A CRITERION ON THE HYPONORMALITY OF TOEPLITZ OPERATORS WITH POLYNOMIAL SYMBOLS VIA SCHUR NUMBERS

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Abstract. In this paper we give a complete characterization on the hyponormality of the Toeplitz operators T_φ with trigonometric polynomial symbols $\varphi = \overline{g} + h$ $(g, h \in H^2)$ when g divides h. This is accomplished by using a criterion on the contractivity of the inverse of a lower triangular (finite) Toeplitz matrix via Schur numbers of an induced analytic polynomial.

1. Introduction

Let $L^2 \equiv L^2(\mathbb{T})$ be the set of all square-integrable measurable functions on the unit circle $\mathbb{T} \equiv \partial\,\mathbb{D}$ in the complex plane and $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$, that is, H^∞ is the set of bounded analytic functions on $\mathbb{D}$. Given $\varphi \in L^\infty \equiv L^\infty(\mathbb{T})$, the Toeplitz operator T_φ with symbol φ is defined by

$$T_\varphi g := P(\varphi g) \quad (g \in H^2),$$

where P denotes the orthogonal projection that maps from L^2 onto H^2. Normal Toeplitz operators were characterized by a property of their symbols in the early 1960's by A. Brown and P.R. Halmos [BH] and the hyponormality of Toeplitz operators was completely solved in terms of their symbols by C. Cowen [Co2] in 1988.

Cowen's Theorem. ([Co2], [NT]) *For $\varphi \in L^\infty$, write*

$$\mathcal{E}(\varphi) := \Big\{ k \in H^\infty : \ ||k||_\infty \le 1 \ \textit{and} \ \varphi - k\overline{\varphi} \in H^\infty \Big\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

The elegant and useful theorem of C. Cowen is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution to a certain functional equation involving the operator's symbol. Cowen's theorem was extensively used in the works [CCL], [Co1], [Co2], [CL1], [CL2], [FL1], [FL2], [Gu], [GS], [HKL1], [HKL2], [HL1], [HL2], [HL3], [Le], [NT], [Zhu], and etc to study the hyponormality of Toeplitz operators. When we study hyponormality of the Toeplitz operator T_φ with symbol φ we may, without loss of generality, assume that $\varphi(0) = 0$ because the hyponormality of an operator is invariant under translation by scalars.

If φ is a trigonometric polynomial, say $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero, then the nonnegative integer N and m denote the analytic and the co-analytic degrees of φ. For arbitrary trigonometric polynomials, K. Zhu [Zhu] has applied Cowen's criterion and used a method based on the classical interpolation theorems of Schur to obtain an abstract characterization of those trigonometric polynomial symbols corresponding to hyponormal Toeplitz operators. In [FL1], the hyponormality of T_φ was completely characterized in terms of the Fourier coefficients of φ when

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2010 *Mathematics Subject Classification.* 15B05, 47B35, 47B20, 15A60, 47A57

Key words and phrases. Carathéodory interpolation problem, Schur numbers, Toeplitz matrices, Toeplitz operators, trigonometric polynomials, hyponormal.

This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST)(2011-0001250).

the outer coefficients have the same modulus, i.e., $|a_{-m}| = |a_N|$. However, with polynomials of higher degrees with $|a_{-m}| < |a_N|$, the analogous explicit criterion would be too complicated to be of much value, even though it was in principle solved via the solution of the interpolation problems. The authors of [HKL2] considered the hyponormality of T_φ with polynomial symbol $\varphi = \overline{g} + h$ ($g, h \in H^2$) when g divides h and gave an explicit criterion in terms of the Fourier coefficients of the quotient $\psi := \frac{h}{g}$ when φ enjoys a certain extremal condition: its advantage is that this criterion depends only on ψ regardless the degree of g. But if φ does not satisfy the extremal condition, the criterion cannot be applied. The purpose of this paper is to get a complete criterion on the hyponormality of T_φ with polynomial symbol $\varphi \equiv \overline{g} + h$ ($g, h \in H^2$) when g divides h (Corollary 2.4). Here, our approach we take is to use a criterion on the contractivity of the inverse of a lower triangular (finite) Toeplitz matrix via Schur numbers of an induced analytic polynomial - the quotient of its division (Theorem 2.3).

2. The main result

Let $\varphi \in L^\infty$ be a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$. If a function $k \in H^\infty$ satisfies $\varphi - k\overline{\varphi} \in H^\infty$, then k necessarily satisfies

$$k \sum_{n=1}^{N} \overline{a_n} z^{-n} - \sum_{n=1}^{m} a_{-n} z^{-n} \in H^\infty. \tag{2.1}$$

If we write $k(z) = \sum_{n=0}^{\infty} c_n z^n$, then from (2.1), $c_0, c_1, \ldots, c_{N-1}$ are determined uniquely from the coefficients of φ by the recurrence relation:

$$\begin{cases} c_0 = c_1 = \ldots = c_{N-m-1} = 0 \\ c_{N-m}\overline{a_N} = a_{-m} \\ c_n\overline{a_N} + \sum_{j=N-m}^{n-1} c_j \overline{a_{N-n+j}} \quad \text{for } n = N-m+1, \ldots, N-1. \end{cases} \tag{2.2}$$

Thus (2.2) can be written in the following matrix form:

$$c_0 = c_1 = \ldots = c_{N-m-1} = 0;$$

$$\begin{bmatrix} c_{N-m} \\ c_{N-m+1} \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} \overline{a_{N-m+1}} & \overline{a_{N-m+2}} & \ldots & \overline{a_{N-1}} & \overline{a_N} \\ \overline{a_{N-m+2}} & \overline{a_{N-m+3}} & \ldots & \overline{a_N} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{a_N} & 0 & \ldots & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-m} \end{bmatrix}.$$

Thus $k_p(z) := \sum_{j=N-m}^{N-1} c_j z^j$ is a unique (analytic) polynomial of degree less than N satisfying $\varphi - k\overline{\varphi} \in H^\infty$. Thus the problem of finding a solution in the set $\mathcal{E}(\varphi)$ is to find an analytic function k in the closed unit ball of H^∞ interpolating k_p. This is exactly the Carathéodory interpolation problem (cf. [FF, Theorem VIII.1.3]). Thus by the Cowen's theorem, T_φ is hyponormal if and only if the Toeplitz matrix

$$C = \begin{bmatrix} c_{N-m} & 0 & \ldots & \ldots & 0 \\ c_{N-m+1} & c_{N-m} & \ddots & & \vdots \\ \vdots & c_{N-m} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ c_{N-1} & \ldots & \ldots & c_{N-m+1} & c_{N-m} \end{bmatrix} \tag{2.3}$$

is a contraction (see [FF], [CCL], [FL1], [Zhu]). In this paper we consider the hyponormality of T_φ with trigonometric polynomial symbols $\varphi = \overline{g} + h$ ($g, h \in H^\infty$) satisfying that g divides h. The condition "g divides h" seems to be rigid. However the following lemma shows that if $\varphi \equiv \overline{g} + h$

is a trigonometric polynomial then we may, without loss of generality, assume that the co-analytic part g of φ divides the analytic part h of φ whenever we consider the hyponormality of T_φ.

Lemma 2.1. ([HKL2, Lemma 2.4]) *Let $\varphi \equiv \overline{g} + h \in L^\infty$, where g and h are analytic polynomials of degrees m and N ($m \le N$), respectively. If we let*

$$\widetilde{h} := z^m T_{\overline{z}^{N-m}} h - d,$$

where d is the remainder in the division of $z^m T_{\overline{z}^{N-m}} h$ by g, put $\widetilde{\varphi} := \overline{g} + \widetilde{h}$. We then have:

(i) *T_φ is hyponormal if and only if $T_{\widetilde{\varphi}}$ is;*
(ii) *g divides $\widetilde{h}$.*

We then have:

Theorem 2.2. *Let $\varphi \equiv \overline{g} + h$, where g and h are analytic polynomials of degrees m and N ($m \le N$), respectively. Suppose g divides f and*

$$\frac{h}{g} := \sum_{j=0}^{r} b_j z^j \quad (r := N - m).$$

Let B be a finite Toeplitz matrix of the form

$$B \equiv \begin{bmatrix} b_r & 0 & \cdots & 0 \\ b_{r-1} & b_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_{r-m+1} & \cdots & b_{r-1} & b_r \end{bmatrix},$$

where $b_j = 0$ if $j < 0$ for notational convenience. Then T_φ is hyponormal if and only if each eigenvalue of B^*B is greater than or equal to 1.

Proof. Write $g(z) := \sum_{j=1}^m a_j z^j$. If $k \in H^\infty$ satisfies $\varphi - k\overline{\varphi} \in H^\infty$, then k satisfies

$$\sum_{j=1}^{m} \overline{a_j} z^{-j} - k \left(\sum_{j=0}^{r} \overline{b_j} z^{-j} \right) \left(\sum_{j=1}^{m} \overline{a_j} z^{-j} \right) \in H^2. \tag{2.4}$$

If we write $k(z) := \sum_{j=0}^\infty c_j z^j$,then

$$\left(1 - \Big(\sum_{j=0}^{\infty} c_j z^j \Big) \Big(\sum_{j=0}^{r} \overline{b_j} z^{-j} \Big) \right) \left(\sum_{j=1}^{m} \overline{a_j} z^{-j} \right) \in H^2. \tag{2.5}$$

From (2.5) we can see that $c_0 \overline{b_r}\,\overline{a_m} = 0$. It thus follows that

$$c_0 = 0. \tag{2.6}$$

By repeating the argument we can show that

$$c_0 = c_1 = \ldots = c_{r-1} = 0.$$

Thus (2.5) can be written as

$$\left(1 - \Big(\sum_{j=r}^{\infty} c_j z^j \Big) \Big(\sum_{j=0}^{r} \overline{b_j} z^{-j} \Big) \right) \left(\sum_{j=1}^{m} \overline{a_j} z^{-j} \right) \in H^2. \tag{2.7}$$

From (2.7) we can again see that $(1 - c_r \overline{b_r})\overline{a_m} = 0$, which implies

$$c_r \overline{b_r} = 1. \tag{2.8}$$

In turn we have $(c_{r+1}\overline{b_r} + c_r\overline{b_{r-1}})\overline{a_m} = 0$, which implies

$$c_{r+1}\overline{b_r} + c_r\overline{b_{r-1}} = 0.$$

If we continue to employ the telescoping method for (2.7), then we get the following equations:

$$\begin{cases} c_r\overline{b_r} = 1 \\ c_{r+1}\overline{b_r} + c_r\overline{b_{r-1}} = 0 \\ c_{r+2}\overline{b_r} + c_{r+1}\overline{b_{r-1}} + c_r\overline{b_{r-2}} = 0 \\ \quad\cdots \\ c_{r+m-1}\overline{b_r} + c_{r+m-2}\overline{b_{r-1}} + \cdots + c_r\overline{b_{r-m+1}} = 0, \end{cases} \tag{2.9}$$

(where $b_j = 0$ if $j < 0$ for notational convenience) or in matrix form

$$\begin{bmatrix} c_r & c_{r+1} & \cdots & \cdots & c_{r+m-1} \\ 0 & c_r & c_{r+1} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & c_{r+1} \\ 0 & \cdots & \cdots & 0 & c_r \end{bmatrix} \begin{bmatrix} \overline{b_r} & \overline{b_{r-1}} & \cdots & \cdots & \overline{b_{r-m+1}} \\ 0 & \overline{b_r} & \overline{b_{r-1}} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \overline{b_{r-1}} \\ 0 & \cdots & \cdots & 0 & \overline{b_r} \end{bmatrix} = I\,. \tag{2.10}$$

If we let

$$C := \begin{bmatrix} c_r & c_{r+1} & \cdots & \cdots & c_{r+m-1} \\ 0 & c_r & c_{r+1} & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & c_{r+1} \\ 0 & \cdots & \cdots & 0 & c_r \end{bmatrix},$$

then by the interpolation argument of (2.3), T_φ is hyponormal if and only if and C is a contraction. But since by (2.10), $CB^* = I$, it follows that

$$\sigma(B^*B) = \sigma\big((C^*C)^{-1}\big) = \Big\{\frac{1}{\lambda} :\ \lambda \in \sigma(C^*C)\Big\},$$

which implies that T_φ is hyponormal if and only if and $\sigma(B^*B) \subset [1, \infty)$, where $\sigma(\cdot)$ denotes the set of eigenvalues. This gives the result. □

Theorem 2.2 essentially asserts that under the assumption of the theorem, T_φ is hyponormal if and only if

$$B \equiv \begin{bmatrix} b_r & 0 & \cdots & 0 \\ b_{r-1} & b_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_{r-m+1} & \cdots & b_{r-1} & b_r \end{bmatrix}$$

is an inverse of a contractive matrix C^*. Theorem 2.2 also asserts that B is an inverse of a contractive matrix if and only if all eigenvalues of B^*B are greater than or equal to 1. If the size of the matrix B grows bigger, the calculation of the eigenvalues of B^*B might be another heavy task. On the other hand, we may determine whether B is an inverse of a contractive matrix by using the Schur numbers of the analytic polynomial $f(z) := b_r + b_{r-1}z + \cdots + b_{r-m+1}z^{m-1}$ $(b_j \in \mathbb{C})$.

For an analytic function $f_0(z) \equiv f(z)$ in the open unit disk $\mathbb{D}$, define a sequence $\{f_n\}$ by

$$f_{n+1}(z) = \frac{f_n(z) - f_n(0)}{z\big(1 - \overline{f_n(0)}f_n(z)\big)}, \quad |z| < 1,\ n = 0, 1, 2, \ldots.$$

We then define the *n-th Schur number* $s_n(f)$ of f by

$$s_n(f) := f_n(0).$$

For example, if $f(z) = \sum_{n=0}^{\infty} c_n z^n$ then

$$s_0(f) = c_0, \quad s_1(f) = \frac{c_1}{1-|c_0|^2}, \quad s_2(f) = \frac{c_2(1-+c_0+^2)+\overline{c_0}c_1^2}{(1-|c_0|^2)^2-|c_1|^2}.$$

We note that the n-th Schur number of f depends only on the first n coefficients of f. We would also remark that even though the function $f_n(z)$ is analytic in a neighborhood of 0, the n-th Schur number $s_n(f)$ is well determined whenever none of the numbers $|s_0(f)|, |s_1(f)|, \ldots, |s_{n-1}(f)|$ is equal to 1.

The *Carathéodory interpolation problem* says that given an analytic polynomial $p(z) := c_0 + c_1 z + \cdots + c_n z^n$, find an analytic function k on the open unit disk $\mathbb{D}$ such that

(i) $\widehat{k}(j) = c_j$ for $j = 0, 1, \ldots, n$ ($\widehat{k}(j)$ denotes the j-th Fourier coefficient of k)
(ii) $||k||_\infty \le 1$.

I. Schur gave a solution to the Carathéodory interpolation problem:

Schur's Theorem. ([Sch]) *The above Carathéodory interpolation problem is solvable if and only if $f(z) := c_0 + c_1 z + \cdots + c_n z^n$ satisfies one of the following two conditions:*

(i) $|s_j(f)| < 1$ *for* $0 \le j \le n$;
(ii) *There exists* n_0 $(0 \le n_0 \le n)$ *such that* $|s_j(f)| < 1$ *for* $0 \le j \le n_0 - 1$, $|s_{n_0}(f)| = 1$ *and the coefficients of* f *coincide with the first* $(n+1)$ *Fourier coefficients of*

$$\widetilde{f} := w_{s_0(f)}(zw_{s_1(f)}(\cdots(zw_{s_{n_0-1}(f)}(zs_{n_0}(f)))\cdots)),$$

where $w_s(z) := \frac{z+s}{1+\bar{s}z}$ $(|s| < 1)$.

We then have:

Theorem 2.3. *Let* $b_j \in \mathbb{C}$ $(0 \le j \le r)$ *and* $b_r \ne 0$. *If* B *is a lower triangular Toeplitz matrix of the form*

$$B \equiv \begin{bmatrix} b_r & 0 & \cdots & 0 \\ b_{r-1} & b_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_0 & \cdots & b_{r-1} & b_r \end{bmatrix},$$

we put

$$f(z) = b_r + b_{r-1}z + \cdots + b_0 z^r.$$

Then B *is an inverse of a contractive matrix if and only if* f *satisfies one of the following conditions:*

(i) $|s_0(f)| > 1$ *and* $|s_j(f)| < 1$ *for* $1 \le j \le r$;
(ii) *There exists* n_0 $(0 \le n_0 \le r)$ *such that* $|s_0(f)| > 1$, $|s_j(f)| < 1$ *for* $1 \le j \le n_0 - 1$, $|s_{n_0}(f)| = 1$ *and the coefficients of* f *coincide with the first* $(r+1)$ *Fourier coefficients of*

$$\widetilde{f} := w_{s_0(f)}(zw_{s_1(f)}(\cdots(zw_{s_{n_0-1}(f)}(zs_{n_0}(f)))\cdots)).$$

Proof. As in the proof of Theorem 2.2, a straightforward calculation shows that the inverse of B is of the form

$$\begin{bmatrix} k_0 & 0 & \cdots & 0 \\ k_1 & k_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ k_r & \cdots & k_1 & k_0 \end{bmatrix},$$

where the k_i satisfy the following equation:

$$\begin{cases} k_0 b_r = 1 \\ k_0 b_{r-1} + k_1 b_r = 0 \\ \qquad \cdots \\ k_0 b_0 + k_1 b_1 + \cdots + k_r b_r = 0. \end{cases} \tag{2.11}$$

Since $b_r \neq 0$, we have $k_0 \neq 0$. Let $q(z) := \frac{1}{f(z)}$. Since $f(0) = b_r \neq 0$, there exists a neighborhood $\mathfrak{N}$ of 0 such that f has no zero in $\mathfrak{N}$. Thus $q(z)$ represents an analytic function on $\mathfrak{N}$. Further, (2.11) implies that the power series expansion of q in $\mathfrak{N}$ can be written as

$$q(z) = k_0 + k_1 z + \cdots + k_r z^r + o(z^r).$$

If we define a polynomial

$$q^{(r)}(z) := k_0 + k_1 z + \cdots + k_r z^r,$$

then we can see that

$$s_j(q) = s_j(q^{(r)}) \quad \text{for all } j = 0, \cdots, r.$$

Then the Schur's solution to the Carathéodory interpolation problem implies that

$$\widehat{K} := \begin{bmatrix} k_0 & 0 & \cdots & 0 \\ k_1 & k_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ k_r & \cdots & k_1 & k_0 \end{bmatrix} \text{ is a contraction} \tag{2.12}$$

if and only if $q^{(r)}$ satisfies one of two conditions of the Schur's Theorem. Observe that

$$|s_0(q^{(r)})| = |s_0(q)| = |k_0| = \frac{1}{|b_r|} = \frac{1}{|s_0(f)|}. \tag{2.13}$$

Let $f_0 := f$, $q_0 := q$,

$$f_{n+1}(z) := \frac{f_n(z) - f_n(0)}{z\big(1 - \overline{f_n(0)} f_n(z)\big)} \quad \text{and} \quad q_{n+1}(z) := \frac{q_n(z) - q_n(0)}{z\big(1 - \overline{q_n(0)} q_n(z)\big)} \quad (n \geq 0).$$

Then we have

$$\begin{aligned} f_1(z) &= \frac{f(z) - f(0)}{z\big(1 - \overline{f(0)} f(z)\big)} = \frac{\frac{1}{q(z)} - \frac{1}{q(0)}}{z\left(1 - \frac{1}{\overline{q(0)}}\frac{1}{q(z)}\right)} \\ &= \frac{\overline{q(0)}}{q(0)} \frac{q(z) - q(0)}{z\big(1 - \overline{q(0)} q(z)\big)} \\ &= e^{i\theta} q_1(z) \qquad (e^{i\theta} := \overline{q(0)} q(0)^{-1}). \end{aligned}$$

In turn,

$$\begin{aligned} f_2(z) &= \frac{f_1(z) - f_1(0)}{z\big(1 - \overline{f_1(0)} f_1(z)\big)} = \frac{e^{i\theta} q_1(z) - e^{i\theta} q(0)}{z\big(1 - \overline{e^{i\theta} q_1(0)} e^{i\theta} q_1(z)\big)} \\ &= e^{i\theta} \frac{q_1(z) - q_1(0)}{z\big(1 - \overline{q_1(0)} q_1(z)\big)} \\ &= e^{i\theta} q_2(z). \end{aligned}$$

Inductively we can see that $f_j(z) = e^{i\theta} q_j(z) = \alpha q_j(z)$ for $j = 1, 2, \ldots$, where $\alpha := \frac{b_r}{\overline{b_r}}$. Therefore we have

$$s_j(f) = \alpha s_j(q) = \alpha s_j(q^{(r)}), \quad \text{and hence } \ |s_j(f)| = |s_j(q^{(r)})| \quad \text{for } j = 1, 2, \ldots. \tag{2.14}$$

We now have

$$|s_j(q^{(r)}| < 1 \text{ for } 0 \le j \le r \iff |s_0(f)| > 1 \text{ and } |s_j(f)| < 1 \text{ for } 1 \le j \le r,$$

which says that the first condition of the Schur's solution to the Carathéodory interpolation problem for $q^{(r)}$ is equivalent to the condition (i) of the theorem. Now consider the case when the condition (ii) of the theorem holds. Since $|s_{n_0}(f)| = 1$ implies $|s_{n_0}(q^{(r)})| = 1$, it follows from the Schur's Theorem that the Carathéodory interpolation problem for $q^{(r)}$ has a solution if and only if the coefficients of $q^{(r)}$ coincide with the first $(r+1)$ coefficients of the function

$$\widetilde{g} := w_{s_0(q^{(r)})}\big(z w_{s_1(q^{(r)})}(\cdots (z w_{s_{n_0-1}(q^{(r)})}(z s_{n_0}(q^{(r)}))) \cdots)\big).$$

Since the coefficients of f and $q^{(r)}$ are related by (2.11), we can see that the m coefficients of $q^{(r)}$ coincide with the first m coefficients of $\widetilde{g}$ if and only if the m coefficients of f coincide with the first m coefficients of $\frac{1}{\widetilde{g}}$. Using the relations $s_0(f) = \frac{1}{s_0(q^{(r)})}$ and $s_j(f) = \alpha s_j(q^{(r)})$ (where $\alpha := \frac{b_r}{\overline{b_r}}$), a straightforward calculation shows that

$$\widetilde{g} = \frac{1}{w_{s_0(f)}\big(z w_{s_1(f)}(\cdots (z w_{s_{n_0-1}(f)}(z s_{n_0}(f))) \cdots)\big)},$$

that is,

$$\widetilde{f} \equiv \frac{1}{\widetilde{g}} = w_{s_0(f)}\big(z w_{s_1(f)}(\cdots (z w_{s_{n_0-1}(f)}(z s_{n_0}(f))) \cdots)\big).$$

Therefore we can conclude that the second condition of the Schur's solution to the Carathéodory interpolation problem for $q^{(r)}$ is equivalent to the condition (ii) of the theorem. This completes the proof. □

The following corollary provides a complete criterion on the hyponormality of T_φ with polynomial symbol $\varphi = \overline{g} + h$ $(g, h \in H^2)$ when g divides h.

Corollary 2.4. Let $\varphi \equiv \overline{g} + h$ $(g, h \in H^2)$ be a trigonometric polynomial such that g divides h. If $\frac{h}{g} := b_r z^r + \cdots + b_1 z + b_0$, put

$$f(z) := b_r + b_{r-1} z + \cdots + b_{r-m+1} z^{m-1} \quad (m := \text{the degree of } g),$$

where $b_j = 0$ if $j < 0$ for notational convenience. If $s_j(f)$ denotes the j-th Schur number of f, then T_φ is hyponormal if and only if f satisfies one of the following conditions:

(i) $|s_0(f)| > 1$ and $|s_j(f)| < 1$ for $1 \le j \le m-1$;
(ii) There exists n_0 $(0 \le n_0 \le m-1)$ such that $|s_0(f)| > 1$, $|s_j(f)| < 1$ for $1 \le j \le n_0 - 1$, $|s_{n_0}(f)| = 1$ and the coefficients of f coincide with the first m Fourier coefficients of

$$\widetilde{f} := w_{s_0(f)}(z w_{s_1(f)}(\cdots (z w_{s_{n_0-1}(f)}(z s_{n_0}(f))) \cdots)).$$

Proof. Remembering that the matrix B in Theorem 2.2 is an inverse of a contractive matrix if and only if all eigenvalues of B^*B are greater than or equal to 1, this follows at once from Theorems 2.2 and 2.3. □

Using Corollary 2.4, we can give a short proof of [HKL2, Lemma 2.6 and Theorem 2.7].

Corollary 2.5. *Let* $\varphi \equiv \overline{g} + g\psi = \sum_{j=1}^{m} \overline{a_j} z^{-j} + \left(\sum_{j=1}^{m} a_j z^j\right)\left(\sum_{j=0}^{n} b_j z^j\right)$. *If* T_φ *is hyponormal then*

$$\left|\sum_{\xi\in\mathcal{Z}(\psi)} \xi\right| \le |b_n| - \frac{1}{|b_n|}, \tag{2.15}$$

where $\mathcal{Z}(\psi)$ *denotes the set of zeros of* ψ. *Moreover if the equality of* (2.15) *holds, then* T_φ *is hyponormal if and only if* $n \ge m-1$ *and*

$$b_{j+1} = b_j \left(\frac{|b_n|^2-1}{b_{n-1}\overline{b_n}}\right) \quad \text{for } j = n-m+1, \dots, n-2. \tag{2.16}$$

Proof. Let $f(z) := b_n + b_{n-1}z + \cdots + b_{n-m+1}z^{m-1}$ ($b_j = 0$ if $j < 0$) and suppose that T_φ is hyponormal. If $|s_0(f)| = |b_n| = 1$, then by Corollary 2.4 we have $f(z) = b_n$, i.e., $b_j = 0$ for $n-m-1 \le j \le n-1$. Observe that $\frac{b_{n-1}}{b_n}$ is equal to $-\sum_{\xi\in\mathcal{Z}(\psi)} \xi$. Therefore if $|s_0(f)| = |b_n| = 1$, then (2.15) and (2.16) are automatically satisfied. Assume instead $|b_n| \ne 1$, so that $s_1(f)$ is well defined. If T_φ is hyponormal then by Corollary 2.4,

$$|s_1(f)| = \left|\frac{b_{n-1}}{1-|b_n|^2}\right| \le 1, \quad \text{so that} \quad \left|\frac{b_{n-1}}{b_n}\right| \le \left|\frac{1}{|b_n|} - |b_n|\right| = |b_n| - \frac{1}{|b_n|}.$$

Again, $\frac{b_{n-1}}{b_n} = -\sum_{\xi\in\mathcal{Z}(\psi)} \xi$ gives (2.15).

For the second assertion we assume that the equality holds in (2.15). The preceding calculation shows that $|s_1(f)| = 1$. By Corollary 2.4, T_φ is hyponormal if and only if the second condition of Corollary 2.4 is satisfied by f, i.e., $|s_0(f)| = |b_n| > 1$ and the m coefficients of f coincide with the first m coefficients of

$$\begin{aligned} w_{s_0(f)}(zw_{s_1(f)}) &= \frac{s_0 + zs_1}{1+\overline{s_0}zs_1} \\ &= (s_0 + zs_1)\sum_{j=0}^{\infty}(-1)^j(\overline{s_0}s_1 z)^j \\ &= s_0 + (s_1 - |s_0|^2 s_1)z - (s_1 - |s_0|^2 s_1)(\overline{s_0}s_1)z^2 + (s_1 - |s_0|^2 s_1)(\overline{s_0}s_1)^2 z^3 - \cdots . \end{aligned}$$

Since (2.15) implies $|b_n| \ge 1$, we have $|b_n| > 1$ because $|b_n| \ne 1$. Therefore we can conclude that

$$T_\varphi \text{ is hyponormal} \iff n \ge m-1 \quad \text{and} \quad \frac{b_{n-j-1}}{b_{n-j}} = -\overline{s_0}s_1 = \frac{\overline{b_n}b_{n-1}}{|b_n|^2-1} \quad \text{for } 1 \le j \le m-2,$$

which implies (2.16). This completes the proof. □

We conclude with a revealing example.

Example 2.6. Consider a trigonometric polynomial

$$\varphi(z) := \sum_{j=1}^{4} \overline{a_j} z^{-j} + (\sum_{j=1}^{4} a_j z^j)(2z^4 - 2z^3 + 3z^2 - 3z + 4) \quad (a_4 \ne 0).$$

If we put

$$f(z) = 2 - 2z + 3z^2 - 3z^3,$$

then in view of Corollary 2.4, we need to check the Schur numbers $s_j(f)$ for $j = 0, 1, 2, 3$. Let $f_0(z) = f(z)$. Then $s_0(f) = f_0(0) = 2$ and

$$f_1(z) = \frac{f(z) - f_0(0)}{z(1 - \overline{f_0(0)}f(z))} = \frac{-2 + 3z - 3z^2}{-3 + 4z - 6z^2 + 6z^3},$$

which implies that $s_1(f) = f_1(0) = \frac{2}{3}$. Also a straightforward calculation shows that

$$f_2(z) = \frac{1 + 3z - 12z^2}{-5 + 6z - 12z^2 + 18z^3},$$

which implies that $s_2(f) = f_2(0) = -\frac{1}{5}$. Similarly, $s_3(f) = f_3(0) = -\frac{7}{8}$. Therefore by Corollary 2.4, we can conclude that T_φ is hyponormal, regardless of the values of a_1, a_2, a_3, a_4 $(a_4 \neq 0)$.

We next consider a trigonometric polynomial

$$\varphi'(z) := \sum_{j=1}^{5} \overline{a_j} z^{-j} + (\sum_{j=1}^{5} a_j z^j)(2z^4 - 2z^3 + 3z^2 - 3z + 4) \quad (a_5 \neq 0).$$

If we put

$$f'(z) := 2 - 2z + 3z^2 - 3z^3 + 4z^4,$$

then a straightforward calculation as in the above shows that $s_4(f') = f_4'(0) = -3$. Since $|s_4(f')| = 3 > 1$, we can conclude that T_{φ_1} is not hyponormal. This example shows that if $\varphi = \overline{g} + g\psi$ (g and ψ are analytic polynomials) then the hyponormality of T_φ depends heavily on the *degree* of the co-analytic part g.

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