THE MEAN TRANSFORM OF BOUNDED LINEAR OPERATORS

SANG HOON LEE, WOO YOUNG LEE, AND JASANG YOON

Abstract. In this paper we introduce the mean transform of bounded linear operators acting on a complex Hilbert space and then explore how the mean transform of weighted shifts behaves, in comparison with the Aluthge transform.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, let $T = U|T|$ be the polar decomposition of T. The *Aluthge transform* $\widetilde{T}$ of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. This transform was first studied in [1] and has received much attention in recent years, in particular, in relation to the invariant subspace problem. The *Duggal transform* $\widetilde{T}^D$ of T is defined by $\widetilde{T}^D = |T|U$, which is first referred to in [12]. Clearly, the spectrum of $\widetilde{T}$ $\left(\text{resp. } \widetilde{T}^D\right)$ equals that of T. For $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \cdots) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *(unilateral) weighted shift*, defined by $W_\alpha e_k := \alpha_k e_{k+1}$ (all $k \geq 0$), where $\{e_k\}_{k=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. If $\widetilde{W_\alpha}$ is the Aluthge transform of W_α, then we can see that $\widetilde{W_\alpha} = \text{shift}(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \cdots)$, where we note that each term of weights of $\widetilde{W_\alpha}$ consists of the geometric mean of two consecutive terms of W_α. In this paper we introduce a new transform: if $T = U|T|$ is the polar decomposition of T, then we define

$$\widehat{T} := \frac{1}{2}\left(U|T| + |T|\,U\right) \equiv \frac{1}{2}\left(T + \widetilde{T}^D\right),$$

which will be called the *mean transform* of T and then examine various questions on the mean transform. In particular we will focus on the mean transform of weighted shifts. If $\widehat{W_\alpha}$ is the mean transform of the weighted shift $W_\alpha = \text{shift}(\alpha_0, \alpha_1, \cdots)$, then we can see that $\widehat{W_\alpha} = \text{shift}\left(\frac{\alpha_0+\alpha_1}{2}, \frac{\alpha_1+\alpha_2}{2}, \cdots\right)$ (see Proposition 2.2 below). In comparison with the Aluthge transform of weighted shifts, the weights of the mean transform of weighted shifts consist of the arithmetic means of two consecutive weights of W_α. This suggests there would be a significant difference or resemblance between the Aluthge transform and the mean transform. First of all, we list problems in which we are interested:

Problem 1.1. Does the spectrum of $\widehat{T}$ equal that of T ?

Problem 1.2. Given the mean transform map $T \to \widehat{T}$, (i) is it $(\|\cdot\|, \|\cdot\|)-$ continuous on $\mathcal{B}(\mathcal{H})$?; (ii) is it $(\|\cdot\|, SOT)-$ continuous on $\mathcal{B}(\mathcal{H})$?

Problem 1.3. For $k \geq 1$, if W_α is k-hyponormal, does it follow that the mean transform $\widehat{W_\alpha}$ is also k-hyponormal ?

Problem 1.4. If W_α is subnormal with Berger measure μ, does it follow that $\widehat{W_\alpha}$ is subnormal ? If it does, what is the Berger measure of $\widehat{W_\alpha}$?

MSC(2010): Primary 47B49, 47B20, 47B37.
Keywords: The mean transform; the Aluthge transform; k-hyponormal; subnormal; weighted shifts
The first named author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No.2010-0028171). The second named author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No.2009-0083521).

In Section 2 we provide basic properties of the mean transform $\widehat{T}$. In Section 3 we consider the k-hyponormality and the subnormality for the mean transform of the weighted shifts and moreover the continuity properties of the mean transform.

2. Basic properties of the mean transform $\widehat{T}$

If $\widehat{T}$ is the mean transform of T, then we can easily check that $||\widehat{T}|| \leq ||T||$ in general. How about the spectrum of $\widehat{T}$? It is well known that the spectrum of the Aluthge transform $\widetilde{T}$ (resp. the Duggal transform) equals that of T. We may ask what happens for the spectrum of the mean transform $\widehat{T}$ of T. We first give an answer for Problem 1.1. For this, we let $P \in \mathcal{B}(\mathcal{H})$ be a positive operator and consider an operator matrix $T := \left(\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then $\sigma(T) = \{0\}$. A direct calculation shows that $T = U|T|$, with $U := \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ and $|T| := \left(\begin{smallmatrix} 0 & 0 \\ 0 & P \end{smallmatrix}\right)$. We thus have that $\widehat{T} = \frac{1}{2}\left(\begin{smallmatrix} 0 & P \\ P & 0 \end{smallmatrix}\right)$. Observe that

$$\widehat{T}^2 = \frac{1}{4}\begin{pmatrix} P^2 & 0 \\ 0 & P^2 \end{pmatrix}, \text{ and hence } \sigma(\widehat{T}^2) = \left\{\frac{\sigma\left(P^2\right)}{4}\right\},$$

which implies $\sigma(\widehat{T}) = \left\{\pm\frac{\sigma(P)}{2}\right\}$. Thus we obtain:

Example 2.1. Let $T := \left(\begin{smallmatrix} 0 & P \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, where $P \in \mathcal{B}(\mathcal{H})$ is a positive operator. Then we have
(i) $\sigma(T) = \{0\}$;
(ii) $\sigma(\widehat{T}) = \left\{\pm\frac{\sigma(P)}{2}\right\}$.
Hence, in particular, $\sigma(T) \neq \sigma(\widehat{T})$ if $P \neq 0$, while $||\widehat{T}|| \leq ||T||$.

Since the Duggal transform $\widetilde{T}^D$ shares many spectral properties with T (besides $\sigma(T) = \sigma(\widetilde{T}^D)$) and $\widehat{T} = \frac{1}{2}(T + \widetilde{T}^D)$, one might be tempted to guess that $\sigma(T) \subseteq \sigma(\widehat{T})$. But Example 2.1 illustrates that this is not such a case: consider the case $P = I$. On the other hand, we note that if we define $d(T)$ for the deviation from the normaloid-ness (normaloid means that norm equals spectral radius) by

$$d(T) := ||T|| - r(T) \quad \text{(where } r(T) \text{ denotes the spactral radius of } T\text{)},$$

then T in Example 2.1 has $d(\widehat{T}) = 0$, i.e., $\widehat{T}$ is normaloid, even though $d(T) = ||P||$. Thus it may happen that $\widehat{T}$ becomes a nice operator (i.e., normaloid) by filling out something (i.e., $r(\widehat{T}) = \frac{r(P)}{2}$, but $r(T) = 0$), but by contrast, the Aluthge transform $\widetilde{T}$ becomes a nice operator by collapsing something (i.e., $\widetilde{T} = 0$, but $T \neq 0$).

The iterated mean transforms (or mean iterates) of an operator T are the operators $\widehat{T}^{(n)}$ $(n \geq 0)$, defined by setting $\widehat{T}^{(0)} = T$ and letting $\widehat{T}^{(n+1)}$ be the mean transform of $\widehat{T}^{(n)}$.

We then have:

Proposition 2.2. *For a weighted shift W_α, the mean iterates $\widehat{W_\alpha}^{(n)}$ are also weighted shifts with weight sequences*

$$\alpha^{(n)} \equiv \left\{\alpha_i^{(n)}\right\}_{i=0}^{\infty} := \left\{\frac{\sum_{j=0}^{n}\binom{n}{j}\alpha_{i+j}}{2^n}\right\}_{i=0}^{\infty}, \tag{2.1}$$

where $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.

Proof. We use an induction on n. For $n = 1$, we note that the polar decomposition of W_α is $U_+ D_\alpha$, where $D_\alpha := diag(\alpha_0, \alpha_1, \cdots)$ and U_+ is the unilateral shift. Thus we have that $\widehat{W}_\alpha = \frac{1}{2}(U_+ D_\alpha + D_\alpha U_+)$. For $n \geq 0$ and the orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\ell^2(\mathbb{Z}_+)$, we can see that

$$U_+ D_\alpha (e_n) = \alpha_n U_+ (e_n) = \alpha_n e_{n+1}$$

and

$$D_\alpha U_+ (e_n) = D_\alpha (e_{n+1}) = \alpha_{n+1} e_{n+1}.$$

Therefore we have that

$$\widehat{W}_\alpha (e_n) = \frac{1}{2}(U_+ D_\alpha + D_\alpha U_+)(e_n) = \frac{1}{2}(\alpha_n + \alpha_{n+1}) e_{n+1}.$$

Thus (2.1) holds for $n = 1$. We assume that (2.1) holds for $n = m$. Then we have that

$$\alpha_i^{(m+1)} = \frac{\alpha_i^{(m)} + \alpha_{i+1}^{(m)}}{2} = \frac{\frac{\sum_{j=0}^{m}\binom{m}{j}\alpha_{i+j}}{2^m} + \frac{\sum_{j=0}^{m}\binom{m}{j}\alpha_{i+1+j}}{2^m}}{2} = \frac{\sum_{j=0}^{m}\binom{m}{j}(\alpha_{i+j} + \alpha_{i+1+j})}{2^{m+1}}.$$

Since $\binom{m}{j} + \binom{m}{j+1} = \binom{m+1}{j+1}$, $\binom{m}{0} = \binom{m+1}{0}$ and $\binom{m}{m} = \binom{m+1}{m+1}$, we can see that

$$\sum_{j=0}^{m}\binom{m}{j}(\alpha_{i+j} + \alpha_{i+1+j}) = \sum_{j=0}^{m+1}\binom{m+1}{j}\alpha_{i+j}.$$

Thus (2.1) holds for $n = m + 1$. This completes the proof. □

Remark 2.3. By Proposition 2.2, we can see that if $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \cdots)$, then the mean transform $\widehat{W}_\alpha$ of W_α is

$$\text{shift}\left(\frac{\alpha_0 + \alpha_1}{2}, \frac{\alpha_1 + \alpha_2}{2}, \cdots\right). \tag{2.2}$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is called *normal* if $T^*T = TT^*$, *hyponormal* if its self-commutator $[T^*, T] := T^*T - TT^*$ is positive (semi-definite), and *subnormal* if there exists a normal operator N on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant under N and $N|_{\mathcal{H}} = T$. Also $T \in \mathcal{B}(\mathcal{H})$ is called *quasinormal* if T commutes with T^*T. It is well known that normal $\Rightarrow$ quasinormal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal.

For a weighted shift W_α, it is easy to see that W_α is hyponormal if and only if $\alpha_0 \leq \alpha_1 \leq \cdots$. Thus by (2.2), the mean transform $\widehat{W}_\alpha$ of W_α is hyponormal if and only if $\frac{1}{2}(\alpha_n + \alpha_{n+1}) \leq \frac{1}{2}(\alpha_{n+1} + \alpha_{n+2})$ if and only if $\alpha_n \leq \alpha_{n+2}$ (all $n \geq 0$) (in fact, $\widetilde{W}_\alpha$ is hyponormal if and only if $\widehat{W}_\alpha$ is hyponormal, which says that the mean transform often shares some properties with the Aluthge transform). Hence, if W_α is hyponormal, then the mean transform $\widehat{W}_\alpha$ of W_α is hyponormal. However, the converse is not true in general. For example, if $W_\alpha \equiv \text{shift}\left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \cdots\right)$, then W_α is clearly not hyponormal but the mean transform $\widehat{W}_\alpha = U_+$ is subnormal.

Proposition 2.4. *$\widehat{T} = T$ if and only if T is quasinormal.*

Proof. Note that

$$\widehat{T} = T \Longleftrightarrow \tfrac{1}{2}(U|T| + |T|U) = U|T| \Longleftrightarrow |T|U = U|T|,$$

which gives the desired result. □

3. Main results

We first recall that for $k \geq 1$, $T \in \mathcal{B}(\mathcal{H})$ is *k-hyponormal* if the $(k+1) \times (k+1)$ matrix

$$\begin{pmatrix} I & T^* & T^{*^2} & \cdots & T^{*^k} \\ T & T^*T & T^{*^2}T & \cdots & T^{*^k}T \\ T^2 & T^*T^2 & T^{*^2}T^2 & \cdots & T^{*^k}T^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ T^k & T^{*^2}T^k & T^{*^2}T^k & \cdots & T^{*^k}T^k \end{pmatrix}$$

is positive semi-definite. The Bram-Halmos characterization of subnormality ([2, III.1.9]) can be rephrased as follow: T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([7, Proposition 1.9]). We recall that the *moments* of W_α are given as

$$\gamma_n \equiv \gamma_n(W_\alpha) := \begin{cases} 1, & \text{if } n = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{n-1}^2, & \text{if } n > 0 \end{cases}. \tag{3.1}$$

It is well known that W_α is subnormal if and only if

$$\gamma_n = \int_{[0,||W_\alpha||^2]} s^n \, d\mu(s) \quad (\text{all } n \geq 0),$$

where μ is a probability measure on the interval $[0, ||W_\alpha||^2]$ (this measure μ is called the *Berger measure* of the subnormal weighted shift W_α) (cf. [2]).

We now consider whether the mean transform $\widehat{W_\alpha}$ of W_α preserves k-hyponormality. Below, we will show that there exists a subnormal weighted shift W_α such that $\widehat{W_\alpha}$ is not 2-hyponormal. For this, we need the following:

Lemma 3.1. ([3]) *Let W_α be a weighted shift with the moments $\{\gamma_n\}$. The following statements are equivalent:*

(i) *W_α is k-hyponormal.*
(ii) *The Hankel matrix*

$$H(k;n) := (\gamma_{n+i+j-2})_{i,j=1}^{k+1}$$

is positive semi-definite for all $n \geq 0$.

Lemma 3.2. (cf. [15]) *Let $M \equiv \begin{pmatrix} C & B^* \\ B & A \end{pmatrix}$ be a 2×2 operator matrix, where A and C are square matrices and B is a rectangular matrix. Then*

$$M \geq 0 \iff \text{there exists } W \text{ such that } \begin{cases} C \geq 0 \\ B = CW \\ A \geq W^*CW. \end{cases}$$

For matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their *Schur product*, i.e., $(A \circ B)_{ij} := A_{ij}B_{ij}$ $(1 \leq i, j \leq n)$. The following result is well known: If $A \geq 0$ and $B \geq 0$, then $A \circ B \geq 0$ ([14]). For $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ and $\beta \equiv \{\beta_n\}_{n=0}^\infty$, the Schur product of α and β is defined by $\alpha \circ \beta := \{\alpha_n \beta_n\}_{n=0}^\infty$. Given two weighted shifts W_α and W_β, their Schur product, denoted by $W_\alpha \circ W_\beta$, is defined by a weighted shift $W_{\alpha \circ \beta}$. It is known [8] that

$$W_\alpha \text{ and } W_\beta \text{ are subnormal} \implies W_\alpha \circ W_\beta \text{ is subnormal.} \tag{3.2}$$

Moreover, it is also shown that if W_α and W_β are k-hyponormal $(k \geq 1)$, then $W_\alpha \circ W_\beta$ is also k-hyponormal (cf. [8]). Let $a, b, c, d \geq 0$ satisfy $ad - bc > 0$. Write $S(a, b, c, d) :=$ shift$(\alpha_0, \alpha_1, \alpha_2, \cdots)$, where $\alpha_n := \sqrt{\frac{an+b}{cn+d}}$ $(n \geq 0)$. We then have:

Lemma 3.3. ([9]) *Let $a, b, c, d \geq 0$ satisfy $ad - bc > 0$. Then $S(a, b, c, d)$ is subnormal.*

Recall that the Bergman shift $B_+ \equiv \text{shift}\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \cdots\right)$ is subnormal with Berger measure $d\mu(s) = ds$. From [11, Theorem 2.8], we know that the Aluthge transform $\widetilde{B}_+$ of the Bergman shift B_+ is subnormal. We may ask whether the mean transform $\widehat{B}_+$ of B_+ is subnormal. A direct calculation using the software tool *Mathematica* [17] shows that

$$\widehat{B}_+ \equiv \text{shift}\left(\frac{2+\sqrt{3}}{2\sqrt{6}}, \frac{3+2\sqrt{2}}{4\sqrt{3}}, \frac{5\sqrt{3}+4\sqrt{5}}{20}, \cdots\right) \tag{3.3}$$

and

$$H(k;n)(\widehat{B}_+) \geq 0 \text{ for } 0 \leq k, n \leq 5. \tag{3.4}$$

Looking at (3.4), it is natural to conjecture that the mean transform $\widehat{B}_+$ of B_+ is subnormal. However, the proof of it is highly nontrivial. Because the weights of $\widehat{B}_+$ shown in (3.3) contain several irrational numbers without a common pattern, so it becomes unwieldy to check $H(k;n)(\widehat{B}_+) \geq 0$ for all $k, n \geq 0$.

Conjecture 3.4. *The mean transform $\widehat{B}_+$ of B_+ is subnormal.*

However, in Theorem 3.6 below, we will show that the mean transform $\widehat{B_+^\circ}$ of $B_+^\circ := B_+ \circ B_+ \equiv \text{shift}\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right)$ is subnormal. Moreover, since

$$\int_0^1 -s^n \ln s ds = \left[\frac{-s^{n+1}\ln s}{n+1}\right]_0^1 + \int_0^1 \frac{s^n}{n+1} ds = \frac{1}{(n+1)^2} = \gamma_n(B_+^\circ) \quad \text{and} \quad \int_0^1 -\ln s ds = 1, \tag{3.5}$$

we can see that the Berger measure of B_+° is $-\ln s ds$. We thus have:

Theorem 3.5. *The Aluthge transform $\widetilde{B_+^\circ}$ of B_+° is subnormal with Berger measure* $2(1-s)ds$.

Proof. We let $B_+^\circ = \text{shift}(\alpha_0, \alpha_1, \cdots)$ and $\widetilde{B_+^\circ} = \text{shift}(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \cdots)$. Then for $n \geq 0$, we have

$$\widetilde{\alpha}_n = \sqrt{\alpha_n \alpha_{n+1}} = \left(\sqrt{\tfrac{n+1}{n+2}}\right)\left(\sqrt{\tfrac{n+2}{n+3}}\right)$$

and

$$\widetilde{B_+^\circ} = S(1,1,1,2) \circ S(1,2,1,3).$$

By Lemma 3.3, we can see that $S(1,1,1,2)$ and $S(1,2,1,3)$ are subnormal. Thus by (3.2), we have that the Aluthge transform $\widetilde{B_+^\circ}$ of B_+° is subnormal, as desired. Also a straightforward calculation shows that the Berger measure of $\widetilde{B_+^\circ}$ is $2(1-s)ds$. □

We also have:

Theorem 3.6. *The mean transform $\widehat{B_+^\circ}$ of B_+° is subnormal.*

Proof. We let $\widehat{B_+^\circ} = \text{shift}(\widehat{\alpha}_0, \widehat{\alpha}_1, \cdots)$. Then for $n \geq 0$, we can see that

$$\begin{aligned}
\widehat{\alpha}_n &= \tfrac{\alpha_n + \alpha_{n+1}}{2} \\
&= \tfrac{(n+1)(n+3)+(n+2)^2}{2(n+2)(n+3)} = \tfrac{2(n+2)^2-1}{2(n+2)^2+2(n+2)} = \left(\tfrac{\sqrt{2}(n+2)+1}{2(n+2)+2}\right)\left(\tfrac{\sqrt{2}(n+2)-1}{n+2}\right) \\
&= \left(\tfrac{\sqrt{2}n+2\sqrt{2}+1}{2n+6}\right)\left(\tfrac{\sqrt{2}n+2\sqrt{2}-1}{n+2}\right) \\
&= \left(\sqrt{\tfrac{\sqrt{2}n+2\sqrt{2}+1}{2n+6}}\right)\left(\sqrt{\tfrac{\sqrt{2}n+2\sqrt{2}+1}{2n+6}}\right)\left(\sqrt{\tfrac{\sqrt{2}n+2\sqrt{2}-1}{n+2}}\right)\left(\sqrt{\tfrac{\sqrt{2}n+2\sqrt{2}-1}{n+2}}\right).
\end{aligned}$$

We thus have that

$$\widehat{B_+^\circ} = S(\sqrt{2}, 2\sqrt{2}+1, 2, 6) \circ S(\sqrt{2}, 2\sqrt{2}+1, 2, 6) \circ S(\sqrt{2}, 2\sqrt{2}-1, 1, 2) \circ S(\sqrt{2}, 2\sqrt{2}-1, 1, 2).$$

Since $S(\sqrt{2}, 2\sqrt{2}+1, 2, 6)$ and $S(\sqrt{2}, 2\sqrt{2}-1, 1, 2)$ are subnormal, it follows from (3.2) that the mean transform $\widehat{B_+^\circ}$ of B_+° is also subnormal. □

On the other hand, we need not expect that the mean transform shares so many spectral properties with the Aluthge transform. However, the mean transform has good properties as the Aluthge transform does, at least, for the cases of weighted shifts. The following example shows that both the mean transform and the Aluthge transform may make the given operator a more nicely behaved operator.

Write $\mathcal{L}_n := \bigvee\{e_h : h \geq n\}$ for the invariant subspace obtained by removing the first n vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$. For $n \geq 0$, we let $\text{shift}(\alpha_0, \alpha_1, \alpha_2, \cdots)|_{\mathcal{L}_n} := \text{shift}(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \cdots)$.

Example 3.7. For $0 < x \leq \frac{3}{5}$, let $W_x \equiv \text{shift}\left(x, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\right)$. Then we have:

(i) W_x is never 2-hyponormal for any x;

(ii) $\widetilde{W_x}$ is 2-hyponormal if and only if $0 < x \leq \frac{5}{9} \simeq 0.5556$;

(iii) $\widehat{W_x}$ is 2-hyponormal if and only if

$$0 < x \leq \tfrac{-111492009+6460\sqrt{1102532089}}{185820015} \simeq 0.5543.$$

Proof. We use Lemma 3.1, i.e., W_x is k-hyponormal if and only if for all $k \geq 1$ and $n \geq 0$,

$$H(k;n)\,(W_x) = \left(\gamma_{n+i+j-2}\,(W_x)\right)_{i,j=1}^{k+1} \geq 0. \tag{3.6}$$

A direct calculation shows that the Aluthge transform $\widetilde{W_x}$ of W_x is:

$$\widetilde{W_x} \equiv \text{shift}\left(\sqrt{\frac{3x}{5}}, \sqrt{\frac{2}{5}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{5}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{5}{7}}, \cdots\right)$$

and the mean transform $\widehat{W_x}$ of W_x is:

$$\widehat{W_x} \equiv \text{shift}\left(\frac{x+\frac{3}{5}}{2}, \frac{\frac{3}{5}+\frac{2}{3}}{2}, \frac{\frac{2}{3}+\frac{3}{4}}{2}, \frac{\frac{3}{4}+\frac{4}{5}}{2}, \frac{\frac{4}{5}+\frac{5}{6}}{2}, \frac{\frac{5}{6}+\frac{6}{7}}{2}, \cdots\right).$$

(i) Since $W_x|_{\mathcal{L}_2}$ is subnormal, W_x is 2-hyponormal if and only if $H(2;0)\,(W_x) \geq 0$ and $H(2;1)\,(W_x) \geq 0$. By the Nested Determinants Test (or Choleski's Algorithm), we can see that

$$H(2;1)\,(W_x) \geq 0 \Longleftrightarrow \det H(2;1)\,(W_x) \geq 0\,.$$

A direct calculation shows that

$$\det H(2;1)\,(W_x) = -\frac{481}{6250000}\,x^6 < 0.$$

Therefore W_x can not be 2-hyponormal for any x.

(ii) By Theorem 3.5, we observe that $\widetilde{B_+^\circ}|_{\mathcal{L}_1} \equiv \text{shift}\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{5}}, \sqrt{\frac{2}{3}}, \cdots\right)$ is subnormal. Hence by (3.6), $\widetilde{W_x}$ is 2-hyponormal if and only if $H(2;0)(\widetilde{W_x}) \geq 0$ and $H(2;1)(\widetilde{W_x}) \geq 0$. By the Nested Determinants Test again, we note that

$$H(2;i)(\widetilde{W_x}) \geq 0 \Longleftrightarrow \det H(2;i)(\widetilde{W_x}) \geq 0 \quad \text{for } i = 0, 1\,.$$

By a direct calculation, we have

$$\det H(2;1)(\widetilde{W_x}) \equiv 0 \quad \text{and} \quad \det H(2;0)(\widetilde{W_x}) \geq 0 \Longleftrightarrow 0 < x \leq \frac{5}{9},$$

which gives the result.

(iii) By Theorem 3.6, we can see that $\widehat{B_+^\circ}|_{\mathcal{L}_1} \equiv \text{shift}\left(\frac{\frac{2}{3}+\frac{3}{4}}{2}, \frac{\frac{3}{4}+\frac{4}{5}}{2}, \frac{\frac{4}{5}+\frac{5}{6}}{2}, \cdots\right)$ is subnormal. Hence by (3.6), $\widehat{W}_x$ is 2-hyponormal if and only if $H(2;0)(\widehat{W}_x) \geq 0$ and $H(2;1)(\widehat{W}_x) \geq 0$. Observe that

$$H(2;i)(\widehat{W}_x) \geq 0 \Longleftrightarrow \det H(2;i)(\widehat{W}_x) \geq 0 \quad \text{for } i = 0, 1\,.$$

By a direct calculation, we have

$$\det H(2;1)(\widehat{W}_x) = \frac{9641999505449051}{1393140695040000000000}(3+5x)^6 > 0$$

and

$$\det H(2;0)(\widehat{W}_x) \geq 0 \Longleftrightarrow 0 < x \leq \frac{-111492009 + 6460\sqrt{1102532089}}{185820015}\,,$$

which gives the result. □

The following example shows that the mean transform often behaves well for "bad" operators, in comparison with the Aluthge transform.

Example 3.8. For $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \alpha_2, \cdots)$ with $\alpha_0 = \alpha_1 := \frac{1}{2}$, let

$$\widehat{W}_\alpha \equiv B_+^\circ = \text{shift}\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right).$$

Then we have:

(i) $W_\alpha \equiv \text{shift}\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{6}, \frac{2}{3}, \frac{14}{15}, \frac{11}{15}, \frac{103}{105}, \cdots\right)$ is not hyponormal;

(ii) $\widetilde{W}_\alpha$ is not 2-hyponormal;

(iii) $\widehat{W}_\alpha$ is subnormal.

Proof. (i) and (iii): These are clear from the fact that $\widehat{W}_\alpha = B_+^\circ$ and the following relation:

$$\frac{\alpha_n + \alpha_{n+1}}{2} = \frac{n+1}{n+2} \text{ for all } n \geq 0.$$

(ii): Observe that

$$\widetilde{W}_\alpha \equiv \text{shift}\left(\frac{1}{2}, \sqrt{\frac{5}{12}}, \sqrt{\frac{5}{9}}, \sqrt{\frac{28}{45}}, \sqrt{\frac{154}{225}}, \sqrt{\frac{1133}{1575}}, \cdots\right)$$

and

$$\det H(2;2)(\widetilde{W}_\alpha) = -\frac{889}{7346640384} < 0.$$

Thus by Lemma 3.1, $\widetilde{W}_\alpha$ is not 2-hyponormal. □

The following example shows that for a very nice operator, the mean transform and the Aluthge transform may behave very similarly. To see this, we examine the first-slot perturbations of the Schur product of the Bergman shifts.

Example 3.9. For $0 < x \leq \frac{2}{3}$, let $W_x \equiv \text{shift}\left(x, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\right)$. Then we have

(i) W_x is 2-hyponormal if and only if $0 < x \leq \sqrt{\frac{2511}{7456}} \simeq 0.5803$;

(ii) $\widetilde{W}_x$ is 2-hyponormal if and only if $0 < x \leq \frac{3}{5} = 0.6$;

(iii) $\widehat{W}_x$ is 2-hyponormal if and only if

$$0 < x \leq \frac{-82300568+1581\sqrt{9824630305}}{123450852} \simeq 0.6027;$$

(iv) W_x is 3-hyponormal if and only if $0 < x \leq \sqrt{\frac{1267136}{4189775}} \simeq 0.5499$;

(v) $\widetilde{W}_x$ is 3-hyponormal if and only if $0 < x \leq \frac{5}{9} \simeq 0.5556$;

(vi) $\widehat{W}_x$ is 3-hyponormal if and only if

$$0 < x \leq \tfrac{-762289598222620+527\sqrt{704223757072598183794971 7}}{1143434397333930} \simeq 0.5564.$$

Proof. We use Lemma 3.1, i.e., W_x is k-hyponormal if and only if for all $k \geq 1$ and $n \geq 0$,

$$H(k;n)\,(W_x) = \left(\gamma_{n+i+j-2}\,(W_x)\right)_{i,j=1}^{k+1} \geq 0. \tag{3.7}$$

A direct calculation shows that the Aluthge transform $\widetilde{W}_x$ of W_x is:

$$\widetilde{W}_\alpha \equiv \text{shift}\left(\sqrt{\frac{2x}{3}}, \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{5}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{5}{7}}, \cdots\right)$$

and the mean transform $\widehat{W}_x$ of W_x is:

$$\widehat{W}_x \equiv \text{shift}\left(\frac{x+\frac{2}{3}}{2}, \frac{\frac{2}{3}+\frac{3}{4}}{2}, \frac{\frac{3}{4}+\frac{4}{5}}{2}, \frac{\frac{4}{5}+\frac{5}{6}}{2}, \frac{\frac{5}{6}+\frac{6}{7}}{2}, \cdots\right).$$

(i) Since $W_x|_{\mathcal{L}_1}$ is subnormal, W_x is 2-hyponormal if and only if $H(2;0)\,(W_x) \geq 0$. By the Nested Determinants Test, we can see that

$$H(2;0)\,(W_x) \geq 0 \Longleftrightarrow 0 < x \leq \sqrt{\frac{2511}{7456}}.$$

(ii) By Theorem 3.5, we observe that $\widetilde{B_+^\circ}|_{\mathcal{L}_1} \equiv \text{shift}\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{5}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{5}{7}}, \cdots\right)$ is subnormal. Hence by (3.7),

$$\widetilde{W}_x \text{ is 2-hyponormal } \Longleftrightarrow\; H(2;0)(\widetilde{W}_x) \geq 0 \;\Longleftrightarrow\; 0 < x \leq \frac{3}{5}.$$

(iii) By Theorem 3.6, we can see that $\widehat{B_+^\circ}|_{\mathcal{L}_1} \equiv \text{shift}\left(\frac{\frac{2}{3}+\frac{3}{4}}{2}, \frac{\frac{3}{4}+\frac{4}{5}}{2}, \frac{\frac{4}{5}+\frac{5}{6}}{2}, \cdots\right)$ is subnormal. Hence by (3.7),

$$\widehat{W}_x \text{ is 2-hyponormal } \Longleftrightarrow\; H(2;0)(\widehat{W}_x) \geq 0 \;\Longleftrightarrow\; 0 < x \leq \tfrac{-82300568+1581\sqrt{9824630305}}{123450852}.$$

(iv) Since $W_x|_{\mathcal{L}_1}$ is subnormal, W_x is 3-hyponormal if and only if $H(3;0)\,(W_x) \geq 0$. Observe that

$$H(3;0)(W_x) \geq 0 \Longleftrightarrow \begin{pmatrix} \frac{1}{x^2} & 1 & \frac{4}{9} & \frac{1}{4} \\ 1 & \frac{4}{9} & \frac{1}{4} & \frac{4}{25} \\ \frac{4}{9} & \frac{1}{4} & \frac{4}{25} & \frac{1}{9} \\ \frac{1}{4} & \frac{4}{25} & \frac{1}{9} & \frac{4}{49} \end{pmatrix} \geq 0\,.$$

Thus, by the Nested Determinants Test, we have that

$$W_x \text{ is 3-hyponormal} \Longleftrightarrow 0 < x \leq \sqrt{\frac{1267136}{4189775}}\,.$$

(v) Since $\widetilde{B_+^\circ}|_{\mathcal{L}_1} \equiv \text{shift}\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{5}}, \sqrt{\frac{2}{3}}, \cdots\right)$ is subnormal, $\widetilde{W}_x$ is 3-hyponormal if and only if $H(3;0)(\widetilde{W}_x) \geq 0$. Observe that

$$H(3;0)(\widetilde{W}_x) \geq 0 \Longleftrightarrow \begin{pmatrix} \frac{1}{x} & \frac{2}{3} & \frac{1}{3} & \frac{1}{5} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{5} & \frac{2}{15} \\ \frac{1}{3} & \frac{1}{5} & \frac{2}{15} & \frac{2}{21} \\ \frac{1}{5} & \frac{2}{15} & \frac{2}{21} & \frac{1}{14} \end{pmatrix} \geq 0\,.$$

Thus, by the Nested Determinants Test, we have that

$$\widetilde{W}_x \text{ is 3-hyponormal} \Longleftrightarrow 0 < x \leq \frac{5}{9}.$$

(vi) Since $\widehat{B_+^\circ}|_{\mathcal{L}_1} \equiv \text{shift}\left(\frac{\frac{2}{3}+\frac{3}{4}}{2}, \frac{\frac{3}{4}+\frac{4}{5}}{2}, \frac{\frac{4}{5}+\frac{5}{6}}{2}, \cdots\right)$ is subnormal, $\widetilde{W}_x$ is 3-hyponormal if and only if $H(3;0)(\widehat{W}_x) \geq 0$. Note that

$$\begin{aligned} & H(3;0)(\widehat{W}_x) \geq 0 \\ \Longleftrightarrow & \begin{pmatrix} \frac{1}{(3x+2)^2} & \frac{1}{36} & \frac{289}{20736} & \frac{277729}{33177600} \\ \frac{1}{36} & \frac{289}{20736} & \frac{277729}{33177600} & \frac{666827329}{119439360000} \\ \frac{289}{20736} & \frac{277729}{33177600} & \frac{666827329}{119439360000} & \frac{68601562561}{17199267840000} \\ \frac{277729}{33177600} & \frac{666827329}{119439360000} & \frac{68601562561}{17199267840000} & \frac{13172900043601}{4403012567040000} \end{pmatrix} \geq 0\,. \end{aligned}$$

Thus, by the Nested Determinants Test, we have that

$$\widetilde{W}_x \text{ is 3-hyponormal} \Longleftrightarrow 0 < x \leq \frac{-762289598222620 + 527\sqrt{704223757072598183794971 7}}{1143434397333930}.$$

This completes the proof. □

Remark 3.10. (i) By Example 3.9, we can see that the mean transform preserves the k-hyponormality ($k = 2, 3$), as does the Aluthge transform when $W_\alpha|_{\mathcal{L}_1}$ is subnormal with a continuous Berger measure $-4s \ln s ds$.

(ii) Example 3.9 provides a positive evidence for the conjecture that the Aluthge and the mean transforms preserve the subnormality when W_α is subnormal with a continuous Berger measure.

In [13, Corollary 3.11], we showed that for a subnormal weighted shift W_α with two-atomic Berger measure $a\delta_p + (1-a)\,\delta_q$ $(0 < a < 1,\, p < q)$, the Aluthge transform $\widetilde{W}_\alpha$ of W_α is subnormal if and only if $p = 0$. We may also ask whether $\widehat{W}_\alpha$ is subnormal when W_α is subnormal with a finite atomic Berger measure. To examine this question, we recall *recursively generated weighted shifts* [5], [6]. We briefly recall some key facts about these shifts, specifically the case when there are two coefficients of recursion. In [16], J. Stampfli proved that given three positive numbers $\sqrt{a} < \sqrt{b} < \sqrt{c}$, it is always possible to find a subnormal weighted shift, denoted $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$, whose first three weights are $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$. In this case, the coefficients of recursion (cf. [5, Example 3.12], [6, Section 3], [4, Section 1, p. 81]) are given by

$$\varphi_0 = -\frac{ab(c-b)}{b-a} \text{ and } \varphi_1 = \frac{b(c-a)}{b-a}, \tag{3.8}$$

the atoms t_0 and t_1 are the roots of the equation

$$t^2 - (\varphi_0 + \varphi_1 t) = 0, \tag{3.9}$$

and the densities ρ_0 and ρ_1 uniquely solve the 2×2 system of equations

$$\begin{cases} \rho_0 + \rho_1 & = & 1 \\ \rho_0 t_0 + \rho_1 t_1 & = & \alpha_0^2. \end{cases} \tag{3.10}$$

Thus we get a two-atomic measure

$$\mu = \rho_0 \delta_{t_0} + \rho_1 \delta_{t_1} \tag{3.11}$$

which is the Berger measure of $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$.

Example 3.11. We consider a recursively generated weighted shift $W_\alpha \equiv W_{(1,2,3)^\wedge}$. Then by (3.11), we have that the Berger measure of $W_\alpha \equiv W_{(1,2,3)^\wedge}$ is:

$$\mu := \left(\frac{27}{28}\right)\delta_{\frac{2}{3}} + \left(\frac{1}{28}\right)\delta_{10}.$$

Thus the mean transform $\widehat{W}_\alpha$ of $W_{(1,2,3)^\wedge}$ is:

$$\widehat{W}_\alpha = \text{shift}\left(\frac{3}{2}, \frac{5}{2}, \frac{27+2\sqrt{201}}{18}, \cdots\right).$$

A straightforward calculation shows that

$$H(2;0)(\widehat{W}_\alpha) = \begin{pmatrix} 1 & \frac{9}{4} & \frac{225}{16} \\ \frac{9}{4} & \frac{225}{16} & \frac{25\left(27+2\sqrt{201}\right)^2}{576} \\ \frac{225}{16} & \frac{25\left(27+2\sqrt{201}\right)^2}{576} & \frac{25\left(134+21\sqrt{41}\right)^2\left(27+2\sqrt{201}\right)^2}{4167936} \end{pmatrix} \ngeq 0.$$

Therefore by Lemma 3.1, $\widehat{W}_\alpha$ is not 2-hyponormal, so that $\widehat{W}_\alpha$ is not subnormal.

In fact, we are tempted to guess that this phenomenon is not accidental. We thus have:

Conjecture 3.12. *Let $W_\alpha \equiv \text{shift}(\alpha_0, \alpha_1, \cdots)$ be a subnormal weighted shift with Berger measure*

$$\mu = a\delta_p + (1-a)\,\delta_q \quad (0 < a < 1,\ 0 \le p < q \le 1)$$

and $\widehat{W}_\alpha$ be the mean transform of W_α. Then

$$\widehat{W}_\alpha \text{ is subnormal} \iff p = 0. \tag{3.12}$$

The following provides some evidence that supports Conjecture 3.12.

The implication ($\Leftarrow$) of (3.12) is evident because $W_\alpha = \text{shift}\left(\sqrt{(1-a)\,q}, \sqrt{q}, \sqrt{q}, \cdots\right)$, so that $\widehat{W}_\alpha = \text{shift}\left(\frac{\sqrt{q}\left(\sqrt{1-a}+1\right)}{2}, \sqrt{q}, \sqrt{q}, \cdots\right)$, and hence $\widehat{W}_\alpha$ is subnormal with Berger measure

$$\widehat{\mu} = \left(1 - \left(\frac{\sqrt{1-a}+1}{2}\right)^2\right)\delta_0 + \left(\frac{\sqrt{1-a}+1}{2}\right)^2 \delta_q. \tag{3.13}$$

For the implication ($\Rightarrow$) of (3.12), we note that if W_α is subnormal with Berger measure $\mu = a\delta_p + (1-a)\,\delta_q$, then $\left(\frac{1}{\sqrt{q}}\right) W_\alpha$ is subnormal with Berger measure $a\delta_{\frac{p}{q}} + (1-a)\,\delta_1$. Also if $\widehat{W}_\alpha$ is subnormal, then $\left(\frac{1}{\sqrt{q}}\right)\widehat{W}_\alpha$ is also subnormal. But since $\left(\frac{1}{\sqrt{q}}\right)\widehat{W}_\alpha$ is the mean transform of $\left(\frac{1}{\sqrt{q}}\right) W_\alpha$, it suffices to show that if W_α is a subnormal weighted shift with Berger measure $\mu = a\delta_p + (1-a)\,\delta_1$ $(0 < a < 1,\ 0 \le p < 1)$, then

$$\widehat{W}_\alpha \text{ is subnormal} \implies p = 0 \tag{3.14}$$

Towards (3.14), we guess that if $p \neq 0$ then $\widehat{W}_\alpha$ is not 3-hyponormal. Indeed, from a computation using the software tool *Mathematica* [17] we find considerable numerical and graphical evidence.

We turn our attention to the continuity properties of the mean transform $\widehat{T}$. The following lemma is well known.

Lemma 3.13. ([10]) *The Aluthge transform map $T \to \widetilde{T}$ is $(\|\cdot\|, \|\cdot\|)-$ continuous on $\mathcal{B}(\mathcal{H})$.*

By comparison, we have:

Theorem 3.14. *The mean transform map $T \to \widehat{T}$ is $(\|\cdot\|, SOT)-$ continuous on $\mathcal{B}(\mathcal{H})$.*

Proof. First of all, we will show that

$$\text{the Duggal transform map } T \to \widetilde{T}^D \text{ is } (\|\cdot\|, SOT) - \text{ continuous on } \mathcal{B}(\mathcal{H}). \tag{3.15}$$

To do so, let T_0 be arbitrary in $\mathcal{B}(\mathcal{H})$ and suppose that a sequence $\{T_n\} \equiv \{U_n|T_n|\}$ converges in norm to $T_0 = U_0|T_0|$. Since the mappings $T \to T^*$ and $(S,T) \to ST$ are norm continuous, and so is the mapping $P \to P^{\frac{1}{2}}$ $(P \geq 0)$, it follows that

$$\||T_n| - |T_0|\| \to 0 \text{ and } \left\||T_n|^{\frac{1}{2}} - |T_0|^{\frac{1}{2}}\right\| \to 0. \tag{3.16}$$

By Lemma 3.13 and (3.16), we have that

$$\left\||T_n|U_n|T_n|^{\frac{1}{2}} - |T_n|U_n|T_0|^{\frac{1}{2}}\right\| \leq \||T_n|U_n\| \left\||T_n|^{\frac{1}{2}} - |T_0|^{\frac{1}{2}}\right\| \to 0, \tag{3.17}$$

$$\left\||T_n|U_n|T_n|^{\frac{1}{2}} - |T_n|^{\frac{1}{2}}|T_0|^{\frac{1}{2}}U_0|T_0|^{\frac{1}{2}}\right\| \leq \left\||T_n|^{\frac{1}{2}}\right\| \left\||T_n|^{\frac{1}{2}}U_n|T_n|^{\frac{1}{2}} - |T_0|^{\frac{1}{2}}U_0|T_0|^{\frac{1}{2}}\right\| \to 0\,, \tag{3.18}$$

and

$$\left\||T_n|^{\frac{1}{2}}|T_0|^{\frac{1}{2}}U_0|T_0|^{\frac{1}{2}} - |T_0|U_0|T_0|^{\frac{1}{2}}\right\| \leq \left\||T_0|^{\frac{1}{2}}U_0|T_0|^{\frac{1}{2}}\right\| \left\||T_n|^{\frac{1}{2}} - |T_0|^{\frac{1}{2}}\right\| \to 0. \tag{3.19}$$

Observe that

$$\begin{aligned} &\left\||T_n|U_n|T_0|^{\frac{1}{2}} - |T_0|U_0|T_0|^{\frac{1}{2}}\right\| \\ &= \left\||T_n|U_n|T_0|^{\frac{1}{2}} - |T_n|U_n|T_n|^{\frac{1}{2}} + |T_n|U_n|T_n|^{\frac{1}{2}} - |T_0|U_0|T_0|^{\frac{1}{2}}\right\| \\ &\leq \left\||T_n|U_n|T_n|^{\frac{1}{2}} - |T_n|U_n|T_0|^{\frac{1}{2}}\right\| + \left\||T_n|U_n|T_n|^{\frac{1}{2}} - |T_0|^{\frac{1}{2}}U_0|T_0|^{\frac{1}{2}}\right\| \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\left\||T_n|U_n|T_n|^{\frac{1}{2}} - |T_0|U_0|T_0|^{\frac{1}{2}}\right\| \\ &\leq \left\||T_n|U_n|T_n|^{\frac{1}{2}} - |T_n|^{\frac{1}{2}}|T_0|^{\frac{1}{2}}U_0|T_0|^{\frac{1}{2}}\right\| + \left\||T_n|^{\frac{1}{2}}|T_0|^{\frac{1}{2}}U_0|T_0|^{\frac{1}{2}} - |T_0|U_0|T_0|^{\frac{1}{2}}\right\|. \end{aligned} \tag{3.21}$$

Thus since $\{U_n|T_n|\}$ converges in norm to $U_0|T_0|$, it follows from (3.16) - (3.21) that for each $x \in \operatorname{ran}\left(|T_0|^{\frac{1}{2}}\right)$,

$$\||T_n|U_n|x - |T_0|U_0|x\| \to 0\,. \tag{3.22}$$

Note that the $\operatorname{ran}\left(|T_0|^{\frac{1}{2}}\right)$ is dense in $(\ker T_0)^{\perp}$. It thus follows from (3.22) that $\left\{\widetilde{T}_n^D\right\}$ converges in SOT to $\widetilde{T}_0^D$ on $(\ker T_0)^{\perp}$. Since $|T_0|^{\frac{1}{2}} = 0$ on $\ker T_0$, it follows from (3.16) that

$$\left\|\widetilde{T}_n^D\right\| \to 0 \text{ on } \ker T_0. \tag{3.23}$$

Thus we have that $\left\{\widetilde{T}_n^D\right\}$ converges in SOT to $\widetilde{T}_0^D$. This proves (3.15).

Now since $\widehat{T} = \frac{1}{2}\left(T + \widetilde{T}^D\right)$, the desired result follows at once from (3.15). □

Remark 3.15. Since the mean transform involves the sum of two operators (or, a "big" perturbation of the operator in the sense that the Duggal transform $\widetilde{T}^D$ shares many spectral properties with the unperturbed operator T), we can not expect that the spectral properties of T are preserved under the mean transform map in general. In spite of it, the hyponormality of $\widehat{T}$ is transmitted from that of T if $T \in \mathcal{B}(\mathcal{H})$ belongs to a certain class of operators. To see this, we introduce a new class of operators.

Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{B}(\mathcal{H})$. We can easily check that $U|T| = |T|U$ if and only if T is quasinormal. If instead $U^2|T| = |T|U^2$, then T will be said to be *in the δ-class*, denoted by $T \in \delta(\mathcal{H})$. Clearly, quasinormal operators belong to $\delta(\mathcal{H})$. However, we need not expect that there is a relationship between $\delta(\mathcal{H})$ and hyponormal operators. To see this, we let

$$T = \begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \quad (\text{where } P \geq 0 \text{ and } P \neq I).$$

Then

$$T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} = U\,|T|$$

is the polar decomposition of T. But since $U^2 = I$, it follows that $U^2|T| = |T|\,U^2$, i.e., $T \in \delta(\mathcal{H})$. However since

$$T^*T = \begin{pmatrix} P^2 & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad TT^* = \begin{pmatrix} I & 0 \\ 0 & P^2 \end{pmatrix},$$

it follows that T is never hyponormal.

We now have:

Theorem 3.16. *If $T \in \mathcal{B}(\mathcal{H})$ is a hyponormal operator in the δ-class, then the mean transform $\widehat{T}$ is hyponormal.*

Proof. Let $T = U|T| = |T^*|U$ be the polar decomposition of T. First of all, we assume that T has dense range, so that U can be chosen as a unitary operator. Then

$$|T|\,U^* = U^*|T^*|. \tag{3.24}$$

If T is hyponormal then $TT^* \leq T^*T$, so that

$$U|T|^2U^* \leq |T|^2, \quad \text{and hence} \quad |T|^2 \leq U^*|T|^2U. \tag{3.25}$$

On the other hand, if $T \in \delta(\mathcal{H})$, then we have that $U^2|T| = |T|U^2$. Thus $U(U|T|) = |T|U^2$ implies $U(|T^*|U) = |T|U^2$, which gives

$$U\,|T^*| = |T|\,U. \tag{3.26}$$

Thus it follows from (3.26) that

$$U^*|T^*||T|U = (|T|U^*)\,(|T|U) = (|T|U^*)\,(U|T^*|) = |T||T^*| \tag{3.27}$$

and

$$U^*|T||T^*|U = (|T^*|U^*)\,(|T^*|U) = (|T^*|U^*)\,(U|T|) = |T^*||T|. \tag{3.28}$$

Thus we have

$$\begin{aligned}
\widehat{T}^*\widehat{T} &= \tfrac{1}{4}\left(|T|^2 + |T|U^*|T|U + U^*|T|U|T| + U^*|T|^2U\right) \\
&= \tfrac{1}{4}\left(|T|^2 + U^*|T^*||T|U + U^*|T||T^*|U + U^*|T|^2U\right) \quad (\text{by } (3.24)) \\
&\geq \tfrac{1}{4}\left(U|T|^2U^* + |T||T^*| + |T^*||T| + |T|^2\right) \quad (\text{by } (3.25), (3.27) \text{ and } (3.28)) \\
&= \tfrac{1}{4}\left(U|T|^2U^* + |T|U|T|U^* + U|T|U^*|T| + |T|^2\right) \quad (\text{by } (3.24)) \\
&= \widehat{T}\widehat{T}^*,
\end{aligned}$$

which says that the mean transform $\widehat{T}$ is hyponormal, as desired.

Suppose instead that T does not have dense range. Since T is hyponormal, and hence $\ker T$ reduces T, we can write

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } (\text{ket}\, T)^\perp \oplus \ker T,$$

where T_0 is still hyponormal and evidently, $T_0 \in \delta(\mathcal{H})$. Thus by the preceding argument, $\widehat{T_0}$ is hyponormal. But since

$$\widehat{T} = \begin{pmatrix} \widehat{T_0} & 0 \\ 0 & 0 \end{pmatrix},$$

it follows that $\widehat{T}$ is hyponormal. This completes the proof. □

Remark 3.17. This paper is only a start on the theory of the mean transform of bounded linear operators. In particular, in the view of the practical use, it is so hard to find the Aluthge transform of the given operator because it involves the positive square roots of positive operators, and it is quite difficult to find the positive square roots in general. By contrast, the mean transform involves the sums of two operators, so it is easy to get the mean transforms if we know the polar decompositions of the operators. Thus the mean transform may be useful in the practical use.

Acknowledgment. The authors are deeply indebted to the referee for many helpful comments that helped improved the presentation and mathematical content of the paper.

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Department of Mathematics, Chungnam National University, Daejeon, 305-764, Korea
E-mail address: slee@cnu.ac.kr

Department of Mathematics, Seoul National University, Seoul, 123-456, Korea
E-mail address: wylee@snu.ac.kr

Department of Mathematics, The University of Texas-Pan American, Edinburg, Texas 78539
E-mail address: yoonj@utpa.edu