A CRITERION ON THE HYПОNORMАLITY OF
TOEPLITZ OPERATORS
WITH POLYNOMIAL SYMBOLS VIA SCHUR
NUMBERS

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Abstract. In this paper we give a complete characterization on the hyponormality of the
Toeplitz operators $T_{\phi}$ with trigonometric polynomial symbols $\phi = g + h$ ($g, h \in H^2$) when
$g$ divides $h$. This is accomplished by using a criterion on the contractivity of the inverse of a
lower triangular (finite) Toeplitz matrix via Schur numbers of an induced analytic polynomial.

1. Introduction

Let $L^2 \equiv L^2(T)$ be the set of all square-integrable measurable functions on the unit circle $T \equiv \partial \mathbb{D}$
in the complex plane and $H^2 \equiv H^2(T)$ be the corresponding Hardy space. Let $H^\infty \equiv H^\infty(T) :=
L^\infty(T) \cap H^2(T)$, that is, $H^\infty$ is the set of bounded analytic functions on $\mathbb{D}$. Given $\phi \in L^\infty \equiv L^\infty(T)$,
the Toeplitz operator $T_{\phi}$ with symbol $\phi$ is defined by

$$
T_{\phi} g := P(\phi g) \quad (g \in H^2),
$$

where $P$ denotes the orthogonal projection that maps from $L^2$ onto $H^2$. Normal Toeplitz operators
were characterized by a property of their symbols in the early 1960’s by A. Brown and P.R. Halmos
[BH] and the hyponormality of Toeplitz operators was completely solved in terms of their symbols

Cowen’s Theorem. ([Co2], [NT]) For $\phi \in L^\infty$, write

$$
E(\phi) := \{ k \in H^\infty : ||k||_{\infty} \leq 1 \text{ and } \phi - k\overline{\phi} \in H^\infty \}.
$$

Then $T_{\phi}$ is hyponormal if and only if $E(\phi)$ is nonempty.

The elegant and useful theorem of C. Cowen is to recast the operator-theoretic problem of hyponormality
for Toeplitz operators into the problem of finding a solution to a certain functional equation
involving the operator’s symbol. Cowen’s theorem was extensively used in the works [CCL], [Co1],
[Co2], [CL1], [CL2], [FL1], [FL2], [Gu], [GS], [HKL1], [HKL2], [HL1], [HL2], [HL3], [Le], [NT],
[Zhu], and etc to study the hyponormality of Toeplitz operators. When we study hyponormality of
the Toeplitz operator $T_{\phi}$ with symbol $\phi$ we may, without loss of generality, assume that $\phi(0) = 0$
because the hyponormality of an operator is invariant under translation by scalars.

If $\phi$ is a trigonometric polynomial, say $\phi(z) = \sum_{n=-m}^{N} a_n z^n$, where $a_{-m}$ and $a_N$ are nonzero,
then the nonnegative integer $N$ and $m$ denote the analytic and the co-analytic degrees of $\phi$. For
arbitrary trigonometric polynomials, K. Zhu [Zhu] has applied Cowen’s criterion and used a method
based on the classical interpolation theorems of Schur to obtain an abstract characterization of
those trigonometric polynomial symbols corresponding to hyponormal Toeplitz operators. In [FL1],
the hyponormality of $T_{\phi}$ was completely characterized in terms of the Fourier coefficients of $\phi$ when

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The authors of [HKL2] considered the hyponormality of $T_\psi$ in the closed unit ball of $H^2$ (only if the Toeplitz matrix $(g, h) \in H^2$) when $g$ divides $h$ and gave an explicit criterion in terms of the Fourier coefficients of the quotient $\psi := \frac{g}{h}$ when $\psi$ enjoys a certain extremal condition: its advantage is that this criterion depends only on $\psi$ regardless the degree of $g$. But if $\varphi$ does not satisfy the extremal condition, the criterion cannot be applied. The purpose of this paper is to get a complete criterion on the hyponormality of $T_\varphi$ with polynomial symbol $\varphi \equiv \overline{g} + h$ ($g, h \in H^2$) when $g$ divides $h$ (Corollary 2.4). Here, our approach we take is to use a criterion on the contractivity of the inverse of a lower triangular (finite) Toeplitz matrix via Schur numbers of an induced analytic polynomial - the quotient of its division (Theorem 2.3).

2. The main result

Let $\varphi \in L^\infty$ be a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-\infty}^{N} a_n z^n$. If a function $k \in H^\infty$ satisfies $\varphi - k\overline{\varphi} \in H^\infty$ then $k$ necessarily satisfies

$$k \sum_{n=1}^{N} a_n z^{-n} - \sum_{n=1}^{m} a_{-n} z^{-n} \in H^\infty.$$  

If we write $k(z) = \sum_{n=0}^{\infty} c_n z^n$, then from (2.1), $c_0, c_1, \ldots, c_{N-1}$ are determined uniquely from the coefficients of $\varphi$ by the recurrence relation:

$$
\begin{align*}
\begin{cases}
c_0 = c_1 = \ldots = c_{N-m-1} = 0, \\
c_{N-m}a_N = a_{-m}, \\
c_n a_N + \sum_{j=N-m}^{n-1} c_j a_{N-n+j} & \text{for } n = N - m + 1, \ldots, N - 1.
\end{cases}
\end{align*}
$$

Thus (2.2) can be written in the following matrix form:

$$
\begin{bmatrix}
c_{N-m} \\
c_{N-m+1} \\
\vdots \\
c_{N-1}
\end{bmatrix} =
\begin{bmatrix}
a_{N-m+1} & \overline{a_{N-m+2}} & \ldots & \overline{a_{N-1}} & \overline{a_N} \\
\overline{a_{N-m+2}} & \overline{a_{N-m+3}} & \ldots & \overline{a_N} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\overline{a_N} & 0 & \ldots & 0 & \overline{a_{-m}}
\end{bmatrix}^{-1} \begin{bmatrix}
a_{-1} \\
a_{-2} \\
\vdots \\
\overline{a_{-m}}
\end{bmatrix}
$$

Thus $k_p(z) := \sum_{j=N-m}^{N-1} c_j z^j$ is a unique (analytic) polynomial of degree less than $N$ satisfying $\varphi - k_p \overline{\varphi} \in H^\infty$. Thus the problem of finding a solution in the set $E(\varphi)$ is to find an analytic function $k$ in the closed unit ball of $H^\infty$ interpolating $k_p$. This is exactly the Carathéodory interpolation problem (cf. [FF, Theorem VIII.1.3]). Thus by the Cowen’s theorem, $T_\varphi$ is hyponormal if and only if the Toeplitz matrix

$$
C =
\begin{bmatrix}
c_{N-m} & 0 & \ldots & \ldots & 0 \\
c_{N-m+1} & c_{N-m} & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
c_0 & \ddots & \ddots & \ddots & 0 \\
c_{N-1} & \ldots & 0 & \ldots & c_{N-m+1}
\end{bmatrix}
$$

is a contraction (see [FF], [CCL], [FL1], [Zhu]). In this paper we consider the hyponormality of $T_\varphi$ with trigonometric polynomial symbols $\varphi \equiv \overline{g} + h$ ($g, h \in H^\infty$) satisfying that $g$ divides $h$. The condition “$g$ divides $h$” seems to be rigid. However the following lemma shows that if $\varphi \equiv \overline{g} + h$
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is a trigonometric polynomial then we may, without loss of generality, assume that the co-analytic part \( g \) of \( \varphi \) divides the analytic part \( h \) of \( \varphi \) whenever we consider the hyponormality of \( T_\varphi \).

**Lemma 2.1.** ([HKL2, Lemma 2.4]) Let \( \varphi \equiv \overline{\varphi} + h \in L^\infty \), where \( g \) and \( h \) are analytic polynomials of degrees \( m \) and \( N \) (\( m \leq N \)), respectively. If we let

\[
\tilde{h} := z^m T_{\overline{z}^{N-m}} h - d,
\]

where \( d \) is the remainder in the division of \( z^m T_{\overline{z}^{N-m}} h \) by \( g \), put \( \tilde{\varphi} := \overline{\varphi} + \tilde{h} \). We then have:

(i) \( T_\varphi \) is hyponormal if and only if \( T_{\tilde{\varphi}} \) is;

(ii) \( g \) divides \( \tilde{h} \).

We then have:

**Theorem 2.2.** Let \( \varphi \equiv \overline{\varphi} + h \), where \( g \) and \( h \) are analytic polynomials of degrees \( m \) and \( N \) (\( m \leq N \)), respectively. Suppose \( g \) divides \( f \) and

\[
e_\varphi := \sum_{j=0}^r b_j z^j \quad (r := N - m).
\]

Let \( B \) be a finite Toeplitz matrix of the form

\[
B \equiv \begin{bmatrix}
      b_r & 0 & \cdots & 0 \\
      b_{r-1} & b_r & \ddots & \vdots \\
      \vdots & \ddots & \ddots & 0 \\
      b_{r-m+1} & \cdots & b_{r-1} & b_r
    \end{bmatrix},
\]

where \( b_j = 0 \) if \( j < 0 \) for notational convenience. Then \( T_\varphi \) is hyponormal if and only if each eigenvalue of \( B^* B \) is greater than or equal to 1.

**Proof.** Write \( g(z) := \sum_{j=1}^m a_j z^j \). If \( k \in H^\infty \) satisfies \( \varphi - k\overline{\varphi} \in H^\infty \), then \( k \) satisfies

\[
(2.4) \quad \sum_{j=1}^m \overline{a_j} z^{-j} - k \left( \sum_{j=0}^r b_j z^{-j} \right) \left( \sum_{j=1}^m \overline{a_j} z^{-j} \right) \in H^2.
\]

If we write \( k(z) := \sum_{j=0}^\infty c_j z^j \), then

\[
(2.5) \quad \left( 1 - \left( \sum_{j=0}^\infty c_j z^j \right) \left( \sum_{j=0}^r b_j z^{-j} \right) \right) \left( \sum_{j=1}^m \overline{a_j} z^{-j} \right) \in H^2.
\]

From (2.5) we can see that \( c_0 b_r \overline{a_m} = 0 \). It thus follows that

\[
(2.6) \quad c_0 = 0.
\]

By repeating the argument we can show that

\[
(2.7) \quad c_0 = c_1 = \ldots = c_{r-1} = 0.
\]

Thus (2.5) can be written as

\[
(2.8) \quad \left( 1 - \left( \sum_{j=r}^\infty c_j z^j \right) \left( \sum_{j=0}^r b_j z^{-j} \right) \right) \left( \sum_{j=1}^m \overline{a_j} z^{-j} \right) \in H^2.
\]

From (2.7) we can again see that \( 1 - c_r b_r \overline{a_m} = 0 \), which implies

\[
(2.9) \quad c_r b_r = 1.
\]
In turn we have \((c_{r+1}b_r + c_rb_{r-1})u_m = 0\), which implies
\[c_{r+1}b_r + c_rb_{r-1} = 0.\]

If we continue to employ the telescoping method for (2.7), then we get the following equations:
\[
\begin{aligned}
\begin{bmatrix}
c_r
& c_{r+1} & \cdots & c_{r+m-1} \\
0 & c_r & c_{r+1} & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & c_{r+1} \\
0 & \cdots & \cdots & c_r
\end{bmatrix}
&= 
\begin{bmatrix}
b_r
& b_{r-1} & \cdots & b_{r-m+1} \\
0 & b_r & b_{r-1} & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & b_{r-1} \\
0 & \cdots & \cdots & b_r
\end{bmatrix}
= I.
\end{aligned}
\]

(2.10)

If we let
\[C := \begin{bmatrix}
c_r
& c_{r+1} & \cdots & c_{r+m-1} \\
0 & c_r & c_{r+1} & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & c_{r+1} \\
0 & \cdots & \cdots & c_r
\end{bmatrix}, \]
then by the interpolation argument of (2.3), \(T_{\phi}\) is hyponormal if and only if and \(C\) is a contraction. But since by (2.10), \(CB^* = I\), it follows that
\[\sigma(B^*B) = \sigma((C^*C)^{-1}) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma(C^*C) \right\},\]
which implies that \(T_{\phi}\) is hyponormal if and only if and \(\sigma(B^*B) \subset [1, \infty)\), where \(\sigma(\cdot)\) denotes the set of eigenvalues. This gives the result. \(\square\)

Theorem 2.2 essentially asserts that under the assumption of the theorem, \(T_{\phi}\) is hyponormal if and only if
\[B \equiv \begin{bmatrix}
b_r & 0 & \cdots & 0 \\
& b_{r-1} & b_r & \cdots \\
& & \ddots & \ddots & \vdots \\
& & & \ddots & 0 \end{bmatrix}\]
is an inverse of a contractive matrix \(C^*\). Theorem 2.2 also asserts that \(B\) is an inverse of a contractive matrix if and only if all eigenvalues of \(B^*B\) are greater than or equal to 1. If the size of the matrix \(B\) grows bigger, the calculation of the eigenvalues of \(B^*B\) might be another heavy task. On the other hand, we may determine whether \(B\) is an inverse of a contractive matrix by using the Schur numbers of the analytic polynomial \(f(z) := b_r + b_{r-1}z + \cdots + b_{r-m+1}z^{m-1}\) \((b_j \in \mathbb{C})\).

For an analytic function \(f_0(z) \equiv f(z)\) in the open unit disk \(\mathbb{D}\), define a sequence \(\{f_n\}\) by
\[f_{n+1}(z) = \frac{f_n(z) - f_n(0)}{z(1 - f_n(0)f_n(z))}, \quad |z| < 1, \ n = 0, 1, 2, \ldots\]
There exists an inverse of a contractive matrix if and only if \( f \) coincides with the first \( n \)-th Schur number of \( f \). We then define the \( n \)-th Schur number \( s_n(f) \) of \( f \) by

\[ s_n(f) := f_n(0). \]

For example, if \( f(z) = \sum_{n=0}^{\infty} c_n z^n \) then

\[ s_0(f) = c_0, \quad s_1(f) = \frac{c_1}{1 - |c_0|^2}, \quad s_2(f) = \frac{c_2(1 - |c_0|^2) + c_0 c_1^2}{(1 - |c_0|^2)^2 - |c_1|^2}. \]

We note that the \( n \)-th Schur number of \( f \) depends only on the first \( n \) coefficients of \( f \). We would also remark that even though the function \( f_n(z) \) is analytic in a neighborhood of 0, the \( n \)-th Schur number \( s_n(f) \) is well determined whenever none of the numbers \( |s_0(f)|, |s_1(f)|, \ldots, |s_{n-1}(f)| \) is equal to 1.

The Carathéodory interpolation problem says that given an analytic polynomial \( p(z) := c_0 + c_1 z + \cdots + c_n z^n \), find an analytic function \( k \) on the open unit disk \( \mathbb{D} \) such that

\[
\begin{align*}
(i) & \quad \hat{k}(j) = c_j \text{ for } j = 0, 1, \ldots, n \\
(ii) & \quad \|k\|_{\infty} \leq 1.
\end{align*}
\]

I. Schur gave a solution to the Carathéodory interpolation problem:

**Schur's Theorem.** ([Sch]) The above Carathéodory interpolation problem is solvable if and only if \( f(z) := c_0 + c_1 z + \cdots + c_n z^n \) satisfies one of the following two conditions:

\[
\begin{align*}
(i) & \quad |s_j(f)| < 1 \text{ for } 0 \leq j \leq n; \\
(ii) & \quad \text{There exists } n_0 \text{ (} 0 \leq n_0 \leq n \text{) such that } |s_j(f)| < 1 \text{ for } 0 \leq j \leq n_0 - 1, \quad |s_{n_0}(f)| = 1 \quad \text{and the coefficients of } f \text{ coincide with the first } (n+1) \text{ Fourier coefficients of } \\
& \quad \tilde{f} := w_{n_0}(f)(zw_{s_1}(f)(\cdots(zw_{s_{n_0-1}}(f)(zs_{n_0}(f))\cdots))),
\end{align*}
\]

where \( w_s(z) := \frac{z + s}{1 + zs} \text{ (}|s| < 1|). \]

We then have:

**Theorem 2.3.** Let \( b_j \in \mathbb{C} \) (\( 0 \leq j \leq r \)) and \( b_j \neq 0 \). If \( B \) is a lower triangular Toeplitz matrix of the form

\[
B = \begin{bmatrix}
b_r & 0 & \cdots & 0 \\
b_{r-1} & b_r & \cdots & \\
& \ddots & \ddots & \\
b_0 & \cdots & b_{r-1} & b_r
\end{bmatrix},
\]

we put

\[ f(z) = b_r + b_{r-1}z + \cdots + b_0 z^r. \]

Then \( B \) is an inverse of a contractive matrix if and only if \( f \) satisfies one of the following conditions:

\[
\begin{align*}
(i) & \quad |s_0(f)| > 1 \text{ and } |s_j(f)| < 1 \text{ for } 1 \leq j \leq r; \\
(ii) & \quad \text{There exists } n_0 \text{ (} 0 \leq n_0 \leq r \text{) such that } |s_0(f)| > 1, \quad |s_j(f)| < 1 \text{ for } 1 \leq j \leq n_0 - 1, \quad |s_{n_0}(f)| = 1 \quad \text{and the coefficients of } f \text{ coincide with the first } (r+1) \text{ Fourier coefficients of } \\
& \quad \tilde{f} := w_{n_0}(f)(zw_{s_1}(f)(\cdots(zw_{s_{n_0-1}}(f)(zs_{n_0}(f))\cdots))).
\end{align*}
\]
Proof. As in the proof of Theorem 2.2, a straightforward calculation shows that the inverse of $B$ is of the form

$$
\begin{bmatrix}
  k_0 & 0 & \cdots & 0 \\
  k_1 & k_0 & \cdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  k_r & \cdots & k_1 & k_0 \\
\end{bmatrix}
$$

where the $k_i$ satisfy the following equation:

$$
\begin{align*}
  k_0 b_r &= 1 \\
  k_0 b_{r-1} + k_1 b_r &= 0 \\
  \vdots & \quad \vdots \\
  k_0 b_0 + k_1 b_1 + \cdots + k_r b_r &= 0.
\end{align*}
$$

(2.11)

Since $b_r \neq 0$, we have $k_0 \neq 0$. Let $q(z) := \frac{1}{f(z)}$. Since $f(0) = b_r \neq 0$, there exists a neighborhood $\mathcal{R}$ of 0 such that $f$ has no zero in $\mathcal{R}$. Thus $q(z)$ represents an analytic function on $\mathcal{R}$. Further, (2.11) implies that the power series expansion of $q$ in $\mathcal{R}$ can be written as

$$
q(z) = k_0 + k_1 z + \cdots + k_r z^r + o(z^r).
$$

If we define a polynomial

$$
q^{(r)}(z) := k_0 + k_1 z + \cdots + k_r z^r,
$$

then we can see that

$$
\sigma_j(q) = \sigma_j(q^{(r)}) \quad \text{for all } j = 0, \cdots, r.
$$

Then the Schur’s solution to the Carathéodory interpolation problem implies that

$$
\hat{K} := \begin{bmatrix}
  k_0 & 0 & \cdots & 0 \\
  k_1 & k_0 & \cdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  k_r & \cdots & k_1 & k_0 \\
\end{bmatrix}
$$

is a contraction

(2.12)

if and only if $q^{(r)}$ satisfies one of two conditions of the Schur’s Theorem. Observe that

$$
|s_0(q^{(r)})| = |s_0(q)| = |k_0| = \frac{1}{|b_r|} = \frac{1}{|s_0(f)|}.
$$

Let $f_0 := f$, $q_0 := q$,

$$
f_{n+1}(z) := \frac{f_n(z) - f_n(0)}{z(1 - f_n(0)f_n(z))} \quad \text{and} \quad q_{n+1}(z) := \frac{q_n(z) - q_n(0)}{z(1 - q_n(0)q_n(z))} \quad (n \geq 0).
$$

Then we have

$$
f_1(z) = \frac{f(z) - f(0)}{z(1 - f(0)f(z))} = \frac{1}{q(z)} - \frac{1}{q(0)}
$$

$$
= \frac{q(0) - q(z)}{z(1 - q(0)q(z))}
$$

$$
= e^{i\theta} q_1(z) \quad (e^{i\theta} := \frac{q(0)}{q(0)q(0)^{-1}}).
$$

In turn,

$$
f_2(z) = \frac{f_1(z) - f_1(0)}{z(1 - f_1(0)f_1(z))} = \frac{e^{i\theta} q_1(z) - e^{i\theta} q(0)}{z(1 - e^{i\theta} q_1(0)e^{i\theta} q_1(z))}
$$

$$
= \frac{q_1(z) - q_1(0)}{z(1 - q_1(0)q_1(z))}
$$

$$
= e^{i\theta} q_2(z).
$$
Inductively we can see that \( f_j(z) = e^{i\theta}q_j(z) = \alpha q_j(z) \) for \( j = 1, 2, \ldots \), where \( \alpha := \frac{h}{g} \). Therefore we have
\[
(2.14) \quad s_j(f) = \alpha s_j(q) = \alpha s_j(q^{(r)}), \quad \text{and hence } |s_j(f)| = |s_j(q^{(r)})| \quad \text{for } j = 1, 2, \ldots.
\]
We now have
\[
|s_j(q^{(r)})| < 1 \quad \text{for } 0 \leq j \leq r \iff |s_0(f)| > 1 \quad \text{and } |s_j(f)| < 1 \quad \text{for } 1 \leq j \leq r,
\]
which says that the first condition of the Schur’s solution to the Carathéodory interpolation problem for \( q^{(r)} \) is equivalent to the condition (i) of the theorem. Now consider the case when the condition (ii) of the theorem holds. Since \(|s_{n_0}(f)| = 1\) implies \(|s_{n_0}(q^{(r)})| = 1\), it follows from the Schur’s Theorem that the Carathéodory interpolation problem for \( q^{(r)} \) has a solution if and only if the coefficients of \( q^{(r)} \) coincide with the first \((r+1)\) coefficients of the function
\[
\tilde{g} := w_{s_0(q^{(r)})}(zw_{s_1(q^{(r)})}(\cdots(zw_{s_{n_0-1}(q^{(r)})}(zs_{n_0}(q^{(r)})))\cdots)).
\]
Since the coefficients of \( f \) and \( q^{(r)} \) are related by (2.11), we can see that the \( m \) coefficients of \( q^{(r)} \) coincide with the first \( m \) coefficients of \( \tilde{g} \) if and only if the \( m \) coefficients of \( f \) coincide with the first \( m \) coefficients of \( \frac{1}{\tilde{g}} \). Using the relations \( s_0(f) = \frac{1}{s_0(q^{(r)})} \) and \( s_j(f) = \alpha s_j(q^{(r)}) \) (where \( \alpha := \frac{h}{g} \)), a straightforward calculation shows that
\[
\tilde{g} = \frac{1}{w_{s_0(f)}(zw_{s_1(f)}(\cdots(zw_{s_{n_0-1}(f})(zs_{n_0}(f)))\cdots))},
\]
that is,
\[
\tilde{f} \equiv \frac{1}{\tilde{g}} = w_{s_0(f)}(zw_{s_1(f)}(\cdots(zw_{s_{n_0-1}(f})(zs_{n_0}(f)))\cdots)).
\]
Therefore we can conclude that the second condition of the Schur’s solution to the Carathéodory interpolation problem for \( q^{(r)} \) is equivalent to the condition (ii) of the theorem. This completes the proof.

The following corollary provides a complete criterion on the hyponormality of \( T_{\varphi} \) with polynomial symbol \( \varphi = \overline{g} + h \) \((g, h \in H^2)\) when \( g \) divides \( h \).

**Corollary 2.4.** Let \( \varphi \equiv \overline{g} + h \) \((g, h \in H^2)\) be a trigonometric polynomial such that \( g \) divides \( h \).

If \( \frac{h}{g} := b_r z^r + \cdots + b_1 z + b_0 \), put
\[
f(z) := b_r + b_{r-1} z + \cdots + b_{r-m+1} z^{m-1} \quad (m := \text{the degree of } g),
\]
where \( b_j = 0 \) if \( j < 0 \) for notational convenience. If \( s_j(f) \) denotes the \( j \)-th Schur number of \( f \), then \( T_{\varphi} \) is hyponormal if and only if \( f \) satisfies one of the following conditions:

(i) \(|s_0(f)| > 1 \) and \(|s_j(f)| < 1 \) for \( 1 \leq j \leq m - 1 \);
(ii) There exists \( n_0 \) \((0 \leq n_0 \leq m - 1)\) such that \(|s_0(f)| > 1\), \(|s_j(f)| < 1\) for \( 1 \leq j \leq n_0 - 1\), \(|s_{n_0}(f)| = 1\) and the coefficients of \( f \) coincide with the first \( m \) Fourier coefficients of
\[
\tilde{f} := w_{s_0(f)}(zw_{s_1(f)}(\cdots(zw_{s_{n_0-1}(f})(zs_{n_0}(f)))\cdots)).
\]

**Proof.** Remembering that the matrix \( B \) in Theorem 2.2 is an inverse of a contractive matrix if and only if all eigenvalues of \( B^*B \) are greater than or equal to 1, this follows at once from Theorems 2.2 and 2.3.

Using Corollary 2.4, we can give a short proof of [HKL2, Lemma 2.6 and Theorem 2.7].
Corollary 2.5. Let $\varphi \equiv \eta + g\psi = \sum_{j=1}^{m} \varpi_j z^{-j} + \left( \sum_{j=1}^{m} a_j z^j \right) \left( \sum_{j=0}^{n} b_j z^j \right)$. If $T_\varphi$ is hyponormal then

$$
(2.15) \quad \left| \sum_{\xi \in \mathbb{Z}(\psi)} \xi \right| \leq |b_n| - \frac{1}{|b_n|},
$$

where $\mathbb{Z}(\psi)$ denotes the set of zeros of $\psi$. Moreover if the equality of (2.15) holds, then $T_\varphi$ is hyponormal if and only if $n \geq m - 1$ and

$$
(2.16) \quad b_{j+1} = b_j \left( \frac{|b_n|^2 - 1}{b_{n-1}b_n} \right) \quad \text{for} \quad j = n - m + 1, \ldots, n - 2.
$$

Proof. Let $f(z) := b_n + b_{n-1}z + \cdots + b_{n-m+1}z^{m-1}$ ($b_j = 0$ if $j < 0$) and suppose that $T_\varphi$ is hyponormal. If $|s_0(f)| = |b_n| = 1$, then by Corollary 2.4 we have $f(z) = b_n$, i.e., $b_j = 0$ for $n - m - 1 \leq j \leq n - 1$. Observe that $\frac{b_{n-1}}{b_n}$ is equal to $- \sum_{\xi \in \mathbb{Z}(\psi)} \xi$. Therefore if $|s_0(f)| = |b_n| = 1$, then (2.15) and (2.16) are automatically satisfied. Assume instead $|b_n| \neq 1$, so that $s_1(f)$ is well defined. If $T_\varphi$ is hyponormal then by Corollary 2.4,

$$
|s_1(f)| = \left| \frac{b_{n-1}}{1 - |b_n|^2} \right| \leq 1, \quad \text{so that} \quad \left| \frac{b_{n-1}}{b_n} \right| \leq \left| \frac{1}{|b_n|} - |b_n| \right| = \left| |b_n| - \frac{1}{|b_n|} \right|.
$$

Again, $\frac{b_{n-1}}{b_n} = - \sum_{\xi \in \mathbb{Z}(\psi)} \xi$ gives (2.15).

For the second assertion we assume that the equality holds in (2.15). The preceding calculation shows that $|s_1(f)| = 1$. By Corollary 2.4, $T_\varphi$ is hyponormal if and only if the second condition of Corollary 2.4 is satisfied by $f$, i.e., $|s_0(f)| = |b_n| > 1$ and the $m$ coefficients of $f$ coincide with the first $m$ coefficients of

$$
w_{s_0(f)}(z w_{s_1(f)}) = \frac{s_0 + z s_1}{1 - |s_0|^2} \sum_{j=0}^{\infty} \left( -1 \right)^j \left( |s_0| s_1 \right)^j
$$

$$
= s_0 + (s_1 - |s_0|^2 s_1) z - (s_1 - |s_0|^2 s_1)(|s_0|^2 s_1) z^2 + (s_1 - |s_0|^2 s_1)(|s_0|^2 s_1)^2 z^3 - \cdots .
$$

Since (2.15) implies $|b_n| \geq 1$, we have $|b_n| > 1$ because $|b_n| \neq 1$. Therefore we can conclude that

$$
T_\varphi \text{ is hyponormal} \iff n \geq m - 1 \quad \text{and} \quad \frac{b_{n-j-1}}{b_{n-j}} = - |s_0|^2 s_1 = \frac{b_n b_{n-1}}{|b_n|^2 - 1} \quad \text{for} \quad 1 \leq j \leq m - 2,
$$

which implies (2.16). This completes the proof. \qed

We conclude with a revealing example.

**Example 2.6.** Consider a trigonometric polynomial

$$
\varphi(z) := \sum_{j=1}^{4} \varpi_j z^{-j} + \left( \sum_{j=1}^{4} a_j z^j \right) (2z^4 - 2z^3 + 3z^2 - 3z + 4) \quad (a_4 \neq 0).
$$

If we put

$$
f(z) = 2 - 2z + 3z^2 - 3z^3,
$$

then in view of Corollary 2.4, we need to check the Schur numbers $s_j(f)$ for $j = 0, 1, 2, 3$. Let $f_0(z) = f(z)$. Then $s_0(f) = f_0(0) = 2$ and

$$
f_1(z) = \frac{f(z) - f_0(0)}{z(1 - f_0(0)f(z))} = \frac{-2 + 3z - 3z^2}{-3 + 4z - 6z^2 + 6z^3},
$$

for $j = 1$. The details are omitted.
which implies that \( s_1(f) = f_1(0) = \frac{2}{3} \). Also a straightforward calculation shows that

\[
f_2(z) = \frac{1 + 3z - 12z^2}{-5 + 6z - 12z^2 + 18z^3},
\]

which implies that \( s_2(f) = f_2(0) = -\frac{1}{3} \). Similarly, \( s_3(f) = f_3(0) = -\frac{2}{3} \). Therefore by Corollary 2.4, we can conclude that \( T_\varphi \) is hyponormal, regardless of the values of \( a_1, a_2, a_3, a_4 (a_4 \neq 0) \).

We next consider a trigonometric polynomial

\[
\varphi'(z) := \sum_{j=1}^{5} a_j z^{-j} + \left( \sum_{j=1}^{5} a_j z^j \right) (2z^4 - 2z^3 + 3z^2 - 3z + 4) \quad (a_5 \neq 0).
\]

If we put

\[
f'(z) := 2 - 2z + 3z^2 - 3z^3 + 4z^4,
\]

then a straightforward calculation as in the above shows that \( s_4(f') = f'_4(0) = -3 \). Since \( |s_4(f')| = 3 > 1 \), we can conclude that \( T_{\varphi'} \) is not hyponormal. This example shows that if \( \varphi = \overline{\varphi} + g\psi \) (\( g \) and \( \psi \) are analytic polynomials) then the hyponormality of \( T_\varphi \) depends heavily on the degree of the co-analytic part \( g \).

\[\text{References}\]


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