\zeta_0\,\big(\theta_3 a - \overline{\beta}\theta_1 c\big) \in \mathcal{H}(\zeta_0\theta_1\theta_3)$. By Lemma 5, $T_{\widehat{\psi}}$ is hyponormal, where

$$\widehat{\psi} = \psi_{1,\zeta_0\theta_1\theta_3} = P_{H_0^2}(\theta_0\overline{\zeta_0\theta_1 c}) + \overline{\theta_0\overline{d}}.$$

Therefore we have that

$$\ker H_{\overline{\widehat{\psi}_-}} = \theta_0 H^2 \subseteq \theta_0\overline{\zeta_0} H^2 \subset \ker H_{\overline{\widehat{\psi}_+}},$$

which is a contradiction (see (2.3)). Thus $\theta_0(\overline{\theta_3 a} - \beta\overline{\theta_1 c}) \in \overline{H^2}$. Therefore, $\theta_3 a - \overline{\beta}\theta_1 c \in \theta_0 H^2$. In particular, $\phi - \beta\psi = \theta_1\theta_3\big[\theta_0(\overline{\theta_3 a} - \beta\overline{\theta_1 c})\big] \in zH^2$, where $\overline{\theta_0}(\theta_3 a - \overline{\beta}\theta_1 c) \in \mathcal{H}(\theta_1\theta_3)$. Therefore by Lemma 5, $T_{\psi'}$ is hyponormal, where

$$\psi' := P(\overline{\theta_1\theta_3}\psi_+) + \overline{\psi_-} = P(\theta_0\overline{\theta_1 c}) + \overline{\theta_2}d = \psi_{1,\theta_1\theta_3} + c \quad \text{for a constant } c.$$

The converse is obtained by reversing the above argument.

The proof for the general case can be accomplished by passing to (4.1) with the assumption that θ is the greatest common inner factor of θ_1 and θ_3. $\square$

By comparison with the cases of trigonometric Toeplitz pairs, we are tempted to guess $\theta_1 = \theta_3$ in the criterion of Corollary 2. If this were true then we would conclude that if ψ is a trigonometric polynomial and ϕ is an arbitrary rational symbol then the hyponormality of $\mathbf{T} = (T_\phi, T_\psi)$ forces ϕ to be a trigonometric polynomial. However this is not the case. For example, if $\psi(z) = \frac{1}{6}z^{-1} - z$ and $\phi(z) = \frac{1}{6}z^{-1} + zB\,\overline{(\frac{1}{3}B + \frac{2}{3})}$, where $B = \frac{z-\frac{1}{2}}{1-\frac{1}{2}z}$ then ϕ and ψ satisfy all three conditions in Corollary 2, so that $\mathbf{T} = (T_\phi, T_\psi)$ is hyponormal even though ϕ is not a trigonometric polynomial.

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