Subnormality of Aluthge transforms of weighted shifts

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Abstract. In this note we study the k-hyponormality and the subnormality of Aluthge transforms of weighted shifts. It is shown that Aluthge transforms of weighted shifts need not preserve the k-hyponormality. Moreover, we show that if W_α is a subnormal weighted shift with 2-atomic Berger measure then its Aluthge transform $\widetilde{W}_\alpha$ is subnormal if and only if at least one of two atoms is zero.

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1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. Recall that given a bounded sequence of positive real numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the *(unilateral) weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+)$ (where $\mathbb{Z}_+$ is the set of non-negative integers). In what follows we simply write $W_\alpha \equiv \text{shift}\,(\alpha_0, \alpha_1, \cdots)$. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if

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$\alpha_n \le \alpha_{n+1}$ for all $n \ge 0$. On the other hand, the Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \ge 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([Con, III.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \ge 0 \qquad (\text{all } k \ge 1). \tag{1.1}$$

The positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.1) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{1.2}$$

is positive semi-definite. An application of the Choleski algorithm for operator matrices shows that the positivity of (1.2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (1.1); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \ge 1$ ([CMX, Proposition 1.9]).

If $T \in \mathcal{B}(\mathcal{H})$, write $T = U|T|$ for the polar decomposition of T. The *Aluthge transform* of T is defined by the operator $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$. This transform was first introduced in [Alu] and has received much attention in recent years. For a weighted shift $W_\alpha \equiv \text{shift}\,(\alpha_0, \alpha_1, \cdots)$, we also write $\widetilde{W_\alpha}$ for the Aluthge transform of W_α. In this note we consider the following two problems.

Problem 1.1. *If W_α is k-hyponormal ($k \ge 1$), is the Aluthge transform $\widetilde{W_\alpha}$ k-hyponormal ?*

Problem 1.2. *If W_α is subnormal, is the Aluthge transform $\widetilde{W_\alpha}$ subnormal ? If it does, what is the Berger measure of $\widetilde{W_\alpha}$?*

2. Notations and Preliminaries

For a weighted shift $W_\alpha \equiv \text{shift}\,(\alpha_0, \alpha_1, \cdots)$, the *moments* of W_α, γ_n, are defined by

$$\gamma_n \equiv \gamma_n(W_\alpha) := \begin{cases} 1, & \text{if } n = 0 \\ \alpha_0^2 \cdots \alpha_{n-1}^2, & \text{if } n > 0 \end{cases}. \tag{2.1}$$

It is well-known that W_α is subnormal if and only if

$$\gamma_n = \int_{[0,||W_\alpha||^2]} t^n \, d\mu(t) \quad (\text{all } n \geq 0),$$

where μ is a probability measure on the interval $[0, ||W_\alpha||^2]$ (this measure μ is called the *Berger measure* of the subnormal weighted shift W_α). We consider *recursively generated weighted shifts* [CuFi1], [CuFi2]. We briefly recall some basic facts about these shifts, specifically the case when there are two coefficients of recursion. In [Sta], J. Stampfli proved that given three positive real numbers $\sqrt{a} < \sqrt{b} < \sqrt{c}$, it is always possible to find a subnormal weighted shift, denoted $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$, whose first three weights are $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$. In this case, the coefficients of recursion (cf. [CuFi1, Example 3.12], [CuFi2, Section 3], [Cu2, Section 1, p. 81]) are given by

$$\varphi_0 = -\frac{ab(c-b)}{b-a} \text{ and } \varphi_1 = \frac{b(c-a)}{b-a}, \tag{2.2}$$

the atoms t_0 and t_1 are the roots of the equation

$$t^2 - \varphi_1 t - \varphi_0 = 0, \tag{2.3}$$

and the densities ρ_0 and ρ_1 are uniquely determined by the following equations

$$\begin{cases} \rho_0 + \rho_1 &= 1 \\ \rho_0 t_0 + \rho_1 t_1 &= \alpha_0^2 . \end{cases} \tag{2.4}$$

Then

$$\mu = \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1} \tag{2.5}$$

is the Berger measure of $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$.

3. Main Results

We first provide several well-known auxiliary results which are needed for the proofs of the main results in this note.

Lemma 3.1. *If* $W_\alpha \equiv \text{shift}\,(\alpha_0, \alpha_1, \cdots)$, *then the Aluthge transform* $\widetilde{W}_\alpha$ *of* W_α *is*

$$\text{shift}\,(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \cdots).$$

Proof. Note that the polar decomposition of W_α is $U_+ D_\alpha$, where $D_\alpha := \text{diag}\,(\alpha_0, \alpha_1, \cdots)$. Hence, $\widetilde{W}_\alpha = D_\alpha^{\frac{1}{2}} U_+ D_\alpha^{\frac{1}{2}}$. For $n \geq 0$ and the orthonormal basis $\{e_n\}_{n=0}^\infty$ for $\ell^2(\mathbb{Z}_+)$, we have $U_+ D_\alpha^{\frac{1}{2}}(e_n) = \sqrt{\alpha_n} U_+(e_n) = \sqrt{\alpha_n} e_{n+1}$. Thus we get

$$\widetilde{W}_\alpha(e_n) = D_\alpha^{\frac{1}{2}} U_+ D_\alpha^{\frac{1}{2}}(e_n) = \sqrt{\alpha_n \alpha_{n+1}} e_{n+1},$$

which gives the result. □

Lemma 3.2. ([Cu1]) *Let $W_\alpha e_i = \alpha_i e_{i+1}$ $(i \geq 0)$ be a hyponormal weighted shift and let $k \geq 1$. The following statements are equivalent:*

(i) *W_α is k-hyponormal;*

(ii) *The matrix*

$$(([W_\alpha^{*j}, W_\alpha^i]e_{n+j}, e_{n+i}))_{i,j=1}^k$$

is positive semi-definite for all $n \geq -1$;

(iii) *The matrix*

$$(\gamma_n\gamma_{n+i+j} - \gamma_{n+i}\gamma_{n+j})_{i,j=1}^k$$

is positive semi-definite for all $n \geq 0$, where as (2.1), $\gamma_0 = 1$, $\gamma_n = \alpha_0^2 \cdots \alpha_{n-1}^2$ $(n \geq 1)$;

(iv) *The Hankel matrix*

$$H(k;n)(W_\alpha) := (\gamma_{n+i+j-2})_{i,j=1}^{k+1}$$

is positive semi-definite for all $n \geq 0$.

Lemma 3.3. (Subnormal backward extensions) (cf. [Cu1], [CuYo]) *Let T be a weighted shift whose restriction $T_{\mathcal{M}} := T|_{\mathcal{M}}$ to $\mathcal{M} := \vee\{e_1, e_2, \cdots\}$ is subnormal, with Berger measure $\mu_{\mathcal{M}}$. Then T is subnormal (with Berger measure μ) if and only if*

(i) $\frac{1}{t} \in L^1(\mu_{\mathcal{M}})$;

(ii) $\alpha_0^2 \leq \left(\left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}\right)^{-1}$.

In this case,

$$d\mu(t) = \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(t) + \left(1 - \alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})}\right) d\delta_0(t), \tag{3.1}$$

where δ_0 denotes the Dirac measure at 0. In particular, T is never subnormal when $\mu_{\mathcal{M}}(\{0\}) > 0$.

We now have:

Theorem 3.4. *For $x > 0$, let $W_{\alpha(x)} \equiv \text{shift}\left(\sqrt{\frac{1}{2}}, \sqrt{x}, \left(\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}\right)^\wedge\right)$. Then we have:*

(i) *$W_{\alpha(x)}$ is 2-hyponormal if and only if $4 - \sqrt{6} \leq x \leq 2$;*

(ii) *$W_{\alpha(x)}$ is subnormal if and only if $x = 2$. In this case, the Berger measure μ of $W_{\alpha(2)}$ is given by*

$$\mu = \tfrac{1}{2}\delta_0 + \tfrac{1}{4(2-\sqrt{2})}\delta_{(2-\sqrt{2})} + \tfrac{1}{4(2+\sqrt{2})}\delta_{(2+\sqrt{2})};$$

(iii) *$\widetilde{W}_{\alpha(x)}$ is not 2-hyponormal for any $x > 0$.*

Proof. (i) To check the 2-hyponormality, it suffices to prove, in view of Lemma 3.2, that the Hankel matrix $H(2;n)\left(W_{\alpha(x)}\right)$ in Lemma 3.2 (iv) is positive semi-definite for all $n \geq 0$. Since $W_{\left(\sqrt{3},\sqrt{\frac{10}{3}},\sqrt{\frac{17}{5}}\right)^\wedge}$ is subnormal, it is enough to check the cases of $n = 0, 1$ for the 2-hyponormality of $W_{\alpha(x)}$.

For $n = 0$, we have

$$H(2;0)\left(W_{\alpha(x)}\right) \geq 0 \Longleftrightarrow \det \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2}x \\ \frac{1}{2} & \frac{1}{2}x & \frac{3}{2}x \\ \frac{1}{2}x & \frac{3}{2}x & 5x \end{pmatrix} \geq 0 \Longleftrightarrow 4-\sqrt{6} \leq x \leq 4+\sqrt{6}\,. \tag{3.2}$$

For $n = 1$, we have

$$H(2;1)\left(W_{\alpha(x)}\right) \geq 0 \Longleftrightarrow \det \begin{pmatrix} 1 & x & 3x \\ x & 3x & 10x \\ 3x & 10x & 34x \end{pmatrix} \geq 0 \Longleftrightarrow 0 \leq x \leq 2\,. \tag{3.3}$$

Thus, by (3.2) and (3.3), we have that $W_{\alpha(x)}$ is 2-hyponormal if and only if $4-\sqrt{6} \leq x \leq 2$, as desired.

(ii) By (2.5), we have that the Berger measure of $W_{\left(\sqrt{3},\sqrt{\frac{10}{3}},\sqrt{\frac{17}{5}}\right)^\wedge}$ is

$$\widehat{\mu} := \left(\frac{2-\sqrt{2}}{4}\right)\delta_{(2-\sqrt{2})} + \left(\frac{2+\sqrt{2}}{4}\right)\delta_{(2+\sqrt{2})}.$$

We first note that

$$\left\|\frac{1}{t}\right\|_{L^1(\widehat{\mu})} = \frac{1}{2}.$$

From Lemma 3.3, we observe that shift $\left(\sqrt{x}, \left(\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}\right)^\wedge\right)$ is subnormal if and only if $x \leq 2$. If $\widehat{\mu}_x$ is the corresponding Berger measure then it follows from Lemma 3.3 that

$$\begin{aligned} d\widehat{\mu}_x(t) &= \frac{x}{t}d\widehat{\mu}(t) + \left(1 - x\left\|\frac{1}{t}\right\|_{L^1(\widehat{\mu})}\right)d\delta_0(t) \\ &= \frac{x}{t}d\widehat{\mu}(t) + \left(1-\frac{x}{2}\right)d\delta_0(t)\,. \end{aligned}$$

Thus, if shift $\left(\sqrt{x}, \left(\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}\right)^\wedge\right)$ is back-step extendable, then one get $\widehat{\mu}_x(\{0\}) = 1 - \frac{x}{2} = 0$, and hence $x = 2$. In fact,

$$\text{shift}\left(\sqrt{x}, \left(\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}\right)^\wedge\right)$$

is subnormal and back-step extendable if and only if $x = 2$; in this case, the Berger measure $\widehat{\mu}_2$ is given by

$$\widehat{\mu}_2 = \frac{1}{2}\left(\delta_{(2-\sqrt{2})} + \delta_{(2+\sqrt{2})}\right)$$

because

$$1 - \alpha_0^2 \left\|\frac{1}{t}\right\|_{L^1(\mu_{\mathcal{M}})} = 0 \quad \text{and} \quad \frac{\alpha_0^2}{t} d\mu_{\mathcal{M}}(t) = \frac{1}{2}\left(d\delta_{(2-\sqrt{2})}(t) + d\delta_{(2+\sqrt{2})}(t)\right).$$

We next note that

$$\left\|\frac{1}{t}\right\|_{L^1(\widehat{\mu}_2)} = 1.$$

By Lemma 3.3 and (3.1) again, we can see that $W_{\alpha(2)}$ is subnormal with the Berger measure

$$\mu = \tfrac{1}{2}\delta_0 + \tfrac{1}{4(2-\sqrt{2})}\delta_{(2-\sqrt{2})} + \tfrac{1}{4(2+\sqrt{2})}\delta_{(2+\sqrt{2})}.$$

(iii) By Lemma 3.1, the Aluthge transform $\widetilde{W}_{\alpha(x)}$ of $W_{\alpha(x)}$ is

$$\widetilde{W}_{\alpha(x)} \equiv \text{shift}\left(\sqrt[4]{\frac{x}{2}}, \sqrt[4]{3x}, \sqrt[4]{10}, \sqrt[4]{\frac{34}{3}}, \sqrt[4]{\frac{58}{5}}, \sqrt[4]{\frac{198}{17}}, \sqrt[4]{\frac{338}{29}}, \sqrt[4]{\frac{1154}{99}}, \cdots\right).$$

By Lemma 3.2, we have that $\widetilde{W}_{\alpha(x)}$ is 2-hyponormal if and only if $H(2;n)\left(\widetilde{W}_{\alpha(x)}\right) \geq 0$ for all $n \geq 0$. Note that

$$H(2;4)\left(\widetilde{W}_{\alpha(x)}\right) \geq 0 \Longleftrightarrow N := \begin{pmatrix} 1 & \sqrt{\frac{58}{5}} & 6\sqrt{\frac{319}{85}} \\ \sqrt{\frac{58}{5}} & 6\sqrt{\frac{319}{85}} & 78\sqrt{\frac{22}{85}} \\ 6\sqrt{\frac{319}{85}} & 78\sqrt{\frac{22}{85}} & 52\sqrt{\frac{577}{85}} \end{pmatrix} \geq 0. \quad (3.4)$$

Since $\det N < 0$, we can see that $N \ngeq 0$. Thus, by Lemma 3.2 and (3.4), $\widetilde{W}_{\alpha(x)}$ is not 2-hyponormal. □

Remark 3.5. From Theorem 3.4, we can see that for $k \geq 2$, the Aluthge transforms of weighted shifts need not preserve the k-hyponormality.

For our next results, we recall:

Lemma 3.6. (cf. [Smu]) *Let* $M \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ *be a* 2×2 *operator matrix, where* A *and* C *are square matrices and* B *is a rectangular matrix. Then*

$$M \geq 0 \Longleftrightarrow \text{there exists } W \text{ such that } \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW. \end{cases}$$

The iterated Aluthge transforms (or Aluthge iterates) of an operator T are the operators $\widetilde{T}^{(m)}$ $(m \geq 0)$, defined by setting $\widetilde{T}^{(0)} = T$ and letting $\widetilde{T}^{(m+1)}$ be the Aluthge transform of $\widetilde{T}^{(m)}$. An interesting result for Aluthge iterates (see [Yam]) is the spectral radius of T is equal to the limit of the norm of $\widetilde{T}^{(m)}$ as $m \to \infty$. For weighted shifts, observe that the Aluthge iterate $\widetilde{W}_\alpha^{(m)}$ of a weighted shift W_α is also a weighted shift with weight sequence $\alpha^{(m)} \equiv \{\alpha_i^{(m)}\}_{i=0}^{\infty}$, where $\alpha_i^{(m)} = \left(\prod_{j=0}^{m} (\alpha_{i+j})^{{}_mC_j}\right)^{\frac{1}{2^m}}$ and ${}_mC_j = \frac{m!}{j!(m-j)!}$. Moreover, by a direct calculation we can see that the moments of $\widetilde{W}_\alpha^{(m)}$ are given by

$$\gamma_i^{(m)} = \left(\gamma_i \prod_{j=1}^{m} \left(\frac{\gamma_{i+j}}{\gamma_j}\right)^{{}_mC_j}\right)^{\frac{1}{2^m}}, \tag{3.5}$$

where the γ_i are the moments of W_α.

For weighted shifts, we have:

Theorem 3.7. (i) *For $m \geq 1$, if W_α is subnormal with Berger measure $\mu = \delta_p$ for some $p \geq 0$, then $\widetilde{W}_\alpha^{(m)}$ is subnormal with Berger measure $\widetilde{\mu}^{(m)} = \delta_p$.*

(ii) *If W_α is subnormal with Berger measure $\mu = a\delta_0 + (1-a)\,\delta_p$ for some $p > 0$, then $\widetilde{W}_\alpha^{(m)}$ is subnormal with Berger measure*

$$\widetilde{\mu}^{(m)} = \left(1 - (1-a)^{\frac{1}{2^m}}\right)\delta_0 + (1-a)^{\frac{1}{2^m}}\,\delta_p.$$

(iii) *If W_α is subnormal with Berger measure $\mu = a\delta_p + (1-a)\,\delta_q$ for some $p, q > 0$ $(p \neq q)$, then $\widetilde{W}_\alpha$ need not be subnormal.*

Proof. (i) Since $W_\alpha \equiv \text{shift}\left(\sqrt{p}, \sqrt{p}, \cdots\right)$, by Lemma 3.1, the Aluthge transform $\widetilde{W}_\alpha$ is W_α. Thus, $\widetilde{W}_\alpha$ is subnormal with Berger measure δ_p, as desired.

(ii) A direct calculation shows that $W_\alpha = \text{shift}\left(\sqrt{(1-a)\,p}, \sqrt{p}, \sqrt{p}, \cdots\right)$, so that its Aluthge transform $\widetilde{W}_\alpha$ is given by shift $\left(\sqrt[4]{(1-a)\,p^2}, \sqrt{p}, \sqrt{p}, \cdots\right)$, which is subnormal with Berger measure $\widetilde{\mu} = \left(1 - \sqrt{1-a}\right)\delta_0 + \left(\sqrt{1-a}\right)\delta_p$. Consider

$$\widetilde{\left(\widetilde{W}_\alpha\right)} = \widetilde{W}_\alpha^{(2)} \equiv \text{shift}\left(\sqrt[8]{(1-a)\,p^4}, \sqrt{p}, \sqrt{p}, \cdots\right).$$

We then have that $\widetilde{W}_\alpha^{(2)}$ is subnormal with Berger measure

$$\widetilde{\mu}^{(2)} = \left(1 - \sqrt[4]{1-a}\right)\delta_0 + \left(\sqrt[4]{1-a}\right)\delta_p.$$

Continuing in this way, for $m \geq 1$, we have $\widetilde{W}_\alpha^{(m)}$ is subnormal with Berger measure $\widetilde{\mu}^{(m)} = \left(1 - (1-a)^{\frac{1}{2^m}}\right)\delta_0 + (1-a)^{\frac{1}{2^m}}\,\delta_p$, as desired.

(iii) From the proof of Theorem 3.4, we see that $W_{\left(\sqrt{3},\sqrt{\frac{10}{3}},\sqrt{\frac{17}{5}}\right)^{\wedge}}$ is subnormal with Berger measure

$$\widehat{\mu} = \left(\frac{2-\sqrt{2}}{4}\right)\delta_{(2-\sqrt{2})} + \left(\frac{2+\sqrt{2}}{4}\right)\delta_{(2+\sqrt{2})}.$$

By Lemma 3.1, the Aluthge transform $\widetilde{W}_{\left(\sqrt{3},\sqrt{\frac{10}{3}},\sqrt{\frac{17}{5}}\right)^{\wedge}}$ of $W_{\left(\sqrt{3},\sqrt{\frac{10}{3}},\sqrt{\frac{17}{5}}\right)^{\wedge}}$ is

$$\widetilde{W}_{\left(\sqrt{3},\sqrt{\frac{10}{3}},\sqrt{\frac{17}{5}}\right)^{\wedge}} \equiv \text{shift}\left(\sqrt[4]{10}, \sqrt[4]{\frac{34}{3}}, \sqrt[4]{\frac{58}{5}}, \sqrt[4]{\frac{198}{17}}, \sqrt[4]{\frac{338}{29}}, \sqrt[4]{\frac{1154}{99}}, \cdots\right).$$

From (3.4) and the argument given in the proof of Theorem 3.4 (iii), note that

$$H(2;2)\left(\widetilde{W}_{\left(\sqrt{3},\sqrt{\frac{10}{3}},\sqrt{\frac{17}{5}}\right)^{\wedge}}\right) = N < 0$$

Thus, by Lemma 3.2, $\widetilde{W}_{\left(\sqrt{3},\sqrt{\frac{10}{3}},\sqrt{\frac{17}{5}}\right)^{\wedge}}$ is not 2-hyponormal, so that $\widetilde{W}_\alpha$ is not subnormal. □

Remark 3.8. In Theorem 3.4 (ii), we note that

$$\lim_{n\to\infty} \widetilde{W}_\alpha^{(m)} = \text{shift}\,(\sqrt{p}, \sqrt{p}, \cdots)$$

is subnormal with Berger measure $\lim_{m\to\infty} \widetilde{\mu}^{(m)} = \delta_p$.

We recall that for an $m \times n$ matrix A, the *Moore-Penrose inverse* of A is defined as the unique $n \times m$ matrix $A^\dagger$ satisfying the following four properties:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad \left(AA^\dagger\right)^* = AA^\dagger, \quad \text{and} \quad \left(A^\dagger A\right)^* = A^\dagger A.$$

The following result is a variant of Lemma 3.6.

Lemma 3.9. ([Smu]) *Let* $P \equiv \begin{pmatrix} D & E \\ E^* & F \end{pmatrix}$ *be a finite matrix. Then* $P \geq 0$ *if and only if the following conditions hold:*

(i) $D \geq 0$;
(ii) $\operatorname{ran} E \subseteq \operatorname{ran} D$; *and*
(iii) $F \geq E^* D^\dagger E$, *where* $D^\dagger$ *is the Moore-Penrose inverse of* D.

Proof. Since P is a finite matrix, it has closed range and hence D has the Moore-Penrose inverse $D^\dagger$. The desired result now follows from ([CuLe, Lemma 1.2]). □

We then have:

Theorem 3.10. *Let W_α be the contractive subnormal weighted shift with Berger measure $\mu = a\delta_p + (1-a)\,\delta_1$ $(0 < a < 1$ and $0 \le p < 1)$. Then the Aluthge transform $\widetilde{W}_\alpha$ of W_α is subnormal if and only if $p = 0$.*

Proof. $(\Leftarrow)$ It is clear from Theorem 3.7 (ii).

$(\Rightarrow)$ Suppose that $\widetilde{W}_\alpha$ is subnormal. Then by Lemma 3.2, for all $k \ge 1$ and $n \ge 0$, the Hankel matrix

$$H(k;n)\left(\widetilde{W}_\alpha\right) = (\gamma^{(1)}_{n+i+j-2})_{i,j=1}^{k+1} \ge 0.$$

Observe that $\gamma_i = ap^i + 1 - a$. Hence, by (3.5), we have

$$\gamma_i^{(1)} = \sqrt{\frac{\gamma_i\gamma_{i+1}}{\gamma_1}} = \sqrt{\frac{(ap^i+1-a)(ap^{i+1}+1-a)}{ap+1-a}}.$$

There are two cases to consider.

Case 1 $(p=0)$: This case is clear.

Case 2 $(0<p<1)$: Note that

$$H(3;2)\left(\widetilde{W}_\alpha\right) \ge 0$$

$$\Longleftrightarrow \begin{pmatrix} \sqrt{\gamma_2\gamma_3} & \sqrt{\gamma_3\gamma_4} & \sqrt{\gamma_4\gamma_5} & \sqrt{\gamma_5\gamma_6} \\ \sqrt{\gamma_3\gamma_4} & \sqrt{\gamma_4\gamma_5} & \sqrt{\gamma_5\gamma_6} & \sqrt{\gamma_6\gamma_7} \\ \sqrt{\gamma_4\gamma_5} & \sqrt{\gamma_5\gamma_6} & \sqrt{\gamma_6\gamma_7} & \sqrt{\gamma_7\gamma_8} \\ \sqrt{\gamma_5\gamma_6} & \sqrt{\gamma_6\gamma_7} & \sqrt{\gamma_7\gamma_8} & \sqrt{\gamma_8\gamma_9} \end{pmatrix} \div \begin{pmatrix} D\left(\widetilde{W}_\alpha\right) & E\left(\widetilde{W}_\alpha\right) \\ \left(E\left(\widetilde{W}_\alpha\right)\right)^* & F\left(\widetilde{W}_\alpha\right) \end{pmatrix}$$

$$=: P\left(H(3;2)\left(\widetilde{W}_\alpha\right)\right) \ge 0,$$

where

$$D\left(\widetilde{W}_\alpha\right) := \begin{pmatrix} \sqrt{\gamma_2\gamma_3} & \sqrt{\gamma_3\gamma_4} \\ \sqrt{\gamma_3\gamma_4} & \sqrt{\gamma_4\gamma_5} \end{pmatrix},\ E\left(\widetilde{W}_\alpha\right) := \begin{pmatrix} \sqrt{\gamma_4\gamma_5} & \sqrt{\gamma_5\gamma_6} \\ \sqrt{\gamma_5\gamma_6} & \sqrt{\gamma_6\gamma_7} \end{pmatrix} \text{ and}$$

$$F\left(\widetilde{W}_\alpha\right) := \begin{pmatrix} \sqrt{\gamma_6\gamma_7} & \sqrt{\gamma_7\gamma_8} \\ \sqrt{\gamma_7\gamma_8} & \sqrt{\gamma_8\gamma_9} \end{pmatrix}.$$

Observe that $\sqrt{\gamma_2\gamma_5} > \sqrt{\gamma_3\gamma_4}$. Thus a direct calculation shows that the Moore-Penrose inverse of $D\left(\widetilde{W}_\alpha\right)$ is

$$\begin{pmatrix} \frac{\sqrt{\gamma_4\gamma_5}}{\sqrt{\gamma_2\gamma_3\gamma_4\gamma_5}-\gamma_3\gamma_4} & -\frac{\sqrt{\gamma_3\gamma_4}}{\sqrt{\gamma_2\gamma_3\gamma_4\gamma_5}-\gamma_3\gamma_4} \\ -\frac{\sqrt{\gamma_3\gamma_4}}{\sqrt{\gamma_2\gamma_3\gamma_4\gamma_5}-\gamma_3\gamma_4} & \frac{\sqrt{\gamma_2\gamma_3}}{\sqrt{\gamma_2\gamma_3\gamma_4\gamma_5}-\gamma_3\gamma_4} \end{pmatrix}.$$

By Lemma 3.9 (iii), we can see that

$$H(3;2)\left(\widetilde{W}_\alpha\right) \ge 0 \Longleftrightarrow P\left(H(3;2)\left(\widetilde{W}_\alpha\right)\right) \ge 0$$

$$\Longleftrightarrow F\left(\widetilde{W}_\alpha\right) - \left(E\left(\widetilde{W}_\alpha\right)\right)^* \left(D\left(\widetilde{W}_\alpha\right)\right)^\dagger E\left(\widetilde{W}_\alpha\right) =: R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \ge 0.$$

Note that

$$R \geq 0 \text{ if and only if } r_{11} \geq 0,\ r_{22} \geq 0 \text{ and } \det R \geq 0.$$

Using the software tool *Mathematica* [Wol], we can observe that

$$\det R < 0 \text{ when } 0 < a < 1 \text{ and } 0 < p < 1.$$

Thus $\widetilde{W_\alpha}$ is not 3-hyponormal, that is, $\widetilde{W_\alpha}$ is not subnormal, which is contradict to the assumption. Therefore, by **Case 1** and **Case 2**, the Aluthge transform $\widetilde{W_\alpha}$ is subnormal only if $p = 0$ and now our proof is complete. □

Corollary 3.11. *Let* $W_\alpha \equiv \text{shift}\,(\alpha_0, \alpha_1, \cdots)$ *be a subnormal with Berger measure*

$$\mu = a\delta_p + (1-a)\,\delta_q \ \ (0 < a < 1,\ p < q)\,.$$

Then the Aluthge transform $\widetilde{W_\alpha} \equiv \text{shift}\left(\alpha_0^{(1)}, \alpha_1^{(1)}, \cdots\right)$ *of* W_α *is subnormal if and only if* $p = 0$.

Proof. (⇐) It follows from Theorem 3.7 (ii).

(⇒) Suppose that $\widetilde{W_\alpha}$ is subnormal. A direct calculation shows that for $n \geq 0$,

$$\alpha_n = \sqrt{\frac{ap^{n+1} + (1-a)\,q^n}{ap^{n+1} + (1-a)\,q^n}} \text{ and } \alpha_n^{(1)} = \sqrt{\frac{ap^{n+1} + (1-a)\,q^n}{q\,(ap^{n+1} + (1-a)\,q^n)}}.$$

Thus $\left(\frac{1}{\sqrt{q}}\right)\widetilde{W_\alpha}$ is also subnormal with Berger measure $a\delta_{\frac{p}{q}} + (1-a)\,\delta_1$. Therefore, by Theorem 3.10, we have $p = 0$, as desired. □

Remark 3.12. Looking at Theorem 3.10 and Corollary 3.11, it seems natural to conjecture that a similar method used in the proof of Theorem 3.10 should work for a subnormal weighted shift W_α with finite atomic Berger measure. However, it is highly nontrivial to find the necessary and sufficient conditions for the k-hyponormality (all $k \geq 1$) (resp. subnormality) of $\widetilde{W_\alpha}$ when k is big enough. For example, $D\left(\widetilde{W_\alpha}\right)$ shown in the proof of Theorem 3.10 is just a Hankel matrix without a common well known pattern, so it becomes unwieldy to check its determinant or invertibility.

In view of Theorem 3.4 (iii), 3.7, Corollary 3.11 and Remark 3.12, it is natural to pose:

Conjecture 3.13. Let W_α be a nonzero subnormal weighted shift with finite atomic Berger measure μ. Then the Aluthge transform $\widetilde{W_\alpha}$ of W_α is subnormal if and only if $\mu = a\delta_0 + (1-a)\,\delta_p$ $(0 \leq a < 1;\ p > 0)$.

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