Hyponormal Toeplitz Operators with Matrix-Valued Circulant Symbols

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Abstract. In this paper we are concerned with the hyponormality of Toeplitz operators with matrix-valued circulant symbols. We establish a necessary and sufficient condition for Toeplitz operators with matrix-valued partially circulant symbols to be hyponormal and also provide a rank formula for the self-commutator.

Keywords. Block Toeplitz operators, matrix-valued symbols, bounded type functions, circulant functions, hyponormal.

1. Introduction

Throughout this paper, let $\mathcal{H}$ denote a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators acting on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, T^* denotes the adjoint of T. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *unitary* if $T^*T = TT^* = I$, *hyponormal* if its self-commutator $[T^*, T] \equiv T^*T - TT^*$ is positive semi-definite, and *subnormal* if T has a normal extension N, i.e., there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subseteq \mathcal{H}$ and $T = N|_{\mathcal{H}}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, we write $\ker T$ and $\operatorname{ran} T$ for the kernel and the range of T, respectively. For a set $\mathcal{M}$, $\operatorname{cl} \mathcal{M}$ and $\mathcal{M}^\perp$ denote the closure and the orthogonal complement of $\mathcal{M}$, respectively.

We review a few essential facts for (block) Toeplitz operators and (block) Hankel operators that we will need to begin with, using [Do1], [Do2], [GGK], [Ni], and [Pe]. Let $L^2 \equiv L^2(\mathbb{T})$ be the set of all square-integrable measurable functions on the unit circle $\mathbb{T} \equiv \partial\,\mathbb{D}$ in the complex plane and $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$, that is, H^∞ is the set of bounded analytic functions on $\mathbb{D}$. Given $\varphi \in L^\infty \equiv L^\infty(\mathbb{T})$, the Toeplitz operator T_φ and the Hankel operator H_φ are defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi g := JP^\perp(\varphi g) \qquad (g \in H^2),$$

where P and $P^\perp$ denote the orthogonal projections that map from L^2 onto H^2 and $(H^2)^\perp$, respectively, and J denotes the unitary operator on L^2 defined by $J(f)(z) = \overline{z} f(\overline{z})$.

Normal Toeplitz operators were characterized by a property of their symbols in the early 1960's by A. Brown and P.R. Halmos [BH] and the hyponormality of Toeplitz operators was completely solved in terms of their symbols by C. Cowen [Co2] in 1988.

Cowen's Theorem ([Co2], [NT]) *For $\varphi \in L^\infty$, write*

$$\mathcal{E}(\varphi) := \Big\{ k \in H^\infty : \ ||k||_\infty \le 1 \ \text{and} \ \varphi - k\overline{\varphi} \in H^\infty \Big\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

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The elegant and useful theorem of C. Cowen is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution to a certain functional equation involving the operator's symbol. Tractable and explicit criteria for the hyponormality of Toeplitz operators T_φ with scalar-valued trigonometric polynomials or rational symbols φ were established by many authors (cf. [Co1], [Co2], [CL], [FL1], [Gu], [GS], [HK], [HKL1], [HKL2], [HL1], [HL2], [HL3], [Le], [NT], [Zhu], and etc.). When we study hyponormality (also, normality and subnormality) of the Toeplitz operator T_φ with symbol φ we may, without loss of generality, assume that $\varphi(0) = 0$ because the hyponormality of an operator is invariant under translation by scalars. We recall that a function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are analytic functions $\psi_1, \psi_2 \in H^\infty(\mathbb{D})$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

We write, for an inner function θ,

$$\mathcal{H}(\theta) := H^2 \ominus \theta H^2.$$

It was known [Ab, Lemma 3] that if $\varphi \in H^2$ is such that $\overline{\varphi}$ is of bounded type and $\varphi(0) = 0$ then we can write

$$\varphi = \theta \overline{b}, \tag{1.1}$$

where θ is an inner function and $b \in \mathcal{H}(\theta)$ satisfies that b and θ are coprime. If φ is a rational function then by Kronecker's Lemma [Ni, p.183], θ in (1.1) can be chosen as a finite Blaschke product. It was also [Ab, Lemma 6] known that if T_φ is hyponormal, if $\varphi \notin H^\infty$, and if φ or $\overline{\varphi}$ is of bounded type then both φ and $\overline{\varphi}$ are of bounded type.

We now introduce the notion of block Toeplitz and block Hankel operators. Let M_n denote the set of $n \times n$ complex matrices. For a complex Hilbert space $\mathcal{X}$, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy space. We observe that $L^2_{\mathbb{C}^n} = L^2(\mathbb{T}) \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2(\mathbb{T}) \otimes \mathbb{C}^n$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ $(= L^\infty(\mathbb{T}) \otimes M_n)$ then the block Toeplitz operator T_Φ and the block Hankel operator H_Φ on $H^2_{\mathbb{C}^n}$ are defined by

$$T_\Phi f = P_n(\Phi f) \quad \text{and} \quad H_\Phi f = J P_n^\perp(\Phi f) \quad (f \in H^2_{\mathbb{C}^n}),$$

where P_n and $P_n^\perp$ denote the orthogonal projections that map from $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$ and $\big(H^2_{\mathbb{C}^n}\big)^\perp$, respectively and J denotes the unitary operator on $L^2_{\mathbb{C}^n}$ given by $J(g)(z) = \overline{z} I_n g(\overline{z})$ for $g \in L^2_{\mathbb{C}^n}$ (I_n :=the $n \times n$ identity matrix). For $\Phi \in L^\infty_{M_n}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}). \tag{1.2}$$

An inner (matrix) function $\Theta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is one satisfying $\Theta^*\Theta = I_m$ for almost all $z \in \mathbb{T}$, where $M_{n\times m}$ denotes the set of $n \times m$ complex matrices. The following basic relations can be easily derived:

(1.3) $T^*_\Phi = T_{\Phi^*}$, $H^*_\Phi = H_{\widetilde{\Phi}}$ $\quad (\Phi \in L^\infty_{M_n})$;
(1.4) $T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi$ $\quad (\Phi, \Psi \in L^\infty_{M_n})$;
(1.5) $H_\Phi T_\Psi = H_{\Phi\Psi}$, $H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi$ $\quad (\Phi \in L^\infty_{M_n}, \Psi \in H^\infty_{M_n})$;
(1.6) $H^*_\Phi H_\Phi - H^*_{\Theta\Phi} H_{\Theta\Phi} = H^*_\Phi H_{\Theta^*} H^*_{\Theta^*} H_\Phi$ $\quad (\Theta \in H^\infty_{M_n}$ is inner, $\Phi \in L^\infty_{M_n})$.

For a matrix-valued function $\Phi = [\varphi_{ij}] \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry φ_{ij} is of bounded type and that Φ is *rational* if each entry φ_{ij} is a rational function. A matrix-valued trigonometric polynomial $\Phi \in L^\infty_{M_n}$ is of the form

$$\Phi(z) = \sum_{j=-m}^{N} A_j z^j \ (A_j \in M_n),$$

where A_N and A_{-m} are called the *outer* coefficients of Φ.

For matrix-valued functions $A(z) = \sum_{j=-\infty}^{\infty} A_j z^j \in L^2_{M_n}$ and $B(z) = \sum_{j=-\infty}^{\infty} B_j z^j \in L^2_{M_n}$, we define the inner product of A and B by

$$(A, B) := \int_{\mathbb{T}} \operatorname{tr}(B^* A)\, d\mu = \sum_{j=-\infty}^{\infty} \operatorname{tr}(B_j^* A_j),$$

where $\operatorname{tr}(\cdot)$ means the trace of the matrix and define $||A||_2 := (A, A)^{\frac{1}{2}}$. We also define, for $A \in L^\infty_{M_n}$,

$$||A||_\infty := \operatorname{ess\,sup}_{t\in\mathbb{T}} ||A(t)|| \quad (||\cdot|| \text{ means the spectral norm of the matrix}).$$

The following fundamental result is known as the Beurling-Lax-Halmos Theorem (cf. [FF], [Ni]), which will be useful in the sequel.

The Beurling-Lax-Halmos Theorem. *A nonzero subspace M of $H^2_{\mathbb{C}^n}$ is invariant for the shift operator $S \equiv T_{zI_n}$ on $H^2_{\mathbb{C}^n}$* (i.e., $S(M) \subset M$) *if and only if $M = \Theta H^2_{\mathbb{C}^m}$, where Θ is an inner matrix function in $H^\infty_{M_{n\times m}}$ $(m \le n)$.*

From (1.5) we can see that the kernel of a block Hankel operator H_Φ is an invariant subspace of the shift operator on $H^2_{\mathbb{C}^n}$. Thus, if $\ker H_\Phi \neq \{0\}$, then by the Beurling-Lax-Halmos theorem,

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function Θ. In general, Θ need not be square. We note that if $\Theta \in H^\infty_{M_n}$ is an inner matrix function then $\ker H_{\Theta^*} = \Theta H^2_{\mathbb{C}^n}$.

Recently, Gu, Hendricks and Rutherford [GHR] characterized the hyponormality of block Toeplitz operators in terms of their symbols. In particular they showed that if T_Φ is a hyponormal block Toeplitz operator on $H^2_{\mathbb{C}^n}$, then Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$. Their characterization for hyponormality of block Toeplitz operators resembles the Cowen's theorem except for an additional condition – the normality of the symbol.

Lemma 1.1. (Hyponormality of Block Toeplitz Operators) [GHR] *For each $\Phi \in L^\infty_{M_n}$, let*

$$\mathcal{E}(\Phi) := \Big\{ K \in H^\infty_{M_n} :\ ||K||_\infty \le 1 \ \ \text{and} \ \ \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

However, as in the scalar-valued cases, the case of arbitrary matrix-valued symbol $\Phi \in L^\infty_{M_n}$, though solved by Lemma 1.1, is in practice very difficult. In [GHR] it was shown that, as in the scalar-valued case, if Φ is a matrix-valued trigonometric polynomial with an invertible analytic outer coefficient then the hyponormality of T_Φ can be determined by a matrix-valued Carathéodory interpolation problem. In [HL4] and [HL5], it was shown that if $\Phi \in L^\infty_{M_n}$ is a matrix-valued rational function then the hyponormality of the block Toeplitz operator T_Φ can be determined by the matrix-valued tangential or classical Hermite-Fejér interpolation problem.

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ $(m \le n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We remark that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$. If $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$.

On the other hand, in the preceding, we have remarked that Θ need not be square in the equality $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$. But it was known [GHR] that for $\Phi \in L^\infty_{M_n}$, the following statements are equivalent:

(i) Φ is of bounded type;
(ii) $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some square inner matrix function Θ;
(iii) $\Phi = A\Theta^*$, where $A \in H^\infty_{M_n}$ and A and Θ are right coprime.

For $\Phi \in L^\infty_{M_n}$ we write

$$\Phi_+ := P_n \Phi \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \left(P_n^\perp \Phi\right)^* \in H^2_{M_n},$$

where P_n denotes the orthogonal projection from $L^2_{M_n}$ onto $H^2_{M_n}$. Thus we can write $\Phi = \Phi_-^* + \Phi_+$. Suppose $\Phi_+ = [\varphi_{ij}] \in H^2_{M_n}$ is such that Φ^* is of bounded type. Then we may write $\varphi_{ij} = \theta_{ij}\overline{b_{ij}}$, where θ_{ij} is an inner function and θ_{ij} and b_{ij} are coprime. Thus if θ is the least common multiple of θ_{ij}'s then we can write

$$\Phi_+ = [\varphi_{ij}] = [\theta_{ij}\overline{b_{ij}}] = [\theta\overline{a_{ij}}] = \Theta A^* \quad (\Theta = \theta I_n,\ A \in H^2_{M_n}). \tag{1.7}$$

For brevity, we write I for the identity matrix and

$$I_\zeta := \zeta I \quad (\zeta \in L^\infty).$$

For an inner matrix function $\Theta \in H^2_{M_n}$, we write

$$\mathcal{H}(\Theta) := H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^n}, \quad \mathcal{H}_\Theta := H^2_{M_n} \ominus \Theta H^2_{M_n} \quad \text{and} \quad \mathcal{K}_\Theta := H^2_{M_n} \ominus H^2_{M_n}\Theta.$$

Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then in view of (1.7) we can write

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*, \tag{1.8}$$

where $\Theta_i = I_{\theta_i}$ with an inner function θ_i $(i = 1, 2)$, $A \in \mathcal{K}_{I_z\Theta_1}$ and $B \in \mathcal{K}_{\Theta_2}$. In particular, if $\Phi \in L^\infty_{M_n}$ is rational then the θ_i are chosen as finite Blaschke products as we observed in (1.1).

Before we proceed we remark that by contrast to the scalar-valued case, Φ^* may not be of bounded type even though T_Φ is hyponormal, $\Phi \notin H^\infty_{M_n}$ and Φ is of bounded type. But we have one-way implication: if T_Φ is hyponormal and Φ^* is of bounded type then Φ is also of bounded type (see [GHR]). Thus whenever we deal with hyponormal Toeplitz operators T_Φ with symbols Φ satisfying that both Φ and Φ^* are of bounded type, it suffices to assume that only Φ^* is of bounded type.

In this paper we are concerned with the hyponormality of Toeplitz operators with matrix-valued circulant symbols. In Section 2, we provide some auxiliary lemmas. In Section 3, we prove the main result which gives a necessary and sufficient condition for Toeplitz operators with matrix-valued partially circulant symbols to be hyponormal and also provide a rank formula for the self-commutator.

2. Auxiliary lemmas

If Ω is the greatest common left inner divisor of A and Θ in the representation (1.7):

$$\Phi = \Theta A^* = A^*\Theta \quad (\Theta \equiv I_\theta \text{ for an inner function } \theta),$$

then $\Theta = \Omega\Omega_l$ and $A = \Omega A_l$ for some inner matrix Ω_l (where $\Omega_l \in H^2_{M_n}$ because $\det\Theta$ is not identically zero) and some $A_l \in H^2_{M_n}$. Therefore if $\Phi^* \in L^\infty_{M_n}$ is of bounded type then we can write

$$\Phi = {A_l}^*\Omega_l, \quad \text{where } A_l \text{ and } \Omega_l \text{ are left coprime:}$$

in this case, $A_l^*\Omega_l$ is called the *left coprime decomposition* of Φ and similarly, we can write

$$\Phi = \Omega_r A_r^*, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime } (\Omega_l \in H^2_{M_n}):$$

in this case, $\Omega_r A_r^*$ is called the *right coprime decomposition* of Φ.

In general, it is not easy to check the condition "Θ and A are right coprime" for the representation $\Phi = \Theta A^*$ (Θ is inner and $A \in H^2_{M_n}$) even though $\Theta = I_\theta$ for an inner function θ. But if θ is a finite Blaschke product then we have a more tractable criterion.

Lemma 2.1. *If $A, \Theta \in H^\infty_{M_n}$ ($\Theta := I_\theta$ with a finite Blaschke product θ) then the following are equivalent:*

(a) *$A(\alpha)$ is invertible for each zero α of θ;*
(b) *A and Θ are right coprime;*
(c) *A and Θ are left coprime.*

Remark. Lemma 2.1 extends Lemma 3.10 of [CHL], in which the same result was proved when $A \in H^\infty_{M_n}$ is rational.

Proof. (a) $\Rightarrow$ (b): Suppose $A(\alpha)$ is invertible for each zero α of θ. Observe that

$$\mathfrak{h} \in \ \ker H_{A\Theta^*} \iff A\Theta^*\mathfrak{h} \in \Theta H^2_{\mathbb{C}^n} \iff A\mathfrak{h} \in \Theta H^2_{\mathbb{C}^n}.$$

Let θ be a finite Blaschke product of degree d. Then we can write

$$\theta(z) = e^{i\xi} \prod_{i=1}^N \Big(\frac{z-\alpha_i}{1-\overline{\alpha_i}z} \Big)^{m_i},$$

where $\sum_{i=1}^N m_i = d$. Thus $\mathfrak{h} \in \ker H_{A\Theta^*}$ if and only if for each $i = 1, 2, \cdots, N$

$$\begin{bmatrix} A_{i,0} & 0 & 0 & 0 & \cdots & 0 \\ A_{i,1} & A_{i,0} & 0 & 0 & \cdots & 0 \\ A_{i,2} & A_{i,1} & A_{i,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ A_{i,m_i-2} & A_{i,m_i-3} & \ddots & \ddots & A_{i,0} & 0 \\ A_{i,m_i-1} & A_{i,m_i-2} & \dots & A_{i,2} & A_{i,1} & A_{i,0} \end{bmatrix} \begin{bmatrix} \mathfrak{h}_{i,0} \\ \mathfrak{h}_{i,1} \\ \mathfrak{h}_{i,2} \\ \vdots \\ \mathfrak{h}_{i,m_i-2} \\ \mathfrak{h}_{i,m_i-1} \end{bmatrix} = 0,$$

where

$$A_{i,j} := \frac{A^{(j)}(\alpha_i)}{j!} \quad \text{and} \quad \mathfrak{h}_{i,j} := \frac{\mathfrak{h}^{(j)}(\alpha_i)}{j!}.$$

Since $A(\alpha)$ is invertible for each zero α of θ, $A_{i,0}$ is invertible for each $i = 1, 2, \cdots, N$. Thus

$$\mathfrak{h}_{i,j} = 0 \quad (i = 1, 2, \cdots, N, \ j = 0, 1, 2, \cdots, m_i - 1),$$

which implies that $\ker H_{A\Theta^*} \subseteq \Theta H^2_{\mathbb{C}^n}$. But since evidently $\Theta H^2_{\mathbb{C}^n} \subseteq \ker H_{A\Theta^*}$, it follows that $\ker H_{A\Theta^*} = \Theta H^2_{\mathbb{C}^n}$, which implies that A and Θ are right coprime.

(b) $\Rightarrow$ (a) and (b) $\Leftrightarrow$ (c) : From the proof of [CHL, Lemma 3.10]. $\square$

If $\Phi \in L^\infty_{M_n}$, then by (1.4),

$$[T^*_\Phi, T_\Phi] = H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi + T_{\Phi^*\Phi - \Phi\Phi^*}\,.$$

Since the normality of Φ is a necessary condition for the hyponormality of T_Φ, the positivity of $H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi$ is an essential condition for the hyponormality of T_Φ. Thus it is more convenient for the argument of the hyponormality of T_Φ to define the positivity of $H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi$ as another notion.

Our proof of Lemma 2.1 relies upon interpolation theory. However, we are informed by the referee that the proof of Lemma 2.1 can be simplified with the help of the corona theorem for matrix-valued functions (cf. [Fu], [DD]). The authors are thankful to the referee for the valuable comment.

Definition 2.2. Let $\Phi \in L^\infty_{M_n}$. The *pseudo-selfcommutator* of T_Φ is defined by

$$[T^*_\Phi, T_\Phi]_p := H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi.$$

Then T_Φ is said to be *pseudo-hyponormal* if $[T_\Phi^*, T_\Phi]_p$ is positive semidefinite.

From the definition, we can see that the pseudo-hyponormality of T_Φ is independent of the constant matrix term $\Phi(0)$. Thus whenever we consider the pseudo-hyponormality of T_Φ we may assume that $\Phi(0) = 0$. Observe that if $\Phi \in L^\infty_{M_n}$ then

$$[T_\Phi^*, T_\Phi] = [T_\Phi^*, T_\Phi]_p + T_{\Phi^*\Phi - \Phi\Phi^*}.$$

We thus have

$$T_\Phi \text{ is hyponormal} \iff T_\Phi \text{ is pseudo-hyponormal and } \Phi^*\Phi = \Phi\Phi^*, \text{ i.e., } \Phi \text{ is normal}$$

and (via Theorem 3.3 of [GHR]) T_Φ is pseudo-hyponormal if and only if $\mathcal{E}(\Phi) \neq \emptyset$.

The following lemma shows that the pseudo-hyponormality of T_Φ with a bounded type symbol Φ gives a relationship between the analytic and co-analytic parts of the symbol:

Lemma 2.3. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type of the form*

$$\Phi_+ = \Theta_1 A^* \text{ (right coprime decomposition)} \quad \textit{and} \quad \Phi_- = \Theta_2 B^* \text{ (right coprime decomposition)}.$$

If T_Φ is pseudo-hyponormal then $\Theta_1 = \Theta_2\Theta_0$ for some inner matrix Θ_0.

Proof. Suppose T_Φ is pseudo-hyponormal. Then there exists a matrix function $K \in H^\infty_{M_n}$ such that $||K||_\infty \leq 1$ and $\Phi - K\Phi^* \in H^\infty_{M_n}$. Thus $H_{\Phi_-^*} = H_{K\Phi_+^*} = T^*_{\widetilde{K}} H_{\Phi_+^*}$, which implies $\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}$. Hence $\Theta_1 H^2_{\mathbb{C}^n} \subseteq \Theta_2 H^2_{\mathbb{C}^n}$ and therefore Θ_2 is a left inner divisor of Θ_1 (cf. [FF, Corollary IX.2.2]), which gives the result. □

In view of Lemma 2.3, when we study the pseudo-hyponormality of block Toeplitz operator T_Φ with symbol Φ whose adjoint is of bounded type, we may assume that the symbol $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is of the form

$$\Phi_+ = \Theta\Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta B^*.$$

For a closed subspace $\mathcal{X}$ of a Hilbert space $\mathcal{H}$, write $P_\mathcal{X}$ for the orthogonal projection from $\mathcal{H}$ onto $\mathcal{X}$.

The following is an elementary observation.

Lemma 2.4. *For any inner matrices Θ_1 and Θ_2 in $H^2_{M_n}$, we have*

$$\mathcal{K}_{\Theta_1\Theta_2} = \mathcal{K}_{\Theta_1}\Theta_2 + \mathcal{K}_{\Theta_2}.$$

Proof. For $F \in \mathcal{K}_{\Theta_1\Theta_2}$, we can write

$$F = F_1 + F_2,$$

where $F_1 \in H^2_{M_n}\Theta_2$ and $F_2 = P_{\mathcal{K}_{\Theta_2}} F$. Thus $F_1 = E\Theta_2$ for some $E \in H^2_{M_n}$. Since $F_1 = E\Theta_2 \in \mathcal{K}_{\Theta_1\Theta_2}$, it follows that $E \in \mathcal{K}_{\Theta_1}$. This proves the inclusion $\mathcal{K}_{\Theta_1\Theta_2} \subseteq \mathcal{K}_{\Theta_1}\Theta_2 + \mathcal{K}_{\Theta_2}$. The reverse inclusion is obvious. □

The following lemma shows the pull-back property on the symbols of hyponormal block Toeplitz operators.

Lemma 2.5. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type of the form*

$$\Phi_+ = \Theta\Theta_1 A^* \quad \textit{and} \quad \Phi_- = \Theta B^* \quad (\Theta \text{ and } \Theta_1 \text{ are inner}),$$

where Θ_1 and A are right coprime. Put

$$\Psi = \Phi_-^* + \Theta\,(P_{\mathcal{K}_\Theta} A_1)^*, \tag{2.1}$$

where A_1 is defined by

$$\Theta_1 A^* = A_1^* \Theta_2,$$

where A_1 and Θ_2 are left coprime. Then

$$T_\Phi \text{ is pseudo-hyponormal} \iff T_\Psi \text{ is pseudo-hyponormal.} \tag{2.2}$$

Moreover, if $\Theta_1 = I_{\theta_1}$ for a finite Blaschke product θ_1, then in (2.1)*, A_1 can be chosen as A.*

Proof. Suppose T_Φ is pseudo-hyponormal. Then there exists a matrix function $K \in \mathcal{E}(\Phi)$, i.e., $B\Theta^* - KA\Theta_1^*\Theta^* \in H^2_{M_n}$, which implies that $KA\Theta_1^* \in H^2_{M_n}$. We thus have

$$K\Theta_2^* A_1 \in H^2_{M_n}, \text{ so that, } A_1^T(\Theta_2^*)^T K^T \in H^2_{M_n},$$

where $(\cdot)^T$ means the transpose of the matrix. This implies $H_{A_1^T(\Theta_2^*)^T} T_{K^T} = 0$. We thus have

$$K^T H^2_{\mathbb{C}^n} \subseteq \ker H_{A_1^T(\Theta_2^T)^*} = \Theta_2^T H^2_{\mathbb{C}^n}, \tag{2.3}$$

where the last equality follows from the observation that A_1^T and Θ_2^T are right coprime because A_1 and Θ_2 are left coprime. Thus (2.3) shows that Θ_2^T is a left inner divisor of K^T, i.e., $K^T = \Theta_2^T (K')^T$ for some $K' \in H^2_{M_n}$, so that $K = K'\Theta_2$. Thus we have

$$\begin{aligned} K \in \mathcal{E}(\Phi) &\iff B\Theta^* - KA\Theta_1^*\Theta^* \in H^2_{M_n} \\ &\iff B\Theta^* - K\Theta_2^* A_1 \Theta^* \in H^2_{M_n} \\ &\iff B\Theta^* - K' A_1 \Theta^* \in H^2_{M_n}. \end{aligned}$$

We write

$$A_1' := P_{\mathcal{K}_\Theta} A_1 \quad \text{and} \quad A_1'' := A - A_1' \in H^2_{M_n}\Theta.$$

We thus have

$$K \in \mathcal{E}(\Phi) \iff B\Theta^* - K'(A_1' + A_1'')\Theta^* \in H^2_{M_n}.$$

But since $A_1'' \in H^2_{M_n}\Theta$, and hence $A_1''\Theta^* \in H^2_{M_n}$, it follows that

$$K \in \mathcal{E}(\Phi) \iff B\Theta^* - K'\big[P_{\mathcal{K}_\Theta} A_1\big]\Theta^* \in H^2_{M_n} \iff K' \in \mathcal{E}(\Psi),$$

which gives the result. The second assertion follows at once from the first together with Lemma 2.1. □

Lemma 2.5 guarantees that the analytic part of the symbol Φ can be "pulled back" to a function having the same inner part of the decomposition as that of the co-analytic part without losing the pseudo-hyponormality. However the 'coprime' condition is essential. To see this consider

$$\Phi := \begin{bmatrix} \overline{z} + 2z^2 & 0 \\ 0 & \overline{z} + 2z \end{bmatrix}.$$

Write

$$\Theta = \Theta_1 := I_z, \quad A := \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2z \end{smallmatrix}\right], \quad \text{and} \quad B := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right].$$

Then

$$\Phi_+ = \Theta\Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta B^*.$$

Put

$$\Psi := \Theta_1 (P_{\mathcal{K}_{\Theta_1}} A)^* + B\Theta_1^* = \begin{bmatrix} \overline{z} + 2z & 0 \\ 0 & \overline{z} \end{bmatrix}.$$

Then T_Φ is pseudo-hyponormal (because if $K := \begin{bmatrix} \frac{1}{2}z & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ then $\Phi_-^* - K\Phi_+^* \in H^\infty$ and $||K||_\infty < 1$), whereas T_Ψ is not (because $T_{\overline{z}}$ is not hyponormal). Note that by Lemma 2.1, A and Θ_1 are not right coprime because $A(0)$ is not invertible.

If $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is of bounded type of the form

$$\Phi_+ = \Theta A^* \quad \text{and} \quad \Phi_- = \Theta B^* \quad (\Theta \text{ is inner})$$

and if Θ_0 is a right inner divisor of Θ, we write

$$\Phi_{\Theta_0} := \big[P_{\mathcal{K}_{\Theta_0}} B\big]\,\Theta_0^* + \Theta_0 \big[P_{\mathcal{K}_{\Theta_0}} A\big]^*.$$

We then have:

Lemma 2.6. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type of the form*

$$\Phi_+ = \Theta A^* \quad \textit{and} \quad \Phi_- = \Theta B^*,$$

where Θ is inner. If Θ_0 is a right inner divisor of Θ, then

$$\mathcal{E}(\Phi) \subseteq \mathcal{E}(\Phi_{\Theta_0}).$$

In particular, if T_Φ is pseudo-hyponormal then $T_{\Phi_{\Theta_0}}$ is pseudo-hyponormal.

Proof. Let $\Theta = \Theta_1\Theta_0$ for some inner matrix function Θ_1. If $K \in \mathcal{E}(\Phi)$, then $B\Theta^* - KA\Theta^* \in H^2_{M_n}$, or equivalently,

$$B\Theta_0^* - KA\Theta_0^* \in H^2_{M_n}\Theta_1 .$$

In view of Lemma 2.4, we can write

$$A := P_{\mathcal{K}_{\Theta_0}}A + A_1 \quad \text{and} \quad B := P_{\mathcal{K}_{\Theta_0}}B + B_1,$$

where $A_1 = H_1\Theta_0$ and $B_1 = H_2\Theta_0$ for some $H_1, H_2 \in \mathcal{K}_{\Theta_1}$. We thus have

$$\Big(P_{\mathcal{K}_{\Theta_0}}B - KP_{\mathcal{K}_{\Theta_0}}A\Big)\Theta_0^* + \Big(H_2 - KH_1\Big) \in H^2_{M_n}\Theta_1,$$

so that

$$\left[P_{\mathcal{K}_{\Theta_0}}B\right]\Theta_0^* - K\left[P_{\mathcal{K}_{\Theta_0}}A\right]\Theta_0^* \in H^2_{M_n},$$

which implies that $K \in \mathcal{E}(\Phi_{\Theta_0})$. Thus we have that $\mathcal{E}(\Phi) \subseteq \mathcal{E}(\Phi_{\Theta_0})$, which gives the result. □

3. Toeplitz operators with matrix-valued circulant symbols

To motivate our interest in the circulant symbols, we recall [FKKL, IC, It] that the characterization of finite normal Toeplitz matrices states that every finite normal Toeplitz matrix whose eigenvalues are not collinear must be a generalized circulant, which is a normal matrix of the form

$$\begin{bmatrix} a_0 & e^{i\omega}a_N & \cdots & \cdots & e^{i\omega}a_1 \\ a_1 & a_0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_0 & e^{i\omega}a_N \\ a_N & \cdots & \cdots & a_1 & a_0 \end{bmatrix}.$$

We also recall that a trigonometric polynomial $\varphi(z) = \sum_{n=-N}^N a_n z^n$ is called a circulant polynomial if $a_{-k} = e^{i\omega}a_{N-k+1}$ for every $1 \le k \le N$ and $\omega \in [0.2\pi)$, in other words, the compression of T_φ to $\bigvee\{1, z, \ldots, z^N\}$ is a generalized circulant matrix. In [FL2], the hyponormality of Toeplitz operators with circulant polynomial symbols was completely characterized.

Suppose $\varphi(z) = \sum_{k=-n}^n a_k z^k$ is a circulant polynomial. If $\varphi(0) = 0$, then we may write

$$\varphi = b + e^{i\omega}\overline{z^{n+1}}b \in L^\infty \quad (b \in \mathcal{H}^0(z^{n+1})),$$

where

$$\mathcal{H}^0(\theta) := \big\{h \in \mathcal{H}(\theta) : h(0) = 0\big\}.$$

More generally, a function $\varphi \in L^\infty$ is called a *circulant function* if

$$\varphi = f + \overline{\theta}f \quad (\theta \text{ is inner, } f \in \mathcal{H}^0(\theta)).$$

We introduce:

Definition 3.1. For $\Phi \in L^\infty_{M_n}$, Φ is called a (*matrix-valued*) *circulant function* if

$$\Phi = A + \Theta^* A,$$

where $\Theta := I_\theta$ for an inner function θ, $A \in \mathcal{K}_\Theta^0 \equiv \{B \in \mathcal{K}_\Theta : B(0) = 0\}$, and $\det A$ is not identically zero.

On the other hand, if

$$\Phi = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{bmatrix} \in L^\infty_{M_n},$$

then Φ is a circulant function if and only if each φ_{ij} is a circulant function of the form

$$\varphi_{ij} = f_{ij} + \overline{\theta} f_{ij} \quad (f_{ij} \in \mathcal{H}^0(\theta)).$$

Since $A \in \mathcal{K}_\Theta^0$ we have that $\Phi_+ = A$ and $\Phi_-^* = \Theta^* A$. In particular, if Φ is a circulant function then Φ^* is of bounded type because $\Phi_- = \Theta A^* \in H^2_{M_n}$, so that $A^* = \Theta^* \Phi_-$ and $\Phi^* = A^* + \Theta A^* = \Theta^* \Phi_- + \Phi_- = \Theta^*(\Phi_- + \Theta \Phi_-)$.

The authors of [GHR] characterized the hyponormality of T_Φ with symbol Φ satisfying $||\Phi_+||_2 = ||\Phi_-||_2$: for given $\Phi = \Phi_+ + \Phi_-^* \in L^\infty_{M_n}$, if $||\Phi_+||_2 = ||\Phi_-||_2$ and $\det \Phi_+$ is not identically zero, then T_Φ is hyponormal if and only if $\Phi^*\Phi = \Phi\Phi^*$ and $\Phi_+ = \Phi_- K$ for some inner matrix function $K \in H^\infty_{M_n}$.

The following lemma says that the hyponormality and the pseudo-hyponormality coincide for the cases of circulant symbols.

Lemma 3.2. *Let $\Psi = A + \Theta^* A \in L^\infty_{M_n}$ be a circulant function. Then the following statements are equivalent:*

(a) *T_Ψ is hyponormal;*
(b) *T_Ψ is pseudo-hyponormal;*
(c) *$K := (A^*)^{-1}\Theta^* A$ is the only inner matrix function in $\mathcal{E}(\Psi)$.*

Proof. (a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (c): Suppose T_Ψ is pseudo-hyponormal. Since

$$||\Psi_-||_2^2 = \int_{\mathbb{T}} \operatorname{tr}\, [(\Theta A^*)^* \Theta A^*]\, d\mu = \int_{\mathbb{T}} \operatorname{tr} AA^*\, d\mu = ||\Psi_+||_2^2,$$

it follows from the preceding remark that there exists an inner matrix function $K \in H^\infty_{M_n}$ such that $A = A^* \Theta K$. Thus $K = (A^*)^{-1}\Theta^* A$ because $\Theta = I_\theta$.

(c) $\Rightarrow$ (a): Suppose $K := (A^*)^{-1}\Theta^* A$ is an inner matrix function in $\mathcal{E}(\Psi)$. Then T_Ψ is pseudo-hyponormal and $A = A^* \Theta K$ because $\Theta = I_\theta$. Since K is an inner matrix, it follows that

$$AA^* = A^* \Theta K K^* \Theta^* A = A^* A,$$

which implies that A is normal, and hence Ψ is also normal. Therefore T_Ψ is hyponormal. $\square$

We are ready to prove the main theorem, which is a kind of the extension property of the symbol. It provides a necessary and sufficient condition for the symbol of a hyponormal Toeplitz operators with circulant symbols to be pulled up without losing the hyponormality.

Theorem 3.3. *Let $\Psi = A + \Theta^* A \in L^\infty_{M_n}$ be a circulant function and let $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be of the form*

$$\Phi_+ := A\Theta_0 + B \quad \text{and} \quad \Phi_- := \Theta A^* \Theta_0 + C,$$

where $\Theta_0 = I_{\theta_0}$ for an inner function θ_0 and $B, C \in \mathcal{K}_{I_z \Theta_0}$. Then

$$T_\Phi \text{ is pseudo-hyponormal} \iff (A^*)^{-1}\Theta^* A \in \mathcal{E}(C^* + B).$$

Moreover, if T_Φ is hyponormal then the rank of the self-commutator of T_Φ is computed from the formula

$$\operatorname{rank}\, [T_\Phi^*, T_\Phi] = \deg \Big[\det \big((A^*)^{-1}\Theta^* A\big)\Big], \tag{3.1}$$

where $\deg(k)$ *denotes the degree of* k *- meaning the number of zeros of* k *(in the open unit disk* $\mathbb{D}$*) if* k *is a finite Blaschke product and* ∞ *otherwise.*

Proof. Suppose T_Φ is pseudo-hyponormal. Since $\Theta = I_\theta$, we know that

$$\begin{aligned}\Phi_+ &:= A\Theta_0 + B = (A\Theta^* + B\Theta^*\Theta_0^*)\,\Theta_0\Theta = (\Theta A^* + \Theta_0\Theta B^*)^*\,\Theta_0\Theta\\ \Phi_- &:= \Theta A^*\Theta_0 + C = (A + \Theta_0\Theta C^*)^*\,\Theta_0\Theta.\end{aligned}$$

Thus it follows that

$$\Phi_\Theta = \Theta\,[P_{\mathcal{K}_\Theta}(\Theta A^* + \Theta\Theta_0 B^*)]^* + [P_{\mathcal{K}_\Theta}(A + \Theta\Theta_0 C^*)]\,\Theta^*.$$

But since $B \in \mathcal{K}_{I_z\Theta_0}$, we have that $\langle B,\, H^2_{M_n} I_z\Theta_0\rangle = 0$. Thus $B\Theta_0^* \in L^2_{M_n} \ominus H^2_{M_n} I_z$, and hence $\Theta_0 B^* \in H^2_{M_n}$ and similarly, $\Theta_0 C^* \in H^2_{M_n}$. This implies that $P_{\mathcal{K}_\Theta}(\Theta\Theta_0 B^*) = 0 = P_{\mathcal{K}_\Theta}(\Theta\Theta_0 C^*)$, so that

$$\Phi_\Theta = \Theta\,[P_{\mathcal{K}_\Theta}(\Theta A^*)]^* + [P_{\mathcal{K}_\Theta}(A)]\,\Theta^* = \Theta\,(\Theta A^*)^* + A\Theta^* = A + \Theta^* A = \Psi.$$

By Lemma 2.6, T_Ψ is pseudo-hyponormal and $\mathcal{E}(\Phi) \subseteq \mathcal{E}(\Psi)$. By Lemma 3.2, $K := (A^*)^{-1}\Theta^* A$ is the only inner function in $\mathcal{E}(\Psi)$. Since $\mathcal{E}(\Phi) \subseteq \mathcal{E}(\Psi)$ and $\mathcal{E}(\Psi)$ is a singleton set, it follows that $\mathcal{E}(\Phi) = \{K\}$, so that

$$\begin{aligned}\Phi_-^* - K\Phi_+^* &= \Theta_0^* A\Theta^* + C^* - K(\Theta_0^* A^* + B^*)\\ &= \big(\Theta_0^* A - (A^*)^{-1} A\Theta_0^* A^*\big)\,\Theta^* + \big(C^* - (A^*)^{-1}\Theta^* A B^*\big) \in H^2_{M_n}.\end{aligned}\tag{3.2}$$

Since $\Theta_0 = I_{\theta_0}$, (3.2) reduces to $C^* - (A^*)^{-1}\Theta^* A B^* \in H^2_{M_n}$ because A is normal. But since $K = (A^*)^{-1}\Theta^* A$ is an inner function, it follows that $(A^*)^{-1}\Theta^* A \in \mathcal{E}(C^* + B)$. The converse is evident from (3.2).

Towards the rank formula (3.1), suppose that T_Φ is hyponormal. Since $K \equiv (A^*)^{-1}\Theta^* A \in \mathcal{E}(C^* + B)$, it follows that for some $F \in H^2_{M_n}$,

$$C^* - KB^* = F, \quad \text{i.e.,} \quad B = CK - F^*K. \tag{3.3}$$

We thus have

$$\Phi_+ = \Phi_- K - F^*K, \quad \text{or equivalently,} \quad \Phi_- = \Phi_+ K^* + F^*. \tag{3.4}$$

Observe by (1.6),

$$[T_\Phi^*, T_\Phi] = H^*_{\Phi_+^*} H_{\Phi_+^*} - H^*_{\Phi_-^*} H_{\Phi_-^*} = H^*_{\Phi_+^*} H_{\Phi_+^*} - H^*_{K\Phi_+^*} H_{K\Phi_+^*} = H^*_{\Phi_+^*} H_{K^*} H^*_{K^*} H_{\Phi_+^*}. \tag{3.5}$$

We now claim that

$$\ker H^*_{\Phi_+^*} H_{K^*} = \ker H_{K^*}. \tag{3.6}$$

Towards (3.6), let $g \in \ker H^*_{\Phi_+^*} H_{K^*}$. We write

$$g = g_1 + Kg_2 \quad \text{where } g_1 \in \mathcal{H}(K) \text{ and } g_2 \in H^2_{\mathbb{C}^n}.$$

We then have

$$\begin{aligned}H^*_{\Phi_+^*} H_{K^*} g &= H^*_{\Phi_+^*} J P_n^\perp\big(K^*(g_1 + Kg_2)\big)\\ &= H^*_{\Phi_+^*} J(K^* g_1)\\ &= J P_n^\perp\big(\widetilde{\Phi}_+^*(z) I_{\overline{z}} K^*(\overline{z}) g_1(\overline{z})\big)\\ &= J P_n^\perp\Big[\Big(\Phi_-(\overline{z})K(\overline{z}) - F^*(\overline{z})K(\overline{z})\Big) I_{\overline{z}} K^*(\overline{z}) g_1(\overline{z})\Big] \quad \text{(by (3.4))}\\ &= J P_n^\perp\Big[\Big(\Phi_-(\overline{z}) - C(\overline{z}) + B(\overline{z})K^*(\overline{z})\Big) I_{\overline{z}} g_1(\overline{z})\Big] \quad \text{(by (3.3))}\\ &= P_n J\Big(I_{\overline{z}}(\Theta A^*\Theta_0)(\overline{z}) g_1(\overline{z}) + B(\overline{z})K^*(\overline{z}) I_{\overline{z}} g_1(\overline{z})\Big) \quad \text{(because } J P_n^\perp = P_n J)\\ &= P_n\Big(I_{\overline{z}} I_z \Theta A^*\Theta_0 g_1 \;+\; I_{\overline{z}} B K^* I_z g_1\Big)\\ &= \Theta A^*\Theta_0 g_1 \;+\; P_n(BK^* g_1).\end{aligned}\tag{3.7}$$

But since K^*g_1 is co-analytic and $B \in \mathcal{K}_{I_z\Theta_0}$, it follows that $BK^*g_1 \in L^2_{C^n} \ominus \Theta_0 H^2_{C^n}$: indeed, for any $d \in H^2_{C^n}$,

$$\langle BK^*g_1,\, \Theta_0 d\rangle = \langle (\Theta_0^* B)(K^*g_1),\, d\rangle = 0.$$

Therefore we have that $P_n(BK^*g_1) \in \mathcal{H}(\Theta_0)$. Since $\Theta A^*\Theta_0 g_1 \in \Theta_0 H^2_{C^n}$, it follows from (3.7) that $H^*_{\Phi_+^*} H_{K^*} g$ cannot be zero unless $g_1 = 0$, which says that $g \in \ker H^*_{\Phi_+^*} H_{K^*}$ only if $g = Kg_2$. Consequently, $\ker H^*_{\Phi_+^*} H_{K^*} \subseteq \ker H_{K^*}$, which proves (3.6). Thus by (3.5) and (3.6),

$$\begin{aligned}
\text{rank}\,[T_\Phi^*,\, T_\Phi] &= \text{rank}\,\left(H^*_{\Phi_+^*} H_{K^*}\right)\\
&= \dim\left(H^2_{\mathbb{C}^n} \ominus \ker H^*_{\Phi_+^*} H_{K^*}\right)\\
&= \dim\left(H^2_{\mathbb{C}^n} \ominus \ker H_{K^*}\right)\\
&= \dim\left(H^2_{\mathbb{C}^n} \ominus K\, H^2_{\mathbb{C}^n}\right)\\
&= \deg\,(\det K)\,,
\end{aligned}$$

where the last equality comes from the following observation:

$$\begin{aligned}
\dim\left(H^2_{\mathbb{C}^n} \ominus K\, H^2_{\mathbb{C}^n}\right) &= \dim\ker T_{K^*} = -\text{index}\, T_K = -\text{index}\, T_{\det K}\\
&= \dim\ker T_{\overline{\det K}} = \dim \mathcal{H}(\det K)\\
&= \deg\,(\det K)\,,
\end{aligned}$$

where the third equality comes from the well-known Fredholm theory of block Toeplitz operators since $\det K \neq 0$ (cf. [Pe, Theorem 3.4.8]). This proves the theorem. $\square$

We give a revealing example.

Example 3.4. Let $a_j, b_j \in \mathbb{C}$ for $j = 1, 2$ and consider the matrix-valued trigonometric polynomial

$$\Phi \equiv \begin{bmatrix} z^{-2} + z^{-1} - z + z^2 & -2z^{-2} + b_1 z^{-1} + a_1 z - 2z^2 \\ 2z^{-2} + b_2 z^{-1} + a_2 z + 2z^2 & z^{-2} + z^{-1} - z + z^2 \end{bmatrix}.$$

Write

$$\Phi_0 \equiv \Phi_{I_z} = \begin{bmatrix} z^{-1} + z & -2z^{-1} - 2z \\ 2z^{-1} + 2z & z^{-1} + z \end{bmatrix} = \begin{bmatrix} z & -2z \\ 2z & z \end{bmatrix} + \begin{bmatrix} z^2 & 0 \\ 0 & z^2 \end{bmatrix}^* \begin{bmatrix} z & -2z \\ 2z & z \end{bmatrix}.$$

Evidently, Φ_0 is a circulant function and T_{Φ_0} is normal. Write

$$A := \begin{bmatrix} z & -2z \\ 2z & z \end{bmatrix}, \quad \Theta := I_{z^2}, \quad B := \begin{bmatrix} -z & a_1 z \\ a_2 z & -z \end{bmatrix}, \quad C := \begin{bmatrix} z & \overline{b_2} z \\ \overline{b_1} z & z \end{bmatrix}.$$

By Theorem 3.3, we know that

$$T_\Phi \text{ is pseudo-hyponormal } \iff (A^*)^{-1}\Theta^* A \in \mathcal{E}(C^* + B).$$

Observe that

$$\begin{aligned}
(A^*)^{-1}\Theta^* A &= \begin{bmatrix} \overline{z} & 2\overline{z} \\ -2\overline{z} & \overline{z} \end{bmatrix}^{-1} I_{z^{-2}} \begin{bmatrix} z & -2z \\ 2z & z \end{bmatrix}\\
&= \frac{1}{5}\begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix}.
\end{aligned}$$

Thus T_Φ is pseudo-hyponormal if and only if

$$\begin{bmatrix} z^{-1} & b_1 z^{-1} \\ b_2 z^{-1} & z^{-1} \end{bmatrix} - \frac{1}{5}\begin{bmatrix} -3 & -4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -z^{-1} & \overline{a_2} z^{-1} \\ \overline{a_1} z^{-1} & -z^{-1} \end{bmatrix} \in H^2_{M_n},$$

or equivalently,

$$\begin{cases} z^{-1}-\left(\frac{3}{5}z^{-1}-\frac{4}{5}\overline{a_1}z^{-1}\right)\in H^2\\ b_1z^{-1}-\left(-\frac{3}{5}\overline{a_2}z^{-1}+\frac{4}{5}z^{-1}\right)\in H^2\\ b_2z^{-1}-\left(-\frac{4}{5}z^{-1}-\frac{3}{5}\overline{a_1}z^{-1}\right)\in H^2\\ z^{-1}-\left(\frac{4}{5}\overline{a_2}z^{-1}+\frac{3}{5}z^{-1}\right)\in H^2, \end{cases}$$

or equivalently,

$$a_1=-\frac{1}{2},\quad a_2=\frac{1}{2},\quad b_1=\frac{1}{2},\quad b_2=-\frac{1}{2}.$$

In fact, in this case, a straightforward calculation shows that T_Φ is normal.

The next corollary gives a nice rank formula for the self-commutators of T_Φ if the symbol Φ is a circulant polynomial.

Corollary 3.5. *Let $\Phi\in L^\infty_{M_n}$ be a matrix-valued circulant polynomial of the form*

$$\Phi(z):=\sum_{j=-r}^{r}A_jz^j\equiv\Theta^*A+A,$$

where $\Theta=I_\theta$ with $\theta:=e^{i\xi}z^{r+1}$ for some $\xi\in\mathbb{R}$ and let F denote the analytic matrix polynomial

$$F(z):=\sum_{j=1}^{r}A_jz^{j-1}.$$

If T_Φ is pseudo-hyponormal then for every zero ζ of $\det F$ such that $|\zeta|>1$, the number $1/\overline{\zeta}$ is a zero of $\det F$ in $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ. Moreover, if T_Φ is hyponormal then the rank of the self-commutator of T_Φ is given by

$$\operatorname{rank}\,[T_\Phi^*,T_\Phi]=Z_{\mathbb{D}}-Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}-\Big(n(r-1)-\deg\big(\det F\big)\Big), \tag{3.8}$$

where $Z_{\mathbb{D}}$ and $Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ are the number of zeros of $\det F$ in $\mathbb{D}$ and in $\mathbb{C}\setminus\overline{\mathbb{D}}$ counting multiplicity. In particular, if the analytic outer coefficient A_r is invertible then

$$\operatorname{rank}\,[T_\Phi^*,T_\Phi]=Z_{\mathbb{D}}-Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}\,. \tag{3.9}$$

Proof. Note that $A(z):=\sum_{j=1}^{r}A_jz^j$. If T_Φ is pseudo-hyponormal then by Lemma 3.2, $K\equiv(A^*)^{-1}\Theta^*A$ is the only inner matrix function in $\mathcal{E}(\Phi)$. Thus $\det K=\det A\,\left(\det\Theta\,\overline{\det A}\right)^{-1}$ is a scalar-valued inner function. Observe

$$\det K=\frac{\det A}{\det\Theta\,\overline{\det A}}=\frac{z^n\det F}{e^{in\xi}z^{n(r+1)}\overline{z^n\det F}}=e^{-in\xi}\,\frac{\det F}{z^{n(r-1)}\overline{\det F}}. \tag{3.10}$$

Since $\det K$ is inner and $\det F$ is a polynomial it follows that $\det K$ is a finite Blaschke product. Therefore for every zero ζ of $\det F$ such that $|\zeta|>1$, the number $1/\overline{\zeta}$ is a zero of $\det F$ in $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ. Towards (3.8) suppose that T_Φ is hyponormal. If $\deg(\det F)=m\le n(r-1)$, then we can write

$$\det K=e^{-in\xi}\,\frac{z^p\prod_{j=1}^{m-p}(z-\alpha_j)}{z^{n(r-1)}z^{-m}\prod_{j=1}^{m-p}(1-\overline{\alpha_j}z)}\quad(\alpha_j\ne 0).$$

Since $\det K$ is a finite Blaschke product it follows that $n(r-1)-m\le p$ and

$$\deg(\det K)=Z_{\mathbb{D}}-Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}-\Big(n(r-1)-m\Big).$$

Thus the formula (3.8) follows from Theorem 3.3. On the other hand, if A_r is invertible then $m=n(r-1)$, which together with (3.8) gives (3.9). $\square$

References

[Ab] M.B. Abrahamse, *Sunormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43** (1976), 597–604.

[BH] A. Brown and P.R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213**(1963/1964), 89–102.

[Co1] C. Cowen, *Hyponormal and subnromal Toeplitz operators*, Survey of Some Recent Results in Operator Theory, I (J.B. Conway and B.B. Morrel, eds.), Pitman Research Notes in Mathematics, Volume 171, Longman, 1988, pp. (155-167).

[Co2] C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103**(1988), 809–812.

[CHL] R.E. Curto, I.S. Hwang and W.Y. Lee, *Which subnormal Toeplitz operators are either normal or analytic ?*, (preprint, 2010).

[CL] R.E. Curto and W.Y. Lee, *Joint hyponormality of Toeplitz pairs*, Memoirs Amer. Math. Soc. **712**, Amer. Math. Soc., Providence, 2001.

[DD] V. Derkach and H. Dym, *On linear fractional transformations associated with generalized J-inner matrix functions*, Integral Equations Operator Theory **65**(2009), 1–50.

[Do1] R.G. Douglas, *Banach algenra techniques in operator theory*, Academic Press, New York, 1972.

[Do2] R.G. Douglas, *Banach algenra techniques in the theory of Toeplitz operators*, CBMS 15, Providence, Amer. Math. Soc. 1973.

[FKKL] D.R. Farenick, K. Krupnik, N. Krupnik and W.Y. Lee, *Normal Toeplitz matrices*, SIAM J. Matrix Anal. Appl. **17**(1996), 1037–1043.

[FL1] D.R. Farenick and W.Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348**(1996), 4153–4174.

[FL2] D.R. Farenick and W.Y. Lee, *On hyponormal Toeplitz operators with polynomial and circulant-type symbols*, Integral Equation and Operator Theory **29**(1997), 202–210.

[FF] C. Foias and A. Frazo, *The commutant lifting approach to interpolation problems*, Operator Theory: Adv. Appl. vol 44, Birkhäuser, Boston, 1993.

[Fu] P.A. Fuhrmann, *On the corona theorem and its applications to spectral problems in Hilbert space*, Trans. Amer. Math. Soc. **132**(1968), 55-66.

[GGK] I. Gohberg, S. Goldberg and M.A. Kaashoek, *Classes of linear operators, Vol II*, Basel, Birkhauser, 1993.

[Gu] C. Gu, *A generalization of Cowen's characterization of hyponormal Toeplitz operators*, J. Funct. Anal. **124**(1994), 135–148.

[GHR] C. Gu, J. Hendricks and D. Rutherford, *Hyponormality of block Toeplitz operators*, Pacific J. Math. **223** (2006), 95–111.

[GS] C. Gu and J.E. Shapiro, *Kernels of Hankel operators and hyponormality of Toeplitz operators*, Math. Ann. **319**(2001), 553–572.

[HK] I. S. Hwang and I. H. Kim, *Hyponormality of Toeplitz operators with genralized circulant symbols*, J. Math. Anal. Appl. **349(1)** (2009), 264-271.

[HKL1] I. S. Hwang, I. H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols*, Math. Ann. **313(2)** (1999), 247-261.

[HKL2] I. S. Hwang, I. H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols: An extremal case*, Math. Nach. **231** (2001), 25-38.

[HL1] I. S. Hwang and W.Y. Lee, *Hyponormality of trigonometric Toeplitz operators*, Trans. Amer. Math. Soc. **354** (2002), 2461-2474.

[HL2] I. S. Hwang and W.Y. Lee, *Hyponormality of Toeplitz operators with rational symbols*, Math. Ann. **335**(2006), 405–414.

[HL3] I. S. Hwang and W.Y. Lee, *Hyponormal Toeplitz operators with rational symbols*, J. Operator Theory **56**(2006), 47–58.

[HL4] I. S. Hwang and W.Y. Lee, *Block Toeplitz Operators with rational symbols*, J. Phys. A: Math. Theor. **41(18)**(2008), 185207.

[HL5] I. S. Hwang and W.Y. Lee, *Block Toeplitz Operators with rational symbols (II)*, J. Phys. A: Math. Theor. **41(38)**(2008), 385205.

[IC] Kh.D. Ikramov and V.N. Chugunov, *Normality conditions for a complex Toeplitz matrix*, Zh. Vychisl. Mat. i Mat. Fiz. **36**(1996), 3–10.

[It] T. Ito, *Every normal Toeplitz matrix is either of type(I) or type (II)*, SIAM J. Matrix Anal. Appl. **17**(1996), 998–1006.

[Le] W. Y. Lee, *Cowen sets for Toeplitz operators with finite rank selfcommutators*, J. Operator Theory **54(2)**(2005), 301-307.

[NT] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338**(1993), 753–769.

[Ni] N. K. Nikolskii, *Treatise on the shift operator*, Springer, New York, 1986.

[Pe] V. V. Peller, *Hankel operators and their applications*, Springer, New York, 2003.

[Zhu] K. Zhu, *Hyponormal Toeplitz operators with polynomial symbols*, Integral Equations Operator Theory **21**(1996), 376–381

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