SUBNORMAL TOEPLITZ OPERATORS AND
THE KERNELS OF THE SELF-COMMUTATORS

IN SUNG HWANG AND WOO YOUNG LEE

Abstract. In this note we give a connection between subnormal Toeplitz operators and the kernels of the self-commutators. This is closely related to P.R. Halmos’s Problem 5: Is every subnormal Toeplitz operator either normal or analytic? Our main theorem is as follows: If $\varphi \in L^\infty$ is such that $\varphi$ and $\overline{\varphi}$ are of bounded type (that is, they are quotients of two analytic functions on the open unit disk) and if the kernel of the self-commutator of $T_\varphi$ is invariant for $T_\varphi$ then $T_\varphi$ is either normal or analytic.

1. Introduction
The present note concerns the relationship between subnormal Toeplitz operators and the kernels of their self-commutators. We begin with a brief survey of research related to P.R. Halmos’s Problem 5 (cf. [Hal1], [Hal2]):

(Prob 5) Is every subnormal Toeplitz operator either normal or analytic?

As we know, (Prob 5) was answered in the negative by C. Cowen and J. Long ([CoL]): directly connected with it is the following problem:

(1.1) Which Toeplitz operators are subnormal?

It remains still open to characterize subnormal Toeplitz operators in terms of their symbols. To date, (Prob 5) has been partially answered in the affirmative by many authors (cf. [Ab], [AIW], [Co1], [Co2], [CuL1], [CuL2], [IW], [NT], and etc). The most interesting partial answer of them was given by M. Abrahamse [Ab]. M. Abrahamse gave a general sufficient condition for the answer to (Prob 5) to be affirmative: his assumption relies heavily upon the invariance of the kernel of its self-commutator under the given Toeplitz operator. In this note we examine the effect on the invariance of the kernel of the self-commutator for the Toeplitz operator from the viewpoint of subnormality of Toeplitz operators.

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if its self-commutator $T^*T = T^*T - TT^*$ is positive (semidefinite), and subnormal if $T = N|_H$, where $N$ is normal on some Hilbert space $K \supseteq H$. Recall that the Hilbert space $L^2 \equiv L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2 \equiv H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \ldots\}$. An element $f \in L^2(\mathbb{T})$ is said to be analytic if $f \in H^2(\mathbb{T})$. Let $H^\infty := L^\infty \cap H^2$, i.e., $H^\infty$ is the set of bounded analytic functions on $\mathbb{D}$. If $P$ denotes the orthogonal projection from $L^2$ to $H^2$ and $J$ denotes the unitary operator on $L^2(\mathbb{T})$ defined by $J(f)(z) = \overline{z}f(\overline{z})$, then for every $\varphi \in L^\infty \equiv L^\infty(\mathbb{T})$, the operators $T_\varphi$ and $H_\varphi$ on $H^2$ defined by

$T_\varphi g := P(\varphi g)$ and $H_\varphi(g) := J(I - P)(\varphi g)$ \quad (g \in H^2)$

2000 Mathematics Subject Classification. Primary 47B20, 47B35
Key words and phrases. Toeplitz operators, subnormal, hyponormal, bounded type, self-commutator.
The second author was supported by a grant (KRF-2008-314-C00014) from the Korea Research Foundation.
are called the **Toeplitz operator** and the **Hankel operator**, respectively, with symbol \( \varphi \). It is easy to see that analytic Toeplitz operators are subnormal: indeed, the multiplication operator \( M_{\varphi} \) on \( L^2 \) is a normal extension of \( T_{\varphi} \) for \( \varphi \in H^\infty \). When we study hyponormality of the Toeplitz operator \( T_{\varphi} \) with symbol \( \varphi \) we may without loss of generality assume that \( \varphi(0) = 0 \) because the hyponormality of an operator is invariant under translation by scalars. The following are basic connections between Hankel and Toeplitz operators (cf. [Ni]):

\[
\begin{align*}
(1.2) & \quad T_{\varphi\psi} - T_{\varphi}T_{\psi} = H_{\varphi}^*H_{\psi}(\varphi, \psi \in L^\infty); \\
(1.3) & \quad H_{\varphi}T_h = H_{\varphi h} = T_h^*H_{\varphi} (h \in H^\infty, \varphi \in L^\infty),
\end{align*}
\]

where \( \tilde{h}(z) := H(z) \).

Normal Toeplitz operators were characterized by a property of their symbols in the early 1960’s by A. Brown and P.R. Halmos [BH] and the exact nature of the relationship between the symbol \( \varphi \in L^\infty \) and the positivity of the self-commutator \( [T_{\varphi}^*, T_{\varphi}] \) was understood via Cowen’s theorem [Co3] in 1988. For each \( \varphi \in L^\infty \), let \( \mathcal{E}(\varphi) \equiv \{ k \in H^\infty : ||k||_{\infty} \leq 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty \} \). Then Cowen’s theorem can be stated as follows:

\[ T_{\varphi} \text{ is hyponormal } \iff \mathcal{E}(\varphi) \neq \emptyset. \]

We denote by \([A, B] := AB - BA\) the commutator of two operators \( A \) and \( B \). It is well-known that if \( T \in B(H) \) is a subnormal operator then

\[ \ker [T^*, T] \text{ is invariant for } T. \]

However we need not expect that if \( T \) is hyponormal then \( \ker [T^*, T] \) is invariant for \( T \): for example, if \( T \equiv W_\alpha \) is a (unilateral) weighted shift with weight sequence \( \alpha \equiv \{ \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \ldots \} \) then \( T \) is hyponormal, \( e_1 \equiv \{ 0, 1, 0, 0, \cdots \} \in \ker [T^*, T] \), but \( Te_1 \notin \ker [T^*, T] \). Thus the invariance of \( \ker [T^*, T] \) for \( T \) is not an intrinsic property of hyponormality. We also note that an operator \( T \) need not be subnormal even though \( T \) is a hyponormal operator and \( \ker [T^*, T] \) is invariant for \( T \): an easy example is given by a non-subnormal weighted shift with strictly increasing weight sequence.

Recall that a function \( \varphi \in L^\infty \) is said to of **bounded type** (or in the Nevanlinna class) if there are functions \( \psi_1, \psi_2 \in H^\infty(D) \) such that

\[ \varphi(z) = \frac{\psi_1(z)}{\psi_2(z)} \]

for almost all \( z \) in \( \mathbb{T} \). Evidently, rational functions in \( L^\infty \) are of bounded type. It was known [Ab, Lemma 3] that if \( \varphi \in L^\infty \) then

\[ \varphi \text{ is of bounded type } \iff \ker H_{\varphi} \neq \{0\}. \]

Abrahamse’s theorem can be then stated as:

**Abrahamse’s theorem ([Ab, Theorem]).** If

(i) \( \varphi \) or \( \overline{\varphi} \) is of bounded type;

(ii) \( T_{\varphi} \) is hyponormal;

(iii) \( \ker [T_{\varphi}^*, T_{\varphi}] \) is invariant for \( T_{\varphi} \),

then \( T_{\varphi} \) is normal or analytic.

M. Abrahamse [Ab, Lemma 6] also showed that if \( T_{\varphi} \) is hyponormal, if \( \varphi \notin H^\infty \), and if \( \varphi \) or \( \overline{\varphi} \) is of bounded type then both \( \varphi \) and \( \overline{\varphi} \) are of bounded type. Its proof given in [Ab] is somewhat intricate. However via Cowen’s theorem we can easily see it: indeed, if \( T_{\varphi} \) is hyponormal and \( \varphi \notin H^\infty \) then there exists nonzero \( k \in H^\infty \) such that \( \varphi - k\overline{\varphi} \in H^\infty \), so that by (1.3),

\[ H_{\varphi} = H_{k\overline{\varphi}} = T_{\overline{\varphi}}^*H_{\overline{\varphi}} = H_{\overline{\varphi}}T_k, \]
which implies that $\ker H_\varphi \neq \{0\}$ if and only if $\ker H_\varphi \neq \{0\}$, and therefore if $\varphi$ or $\overline{\varphi}$ is of bounded type then both $\varphi$ and $\overline{\varphi}$ are of bounded type.

The purpose of this note is to show that the assumption “$T_\varphi$ is hyponormal” is superfluous in the Abrahamse’s theorem: this shows an influence on the invariance of the kernel of the self-commutator of $T_\varphi$ for $T_\varphi$. Our main theorem is now stated as:

**Theorem 1.1.** Let $\varphi \in L^\infty$. If

(i) $\varphi$ and $\overline{\varphi}$ are of bounded type;

(ii) $\ker [T_\varphi^*, T_\varphi]$ is invariant for $T_\varphi$,

then $T_\varphi$ is normal or analytic.

We might ask whether $T_\varphi$ is hyponormal if $\varphi$ and $\overline{\varphi}$ are not of bounded type and $\ker [T_\varphi^*, T_\varphi]$ is invariant for $T_\varphi$. However this is not the case. To see this we first observe that if $\varphi = \overline{g} + f \in L^\infty (f, g \in H^2)$ is such that $T_\varphi$ is hyponormal then $||g||_2 \leq ||f||_2$ (this follows at once from the Cowen’s theorem). Now if $f \in H^\infty$ is such that $\overline{f}$ is not of bounded type, define $\varphi := 2 \overline{f} + f$. Then by the preceding observation $T_\varphi$ is not hyponormal, whereas by (1.2) and (1.4),

$$\ker [T_\varphi^*, T_\varphi] = \ker (T_f T_{\overline{f}} - T_{|f|^2}) = \ker H_{\overline{f}} = \{0\},$$

which implies that vacuously, $\ker [T_\varphi^*, T_\varphi]$ is invariant for $T_\varphi$.

2. The proof of the main theorem

In this section we give a proof of Theorem 1.1. We begin with an observation that if $\varphi$ is of bounded type, say $\varphi = \frac{\psi_2}{\psi_1}$ ($\psi_1, \psi_2 \in H^\infty$) then dividing the outer part of $\psi_1$ into $\psi_2$ one obtains $\varphi = \frac{\theta}{b}$, where $\theta$ is inner and $b \in H^\infty$ satisfies that the inner part of $b$ and $\theta$ are coprime. Thus $\varphi = \theta b$. We write, for an inner function $\theta$,

$$\mathcal{H}(\theta) := H^2 \ominus \theta H^2.$$

Thus if $f \in H^2$ is such that $\overline{f}$ is of bounded type and $f(0) = 0$, then we can write

$$f = \theta b,$$

(2.1)

where $b$ is an inner function and $b \in \mathcal{H}(\theta)$ satisfies that the inner part of $b$ and $\theta$ are coprime.

We then have:

**Lemma 2.1.** Let $\varphi = \overline{g} + f \in L^\infty (f, g \in H^2)$. If $\varphi$ and $\overline{\varphi}$ are of bounded type and $\ker [T_\varphi^*, T_\varphi]$ is invariant for $T_\varphi$ then we can write

$$f = \theta_1 \theta_2 b \quad \text{and} \quad g = \theta_1 b,$$

where $\theta_1$ and $\theta_2$ are inner functions, $a \in \mathcal{H}(\theta_1 \theta_2)$ and $b \in \mathcal{H}(\theta_1)$.

**Proof.** In view of (2.1) write

$$f = \theta b \quad \text{and} \quad g = \omega b,$$

where $\theta$ and $\omega$ are inner functions, and $a \in \mathcal{H}(\theta)$ and $b \in \mathcal{H}(\omega)$ satisfy that the inner part of $a$ and $\theta$ are coprime and the inner part of $b$ and $\omega$ are coprime. From (1.2) we obtain

$$[T_\varphi^*, T_\varphi] = H_\varphi^* H_{\overline{f}} - H_{\overline{g}} H_{\varphi} = H_{\theta a}^* H_{\theta a} - H_{\omega b}^* H_{\omega b}.$$

Thus $\theta \omega H^2 \subset \ker [T_\varphi^*, T_\varphi]$. Since by assumption $\ker [T_\varphi^*, T_\varphi]$ is invariant for $T_\varphi$ it follows that $T_\varphi(\theta \omega) \in \ker [T_\varphi^*, T_\varphi]$, and hence $f \theta \omega + \theta b = P((f + \omega b) \theta \omega) \in \ker [T_\varphi^*, T_\varphi]$. We thus have $(H_{\theta a}^* H_{\theta a} - H_{\omega b}^* H_{\omega b})(f \theta \omega + \theta b) = 0$, which implies that $H_{\omega b}(\theta b) = 0$. Thus $\omega \theta b^2 \in H^2$, and hence $\theta b^2 \in \omega H^2$. But since the inner part of $b$ and $\omega$ are coprime it follows that $\omega$ divides $\theta$. This proves (2.2). □
Lemma 2.2. Let $\theta$ be an inner function and $\psi \in H^\infty$. Put

$$A_\psi := P_{H(\theta)}M_\psi|_{H(\theta)}.$$

If $\theta$ and $\psi_1$ are coprime ($\psi_1$ is the inner part of $\psi$) then $A_\psi$ is one-one and has dense range.

Proof. Let $f \in H(\theta)$ be such that $A_\psi f = 0$. Then $P_{H(\theta)}(\psi f) = 0$ and hence $\psi f = \theta g$ for some $g \in H^2$. Since $\theta$ and $\psi_1$ are coprime we have $f = \theta g_1$ for some $g_1 \in H^2$. Thus $f \in H(\theta) \cap \theta H^2 = \{0\}$, and hence $A_\psi$ is one-one. Also suppose $f \in H(\theta)$ is such that $A_\psi f = 0$. Then $P_{H(\theta)}(\overline{\psi} f) = 0$, so that $\overline{\psi} f = \theta h + \overline{\theta} g$ for some $g, h \in H^2$. Thus $\overline{\theta} \overline{\psi} f = h + \overline{\theta} g$. Since $f \in H(\theta)$ it follows that $\overline{\theta} f \in (H^2)^\perp$, and hence $h + \overline{\theta} g = \overline{\theta} \overline{\psi} f \in (H^2)^\perp$. Therefore $h = 0$ and hence $\overline{\psi} f = \overline{\theta} g \in (H^2)^\perp$. If $\psi = \psi_1 \psi_2$ is the inner-outer factorization of $\psi$ then since $\psi_1 H^2$ is dense in $H^2$, it follows that

$$(\psi H^2)^\perp = (\psi_1 H^2)^\perp = H(\psi_1).$$

Thus we have $f \in H(\psi_1)$. If $\theta$ and $\psi_1$ are coprime then it follows that $f \in H(\theta) \cap H(\psi_1) = \{0\}$ because if $G$ be a family of inner functions then $\bigcap_{\theta \in G} H(\theta) = H(\omega)$, where $\omega$ is the greatest common inner factor of elements in $G$. Therefore $A_\psi$ has dense range. \hfill $\square$

Lemma 2.3. Let $\varphi = \overline{\theta} f + g \in L^\infty$. If $f = \theta \varphi$ and $g = \omega \overline{\varphi}$ for $\alpha \in H(\theta)$ and $b \in H(\omega)$, then $H(\theta) \cap \omega H^2 \subseteq \text{cl ran} [T^*_\varphi, T_\varphi] \subseteq H(\zeta)$, where $\zeta$ is the least common multiple of $\theta$ and $\omega$.

Proof. Observe that

$$[T^*_\varphi, T_\varphi] = H^*_T H_T - H^*_g H_g = H^*_a H_a - H^*_b H_b.$$

Since $\text{cl ran} (H^*_a, H_b) = \text{cl ran} (H^*_a) = \ker (H_b)$ and similarly, $\text{cl ran} (H^*_b H_b) = H(\omega)$, we can see that $H(\theta) \cap \omega H^2 \subseteq \text{cl ran} [T^*_\varphi, T_\varphi] \subseteq H(\zeta)$. \hfill $\square$

We are ready for:

Proof of Theorem 1.1. In view of Lemma 2.1 we may write

$$f = \theta_1 \theta_2 \varphi \quad \text{and} \quad g = \theta_1 \overline{\varphi},$$

where $\theta_i$ ($i = 1, 2$) are inner functions, and $a \in H(\theta_1 \theta_2)$ and $b \in H(\theta_1)$ satisfy that the inner part of $a$ and $\theta_1 \theta_2$ are coprime and the inner part of $b$ and $\theta_1$ are coprime. Suppose $\varphi$ is not analytic, and hence $\theta_1$ is not constant. By Lemma 2.3, $\theta_1 \theta_2 H^2 \subseteq \ker [T^*_\varphi, T_\varphi]$. Observe that for all $h \in H^2$,

$$T_\varphi(\theta_1 \theta_2 h) = P(\overline{\theta_1} b \theta_1 \theta_2 h + f \theta_1 \theta_2 h)$$

(2.3)

$$= b \theta_2 h + \theta_1 \theta_2 h f$$

If $\theta_2 h_1 \in \theta_2 H(\theta_1)$ then $h_1 \in H(\theta_1)$ and

$$P_{H(\theta_1 \theta_2)}(\theta_2 bh) = \theta_2 h_1 \iff \theta_2 bh - \theta_2 h_1 \in \theta_1 \theta_2 H^2$$

$$\iff bh - h_1 \in \theta_1 H^2$$

$$\iff P_{H(\theta_1)}(bh) = h_1.$$
Since \( \ker[T^*_{\varphi}, T^*_{\psi}] \) is invariant for \( T^*_{\varphi} \) and \( \ker[T^*_{\varphi}, T^*_{\psi}] \) is a closed subspace it follows from (2.3) and (2.4) that
\[
(2.5) \quad \theta_1 \theta_2 H^2 \oplus \theta_2 \mathcal{H}(\theta_1) \subseteq \ker[T^*_{\varphi}, T^*_{\psi}].
\]
Since \( \mathcal{H}(\theta_1 \theta_2) = \theta_1 \mathcal{H}(\theta_2) \oplus \mathcal{H}(\theta_1) \), it thus follows from Lemma 2.3 that
\[
(2.6) \quad \theta_1 \mathcal{H}(\theta_2) = \mathcal{H}(\theta_1 \theta_2) \oplus \mathcal{H}(\theta_1) \subseteq \text{cl ran}[T^*_{\varphi}, T^*_{\psi}] \subseteq \mathcal{H}(\theta_1 \theta_2).
\]
Thus by (2.5) and (2.6), \( \theta_1 \theta_2 H^2 \oplus \theta_2 \mathcal{H}(\theta_1) \subseteq \theta_1 \theta_2 H^2 \oplus \mathcal{H}(\theta_1) \), which implies that
\[
(2.7) \quad \theta_2 \mathcal{H}(\theta_1) \subseteq \mathcal{H}(\theta_1).
\]
Since \( \mathcal{H}(\theta_1) = \text{cl ran} H^*_{\theta_1} \) it follow that
\[
\theta_2 \mathcal{H}(\theta_1) = \theta_2 \text{cl ran} H^*_{\theta_1} = \text{cl ran} T_{\theta_2} H^*_{\theta_1}.
\]
Observe that
\[
\theta_2 \mathcal{H}(\theta_1) \subseteq \mathcal{H}(\theta_1) \implies \text{cl ran} H^*_{\theta_1} \supseteq \text{cl ran} T_{\theta_2} H^*_{\theta_1}
\]
\[
\implies \ker H^*_{\theta_1} \subseteq \ker H^*_{\theta_1} T_{\theta_2}
\]
\[
\implies H^*_{\theta_1} T_{\theta_2} \theta_1 = 0
\]
\[
\implies H^*_{\theta_1} T_{\theta_2} \theta_1 = 0.
\]
But since for \( \phi, \psi \in L^\infty, T^*_{\varphi} H^*_{\psi} + H^*_{\psi} T^*_{\varphi} = H^*_{\psi \varphi} \), we thus have that
\[
0 = H^*_{\theta_1} T_{\theta_2} \theta_1 = H^*_{\theta_1} T_{\theta_2} \theta_1 = H^*_{\theta_1} T_{\theta_2} \theta_1 = 0.
\]
which implies that \( H^*_{\theta_2} = T_{\theta_1} H^*_{\theta_2} \), so that \( (I - T^*_{\varphi}) H^*_{\theta_2} = 0 \). We therefore have that \( P_{\mathcal{H}(\theta_2)} H^*_{\theta_2} = 0 \), and hence \( \mathcal{H}(\theta_2) = \text{cl ran} H^*_{\theta_2} = \text{cl ran} H^*_{\theta_2} \subseteq \theta_1 H^2 \). We thus have that
\[
(2.8) \quad \mathcal{H}(\theta_2) \subseteq \theta_1 H^2.
\]
But it is known that \( \mathcal{H}(\theta_2) \) contains at least an outer function if \( \theta_2 \) is not constant. This contradicts (2.8) because \( \theta_1 \) is not constant. We thus conclude that \( \theta_2 \) should be constant. Therefore by (2.5), \( H^2 = \theta_1 H^2 \oplus \mathcal{H}(\theta_1) \subseteq \ker[T^*_{\varphi}, T^*_{\psi}] \) and hence \( T^*_{\varphi} \) is normal. This completes the proof. \( \square \)

We however need not expect that if \( \ker[T^*_{\varphi}, T^*_{\psi}] \) is invariant for \( T^*_{\varphi} \) and if \( \varphi \) or \( \overline{\varphi} \) is of bounded type then both \( \varphi \) and \( \overline{\varphi} \) are of bounded type. In fact, we can prove more:

**Proposition 2.4.** If \( \varphi \) is of bounded type but \( \overline{\varphi} \) is not then \( \ker[T^*_{\varphi}, T^*_{\psi}] = \{0\} \). Hence in particular \( \ker[T^*_{\varphi}, T^*_{\psi}] \) is invariant for \( T^*_{\varphi} \).

**Proof.** Let \( \varphi = \overline{\varphi} + f \in L^\infty (f, g \in H^2) \). If \( g = 0 \), then \( \ker[T^*_{\varphi}, T^*_{\psi}] = H^2_{\overline{\varphi}} H_{\varphi} \). Since \( \overline{\varphi} \) is not of bounded type, this is trivial. Let \( g \neq 0 \). We can then write
\[
g = \theta \overline{b},
\]
where \( \theta \) is an inner function, \( b \in \mathcal{H}(\theta) \) and the inner part of \( b \) and \( \theta \) are coprime. Suppose that \( h \in \ker[T^*_{\varphi}, T^*_{\psi}] \). Since \( \text{cl ran} H^*_{\theta_0} H^*_{\theta_0} = \mathcal{H}(\theta) \), it follows that
\[
H^*_{\overline{\varphi}} H_{\varphi} h = H^*_{\theta_0} H^*_{\theta_0} h = a \in \mathcal{H}(\theta).
\]
Put \( h_1 := H_{\overline{\varphi}} h \). We then have that
\[
H^*_{\overline{\varphi}} h_1 = a \implies J(I - P)(\overline{h_1}) = a
\]
\[
\implies (I - P)(\overline{h_1}) = J(a)
\]
\[
\implies \overline{h_1} = J(a) + h_2 \quad \text{for some } h_2 \in H^2
\]
\[
\implies \overline{\theta h_1} = \overline{\theta J(a)} + \overline{\theta h_2}.
\]
Now we will show that $\tilde{\theta} J(a) \in H^2$. Indeed, if $h \in H^2$ is arbitrary, then we have that

$$\langle \tilde{\theta} J(a), zh \rangle = \langle J(a), \tilde{\theta} zh \rangle = \langle za(z), \tilde{\theta} zh \rangle = \langle \tilde{\theta} h, \tilde{\theta} h \rangle = 0. $$

Thus we have $H^*_f(\tilde{\theta} h) = 0$. But since $\tilde{f}$ is not of bounded type it follows that $\tilde{\theta} h = 0$, and hence $h_1 = 0$. Therefore $H^*_f h = 0$, and so $h = 0$. □

References


In Sung Hwang

DEPARTMENT OF MATHEMATICS, SUNGYUNKWAN UNIVERSITY, SUIWON 440-746, KOREA

E-mail address: ihwang@skku.edu

Woo Young Lee

DEPARTMENT OF MATHEMATICS, SEOUL NATIONAL UNIVERSITY, SEOUL 151-742, KOREA

E-mail address: wylee@math.snu.ac.kr