HYPONORMALITY OF TOEPLITZ OPERATORS WITH POLYNOMIAL SYMBOLS: AN EXTREMAL CASE

In Sung Hwang, In Hyoun Kim and Woo Young Lee

Abstract. If T_φ is a hyponormal Toeplitz operator with polynomial symbol $\varphi = \bar{g} + f$ ($f, g \in H^\infty(\mathbb{T})$) such that g divides f, and if $\psi := \frac{f}{g}$ then

$$\sum_{\zeta \in \mathcal{Z}(\psi)} \zeta \ \le |\mu| - \frac{1}{|\mu|},$$

where μ is the leading coefficient of ψ and $\mathcal{Z}(\psi)$ denotes the set of zeros of ψ. In this paper we present a necessary and sufficient condition for T_φ to be hyponormal when φ enjoys an extremal case in the above inequality, that is, equality holds in the above inequality.

1. Introduction

A bounded linear operator A on a Hilbert space $\mathfrak{H}$ with inner product $(\cdot, \cdot)$ is said to be hyponormal if its selfcommutator $[A^*, A] = A^*A - AA^*$ induces a positive semidefinite quadratic form on $\mathfrak{H}$ via $\xi \mapsto ([A^*, A]\xi, \xi)$, for $\xi \in \mathfrak{H}$. The purpose of this paper is to study hyponormality for Toeplitz operators acting on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$ in the complex plane. In particular, our interest is with Toeplitz operators with polynomial symbols which satisfy certain constraints.

Recall that given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ is the operator T_φ on $H^2(\mathbb{T})$ defined by $T_\varphi f = P(\varphi \cdot f)$, where $f \in H^2(\mathbb{T})$ and P denotes the projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. The problem of determining which symbols induce hyponormal Toeplitz operators was solved by Cowen in [**2**], however here we shall employ an equivalent variant of Cowen's theorem that was first proposed by Nakazi and Takahashi in [**10**]. Suppose that $\varphi \in L^\infty(\mathbb{T})$ is arbitrary and consider the following subset of the closed unit ball of $H^\infty(\mathbb{T})$:

$$\mathcal{E}(\varphi) = \left\{k \in H^\infty(\mathbb{T}) : ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\right\}.$$

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The criterion is that T_φ is hyponormal if and only if the set $\mathcal{E}(\varphi)$ is nonempty [**2,10**]. This theorem is referred to the *Cowen's theorem.* Cowen's method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution of a certain functional equation involving its symbol. This approach has been put to use in the works [**3,5,6,7,9,10,12**] to study Toeplitz operators on the Hardy space of the unit circle.

If φ is a trigonometric polynomial, say $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero, then the nonnegative integers N and m denote the analytic and co-analytic degrees of φ. For arbitrary trigonometric polynomials, Zhu [**12**] has applied Cowen's criterion and adopted a method based on the classical interpolation theorems of Schur to obtain an abstract characterization of those trigonometric polynomial symbols that correspond to hyponormal Toeplitz operators. Furthermore, he was able to use this characterization to give explicit necessary and sufficient conditions for hyponormality in terms of the coefficients of the polynomial φ whenever $m \le 3$. Also, in [**5**], the hyponormality of T_φ was completely characterized for the cases of $|a_{-m}| = |a_N|$. However, with polynomials of higher degree with $|a_{-m}| < |a_N|$, the analogous explicit necessary and sufficient conditions (via properties of coefficients) are not known and in fact would be too complicated to be of much value.

On the other hand, whenever we consider hyponormality of T_φ with polynomial symbols $\varphi = \bar{g} + f$ $(f, g \in H^\infty(\mathbb{T}))$, we may assume, without loss of generality, that g divides f (see Lemma 4 below). If ψ is in $H^\infty(\mathbb{T})$, write $\mathcal{Z}(\psi)$ for the set of zeros of ψ. Then we can show that if T_φ is a hyponormal Toeplitz operator with polynomial symbol $\varphi = \bar{g} + f$ $(f, g \in H^\infty(\mathbb{T}))$ such that g divides f, and if $\psi := \frac{f}{g}$ then

$$\left| \sum_{\zeta \in \mathcal{Z}(\psi)} \zeta \right| \le |\mu| - \frac{1}{|\mu|}, \tag{0.1}$$

where μ is the leading coefficient of ψ (see Lemma 6 below). In this paper we are concerned with hyponormality of T_φ when φ enjoys an extremal case in (0.1), in the sense that equality holds in (0.1). By the preceding consideration it suffices to focus on the cases where $m \ge 3$ and $|\mu| > 1$ (note that if $|\mu| = 1$ then $|a_{-m}| = |a_N|$). Our main result of this paper is as follows. Let $\varphi = \bar{g} + f$, where f and g are analytic polynomials of degrees N and m $(m \ge 3)$, respectively. Suppose that g divides f and the leading coefficient of $\psi := \frac{f}{g}$ is μ with $|\mu| > 1$. If $\left| \sum_{\zeta \in \mathcal{Z}(\psi)} \zeta \right| = |\mu| - \frac{1}{|\mu|}$ then we have:

(i) If $N < 2m - 1$ then T_φ is never hyponormal;

(ii) If $N \ge 2m - 1$ then T_φ is hyponormal if and only if the Fourier coefficients $\hat{\psi}(j)$ $(N - 2m + 1 \le j \le N - m)$ of ψ satisfy the following equation:

$$\begin{pmatrix} \hat{\psi}(N-2m+2) \\ \hat{\psi}(N-2m+3) \\ \vdots \\ \vdots \\ \hat{\psi}(N-m-1) \end{pmatrix} = \frac{\mu - \frac{1}{\bar{\mu}}}{\hat{\psi}(N-m-1)} \begin{pmatrix} \hat{\psi}(N-2m+1) \\ \hat{\psi}(N-2m+2) \\ \vdots \\ \vdots \\ \hat{\psi}(N-m-2) \end{pmatrix}.$$

We will also use this result to show that if $\varphi(z)=\sum_{n=-m}^{N}a_nz^n$ ($3\le m\le N$ and $|a_{-m}|<|a_N|$) satisfies the equality $|a_N|^2-|a_{-m}|^2=\left|\det\begin{pmatrix}\overline{a_{-m}} & \overline{a_{-m+1}}\\ a_N & a_{N-1}\end{pmatrix}\right|$, then

$$T_\varphi \text{ hyponormal } \iff d_{j+1}=\left[\frac{|a_N|^2-|a_{-m}|^2}{\det\begin{pmatrix}\overline{a_{-m}} & \overline{a_{-m+1}}\\ a_N & a_{N-1}\end{pmatrix}}\,\overline{\left(\frac{a_{-m}}{a_N}\right)}\right]\cdot d_j \quad (j=1,\cdots,m-2),$$

where the d_j are given by

$$\begin{pmatrix}d_1\\ d_2\\ \vdots\\ \vdots\\ d_m\end{pmatrix}:=\begin{pmatrix}\overline{a_{-m}} & \overline{a_{-m+1}} & \cdots & \overline{a_{-2}} & \overline{a_{-1}}\\ & \overline{a_{-m}} & \overline{a_{-m+1}} & \cdots & \overline{a_{-2}}\\ & & \ddots & \ddots & \vdots\\ & 0 & & \ddots & \overline{a_{-m+1}}\\ & & & & \overline{a_{-m}}\end{pmatrix}^{-1}\begin{pmatrix}a_{N-m+1}\\ a_{N-m+2}\\ \vdots\\ \vdots\\ a_N\end{pmatrix}.$$

2. MAIN RESULTS

We review Schur's algorithm, due to K. Zhu [**12**], determining hyponormality for Toeplitz operators with polynomial symbols. Suppose that $k(z)=\sum_{j=0}^{\infty}c_jz^j$ is in the closed unit ball of $H^\infty(\mathbb{T})$. If $k_0=k$, define by induction a sequence $\{k_n\}$ of functions in the closed unit ball of $H^\infty(\mathbb{T})$ as follows:

$$k_{n+1}(z)=\frac{k_n(z)-k_n(0)}{z\left(1-\overline{k_n(0)}\,k_n(z)\right)},\quad |z|<1,\ n=0,1,2,\cdots.$$

We write

$$k_n(0)=\Phi_n(c_0,\cdots,c_n),\quad n=0,1,2,\cdots,$$

where Φ_n is a function of $n+1$ complex variables. We call the Φ_n's *Schur's functions.* Then Zhu's theorem can be written as follows: if $\varphi(z)=\sum_{n=-N}^{N}a_nz^n$, where $a_N\neq 0$ and if

$$\begin{pmatrix}\overline{c_0}\\ \overline{c_1}\\ \vdots\\ \overline{c_{N-1}}\end{pmatrix}=\begin{pmatrix}a_1 & a_2 & \dots & a_{N-1} & a_N\\ a_2 & a_3 & \dots & a_N & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ a_N & 0 & \dots & 0 & 0\end{pmatrix}^{-1}\begin{pmatrix}\overline{a_{-1}}\\ \overline{a_{-2}}\\ \vdots\\ \overline{a_{-N}}\end{pmatrix}, \tag{0.2}$$

then T_φ is hyponormal if and only if $|\Phi_n(c_0,\cdots,c_n)|\le 1$ for every $n=0,1,\cdots,N-1$. If $k(z)=\sum_{j=0}^{\infty}c_jz^j$ is a function in H^∞ such that $\varphi-k\overline{\varphi}\in H^\infty$, then $c_0,\cdots,c_{N-1}$ are just the values given in (0.2). Thus Zhu's theorem shows that if $k(z)=\sum_{j=0}^{\infty}c_jz^j$ satisfies $\varphi-k\overline{\varphi}\in H^\infty$, then the hyponormality of T_φ is independent of the values of c_j's for $j\ge N$. On the other hand, Zhu's theorem can be reformulated as follows:

Lemma 1 (Zhu's Theorem [12]). *If* $\varphi(z)=\sum_{n=-m}^{N} a_n z^n$*, where* $m\le N$ *and* $a_N\ne 0$*, then* T_φ *is hyponormal if and only if* $|\Phi_n(c_0,\cdots,c_n)|\le 1$ *for every* $n=0,1,\cdots,N-1$*, where the* c_n *are given by the following recurrence relation:*

$$\begin{cases} c_0=c_1=\cdots=c_{N-m-1}=0\\ c_{N-m}=a_{-m}(\overline{a_N})^{-1}\\ c_n=(\overline{a_N})^{-1}\big(a_{-N+n}-\sum_{j=N-m}^{n-1} c_j\overline{a_{N-n+j}}\big)\ \ \text{for } n=N-m+1,\cdots,N-1. \end{cases} \tag{1.1}$$

Proof. See [**9**, Proposition 1]. □

The following lemma provides a useful criterion of hyponormality for Toeplitz operators T_φ with polynomial symbols φ.

Lemma 2 (Nakazi-Takahashi Theorem [10]). *A Toeplitz operator* T_φ *is hyponormal and the rank of the selfcommutator* $[T_\varphi^*,T_\varphi]$ *is finite (e.g.,* φ *is a trigonometric polynomial) if and only if there exists a finite Blaschke product* $k\in\mathcal{E}(\varphi)$ *of the form*

$$k(z)=e^{i\theta}\prod_{j=1}^{n}\frac{z-\beta_j}{1-\overline{\beta_j}z}\qquad (|\beta_j|<1 \text{ for } j=1,\cdots,n).$$

such that $deg(k)=rank[T_\varphi^*,T_\varphi]$*, where* $deg(k)$ *denotes the degree of* k *– meaning the number of zeros of* k *in the open unit disk* $\mathbb{D}$*.*

To prove the main result we need several auxiliary lemmas. First of all, we record results on the hyponormality of Toeplitz operators with polynomial symbols, which have been recently developed in the literature.

Lemma 3. *Suppose that* φ *is a trigonometric polynomial of the form* $\varphi(z)=\sum_{n=-m}^{N}a_nz^n$*, where* a_{-m} *and* a_N *are nonzero.*

(i) *If* T_φ *is hyponormal then* $m\le N$*,* $|a_{-m}|\le|a_N|$ *and* $N-m\le rank[T_\varphi^*,T_\varphi]\le N$*.*

(ii) *If* $\varphi:=\overline{g}+f$*, where* f *and* g *are in* $H^\infty(\mathbb{T})$ *and if* $\tilde{\varphi}:=\overline{g}+T_{\bar{z}^r}f$ $(r\le N-m)$ *then* T_φ *is hyponormal if and only if* $T_{\tilde{\varphi}}$ *is.*

(iii) *If* $|a_{-m}|=|a_N|$*, then* T_φ *is hyponormal if and only if the following symmetric condition holds:*

$$\overline{a_N}a_{-j}=a_{-m}\overline{a_{N-m+j}}\quad (1\le j\le m). \tag{3.1}$$

In this case, the rank of $[T_\varphi^*,T_\varphi]$ *is* $N-m$ *and* $\mathcal{E}(\varphi)=\{a_{-m}(\overline{a_N})^{-1}z^{N-m}\}$*. In particular,* T_φ *is normal if and only if* $m=N$*,* $|a_{-m}|=|a_N|$*, and (3.1) holds with* $m=N$*.*

(iv) *If* T_φ *is hyponormal then the finite Blaschke product* $k\in\mathcal{E}(\varphi)$ *is of the form*

$$k(z)=e^{i\theta}\,z^{N-m}\prod_{j=1}^{r}\frac{z-\beta_j}{1-\overline{\beta_j}z}\ \ (r\le m;\ \beta_j\ne 0)\quad \text{with } \prod_{j=1}^{r}|\beta_j|=\left|\frac{a_{-m}}{a_N}\right|. \tag{3.2}$$

Proof. The statements (i), (ii) and (iii) are known from [**3**,**5**,**6**,**7**,**8**,**9**,**10**,**12**]. Thus it suffices to focus on the assertion (iv). For (iv) suppose $k(z)=\sum_{j=0}^{\infty} c_j z^j$ is the finite Blaschke product in $\mathcal{E}(\varphi)$. Then by Lemma 1, $c_0=\cdots=c_{N-m-1}=0$. Therefore k is of the form

$$k(z)=e^{i\theta}\, z^{N-m}\prod_{j=1}^{r}\frac{z-\beta_j}{1-\overline{\beta_j}z}\quad (r\le m;\ \beta_j\neq 0).$$

But since

$$\frac{a_{-m}}{\overline{a_N}}=c_{N-m}=e^{i\theta}\prod_{j=1}^{r}(-\beta_j),$$

it follows that $\prod_{j=1}^{r}|\beta_j|=|a_{-m}(a_N)^{-1}|$, which proves (iv). $\square$

Suppose $\varphi=\bar g+f$, where $f=\sum_{n=1}^{N}a_nz^n$ and $g=\sum_{n=1}^{N}b_nz^n$. If T_φ is normal then g divides f: indeed, by Lemma 3 (iii), $g=e^{i\theta}\sum_{n=1}^{N}a_nz^n$ for some $\theta\in[0,2\pi)$, so that g divides f. But if T_φ is hyponormal then g need not divide f. For example, consider the polynomials $g(z)=z^2+2z$ and $f(z)=3z^2+5z$. Using an argument of P. Fan [**4**, Theorem 1] – for every trigonometric polynomial φ of the form $\varphi(z)=\sum_{n=-2}^{2}a_nz^n$,

$$T_\varphi \text{ is hyponormal} \iff \left|\det\begin{pmatrix}\overline{a_{-2}} & \overline{a_{-1}}\\ a_2 & a_1\end{pmatrix}\right|\le |a_2|^2-|a_{-2}|^2, \tag{3.3}$$

a straightforward calculation shows that if $\varphi=\bar g+f$ then T_φ is hyponormal, while g does not divide f.

In view of the preceding example, when $\varphi=\bar g+f$ (f and g are analytic polynomials), the condition "g divides f" seems to be so rigid. However the following lemma shows that we may assume, without loss of generality, that g divides f whenever we consider hyponormality of T_φ.

Lemma 4. *Let $\varphi=\bar g+f$, where g and f are analytic polynomials of degrees m and N ($m\le N$), respectively. If we let*

$$\widetilde{f}(z):=-\sum_{j=0}^{m-1}d_jz^j+z^mT_{\bar z^{N-m}}f,$$

where $\sum_{j=0}^{m-1}d_jz^j$ is the remainder in the division of $z^mT_{\bar z^{N-m}}f$ by g, put $\widetilde{\varphi}:=\bar g+\widetilde{f}$. Then we have:

(i) *T_φ is hyponormal if and only if $T_{\widetilde{\varphi}}$ is;*

(ii) *g divides $\widetilde{f}$.*

Moreover, if $\psi:=\frac{\widetilde{f}}{g}$, then the Fourier coefficients $\hat\psi(j)$ ($0\le j\le m$) of ψ can be obtained from the following equation:

$$\begin{pmatrix}\hat\psi(0)\\ \hat\psi(1)\\ \vdots\\ \vdots\\ \hat\psi(m)\end{pmatrix}=\begin{pmatrix}\overline{a_{-m}} & \overline{a_{-m+1}} & \cdots & \overline{a_{-1}} & \overline{b_0}\\ & \overline{a_{-m}} & \overline{a_{-m+1}} & \cdots & \overline{a_{-1}}\\ & & \ddots & \ddots & \vdots\\ & 0 & & \ddots & \overline{a_{-m+1}}\\ & & & & \overline{a_{-m}}\end{pmatrix}^{-1}\begin{pmatrix}a_{N-m}\\ a_{N-m+1}\\ \vdots\\ \vdots\\ a_N\end{pmatrix}, \tag{4.1}$$

where $f(z) = \sum_{n=0}^{N} a_n z^n$ *and* $g(z) = \overline{b_0} + \sum_{n=1}^{m} \overline{a_{-n}} z^n$*.*

Proof. The assertion (i) follows at once from Lemma 3 (ii). For the assertion (ii), observe that by the division algorithm, there exist unique polynomials ψ and r of degrees m and ℓ ($\ell \le m-1$), respectively, satisfying that $z^m T_{\bar{z}^{N-m}} f = g\psi + r$. Letting $\widetilde{f} := z^m T_{\bar{z}^{N-m}} f - r$ proves (ii). For (4.1), observe that if g divides $\widetilde{f}$ and if $\psi(z) := \sum_{n=0}^{m} \hat{\psi}(n) z^n$ then

$$\begin{cases} a_N &= \overline{a_{-m}}\hat{\psi}(m) \\ a_{N-1} &= \overline{a_{-m}}\hat{\psi}(m-1) + \overline{a_{-m+1}}\hat{\psi}(m) \\ &\vdots \\ a_{N-m} &= \overline{a_{-m}}\hat{\psi}(0) + \overline{a_{-m+1}}\hat{\psi}(1) + \cdots + \overline{b_0}\hat{\psi}(m), \end{cases}$$

which gives (4.1). □

Note that if $\varphi = \bar{g} + f$ (f and g are analytic polynomials), if g divides f and if T_φ is hyponormal then by Lemma 3 (i), the leading coefficient of $\frac{f}{g}$ has modulus ≥ 1. But if its modulus is exactly 1 then this case reduces to the case of Lemma 3 (iii). Therefore if $\varphi = \bar{g} + f$ (f and g are analytic polynomials) then it will suffice to consider hyponormality of T_φ under the assumption "g divides f and the leading coefficient of $\frac{f}{g}$ has modulus bigger than 1."

Lemma 5. *Suppose that* $k(z) = \sum_{j=0}^{\infty} c_j z^j$ *is in the closed unit ball of* $H^\infty(\mathbb{T})$ *and that* $\{\Phi_n\}$ *is a sequence of Schur's functions associated with* $\{c_n\}$*. If* $c_0 = \cdots = c_{n-1} = 0$ *and* $0 < |c_n| < 1$*, then we have that* $\Phi_0 = \cdots = \Phi_{n-1} = 0$*,* $\Phi_n = c_n$*,*

$$\Phi_{n+1} = \frac{c_{n+1}}{1-|c_n|^2} \quad \text{and} \quad \Phi_{n+2} = \frac{(1-|c_n|^2)c_{n+2} + \overline{c_n}c_{n+1}^2}{\big(1-|c_n|^2\big)^2 - |c_{n+1}|^2}. \tag{5.1}$$

Moreover if $|\Phi_{n+1}| = 1$*, then* $k(z)$ *is uniquely determined as follows:*

$$k(z) = \frac{c_{n+1}}{1-|c_n|^2}\, z^n \,\frac{z-\alpha}{1-\bar{\alpha}\, z} \quad \text{with } \alpha = -\frac{c_n\overline{c_{n+1}}}{|c_{n+1}|}. \tag{5.2}$$

Proof. Suppose $k(z) = \sum_{j=0}^{\infty} c_j z^j$. Then $\Phi_0 = k(0) = c_0 = 0$ and

$$k_1(z) = \frac{k(z) - k(0)}{z\big(1 - \overline{c_0}k(z)\big)} = \sum_{j=1}^{\infty} c_j z^{j-1},$$

so that $\Phi_1 = c_1 = 0$. Inductively, $k_m(z) = \sum_{j=m}^{\infty} c_j z^{j-m}$ for $m = 2, \cdots, n-1$, so that $\Phi_m = c_m = 0$ ($m = 2, \cdots, n-1$). Then

$$k_n(z) = \frac{k_{n-1}(z) - k_{n-1}(0)}{z\big(1 - \overline{k_{n-1}(0)}k_{n-1}(z)\big)} = \sum_{j=n}^{\infty} c_j z^{j-n},$$

so that $\Phi_n = c_n$. Also we have

$$k_{n+1}(z) = \frac{k_n(z) - k_n(0)}{z\big(1 - \overline{k_n(0)}k_n(z)\big)} = \frac{\sum_{j=n}^\infty c_j z^{j-n} - c_n}{z\big(1 - \overline{k_n(0)}k_n(z)\big)} = \frac{\sum_{j=n+1}^\infty c_j z^{j-n-1}}{1 - \overline{k_n(0)}k_n(z)},$$

so that

$$\Phi_{n+1} = \frac{c_{n+1}}{1 - |k_n(0)|^2} = \frac{c_{n+1}}{1 - |c_n|^2}.$$

On the other hand, we have

$$\begin{aligned} k_{n+2}(z) &= \frac{k_{n+1}(z) - k_{n+1}(0)}{z\big(1 - \overline{k_{n+1}(0)}k_{n+1}(z)\big)} \\ &= \frac{\frac{1}{1-\overline{c_n}k_n(z)} \sum_{j=n+1}^\infty c_j z^{j-n-1} - \frac{c_{n+1}}{1-|c_n|^2}}{z\big(1 - \overline{k_{n+1}(0)}k_{n+1}(z)\big)} \\ &= \frac{(1-|c_n|^2)\sum_{j=n+2}^\infty c_j z^{j-n-2} + c_{n+1}\overline{c_n}\sum_{j=n+1}^\infty c_j z^{j-n-1}}{(1-|c_n|^2)\big(1-\overline{c_n}k_n(z)\big)\big(1 - \overline{k_{n+1}(0)}k_{n+1}(z)\big)}, \end{aligned} \tag{5.3}$$

so that

$$\Phi_{n+2} = \frac{(1-|c_n|^2)c_{n+2} + c_{n+1}^2\overline{c_n}}{(1-|c_n|^2)^2(1-|k_{n+1}(0)|^2)} = \frac{(1-|c_n|^2)c_{n+2} + c_{n+1}^2\overline{c_n}}{(1-|c_n|^2)^2 - |c_{n+1}|^2}, \tag{5.4}$$

which proves the first assertion. For the second assertion suppose that $|\Phi_{n+1}| = 1$, so that $1 - |c_n|^2 = |c_{n+1}|$. Remember ([**11,12**]) that if $k(z) = \sum_{j=0}^\infty c_j z^j$ is in the closed unit ball of H^∞ then for any $m \in \mathbb{Z}^+$, $|\Phi_j(c_0, \cdots, c_j)| \le 1$ for each $j = 0, \cdots, m$. Thus from (5.4) we must have that $1 - |c_n|^2 c_{n+2} + c_{n+1}^2\overline{c_n} = 0$, or equivalently,

$$c_{n+2} = -\frac{c_{n+1}\overline{c_n}}{|c_{n+1}|}c_{n+1}. \tag{5.5}$$

Substituting (5.5) into (5.3) and multiplying on the denominator and the numerator by z^{-1} give that

$$k_{n+2}(z) = \frac{(1-|c_n|^2)\sum_{j=n+3}^\infty c_j z^{j-n-3} + c_{n+1}\overline{c_n}\sum_{j=n+2}^\infty c_j z^{j-n-2}}{-(1-|c_n|^2)\overline{c_n}\sum_{j=n+1}^\infty c_j z^{j-n-1} - \overline{c_{n+1}}\sum_{j=n+2}^\infty c_j z^{j-n-2}}, \tag{5.6}$$

which forces that $(1-|c_n|^2)c_{n+3} + c_{n+1}\overline{c_n}c_{n+2} = 0$ because the denominator of $k_{n+2}(0)$ is 0. Thus we have

$$c_{n+3} = -\frac{c_{n+1}\overline{c_n}}{|c_{n+1}|}c_{n+2} = \left(\frac{c_{n+1}\overline{c_n}}{|c_{n+1}|}\right)^2 c_{n+1}. \tag{5.7}$$

Repeating this process gives

$$c_{n+j} = \left(-\frac{c_{n+1}\overline{c_n}}{|c_{n+1}|}\right)^{j-1} c_{n+1} \quad \text{for } j = 2, 3, \cdots. \tag{5.8}$$

Thus each c_j $(j = 0, 1, 2, \cdots)$ is uniquely determined. Therefore k should be exactly of the form

$$k(z) = c_n z^n + c_{n+1} z^{n+1} + c_{n+1} \sum_{j=2}^{\infty} \left(-\frac{c_{n+1}\overline{c_n}}{|c_{n+1}|} \right)^{j-1} z^{n+j}.$$

Put $\alpha := -\frac{c_n \overline{c_{n+1}}}{|c_{n+1}|}$. Then $|\alpha| = |c_n| < 1$ and a straightforward calculation shows

$$k(z) = \frac{c_{n+1}}{1 - |c_n|^2}\, z^n\, \frac{z - \alpha}{1 - \bar{\alpha}\, z} \quad \text{with } \alpha = -\frac{c_n \overline{c_{n+1}}}{|c_{n+1}|}.$$

This completes the proof. □

Lemma 6. *Let $\varphi = \bar{g} + f$, where f and g are analytic polynomials of degrees N and m, respectively. Suppose that g divides f and the leading coefficient of $\psi := \frac{f}{g}$ is μ. Then*

$$T_\varphi \text{ hyponormal} \Longrightarrow \left| \sum_{\zeta \in \mathcal{Z}(\psi)} \zeta \right| \leq |\mu| - \frac{1}{|\mu|}. \tag{6.1}$$

In particular, if $|\mu| = 1$ and if T_φ is hyponormal then $\sum_{\zeta \in \mathcal{Z}(\psi)} \zeta = 0$.

Proof. If g divides f, we can write $g(z) = b_0 + \sum_{n=1}^{m} \overline{a_{-n}} z^n = \overline{a_{-m}} \prod_{j=1}^{m} (z - \zeta_j)$ and $f(z) = \sum_{n=0}^{N} a_n z^n = a_N \prod_{j=1}^{N} (z - \zeta_j)$. A straightforward calculation shows that $a_{N-1} = -a_N \sum_{j=1}^{N} \zeta_j$ and $\overline{a_{-m+1}} = -\overline{a_{-m}} \sum_{j=1}^{m} \zeta_j$. By the recurrence relation (1.1) we have

$$\begin{cases} c_0 = \cdots & = c_{N-m-1} = 0; \quad c_{N-m} = \frac{a_{-m}}{\overline{a_N}}; \\ c_{N-m+1} & = (\overline{a_N})^{-1} \big(a_{-m+1} - c_{N-m} \overline{a_{N-1}} \big) \\ & = (\overline{a_N})^{-1} \left(-a_{-m} \sum_{j=1}^{m} \overline{\zeta_j} + \frac{a_{-m}}{\overline{a_N}} \cdot \overline{a_N} \sum_{j=1}^{N} \overline{\zeta_j} \right) \\ & = \frac{a_{-m}}{\overline{a_N}} \sum_{j=m+1}^{N} \overline{\zeta_j}. \end{cases} \tag{6.2}$$

Applying Lemma 5 with $n = N - m$, we have

$$\begin{cases} \Phi_0 = \cdots = \Phi_{N-m-1} = 0; \\ \Phi_{N-m} = c_{N-m} = \frac{a_{-m}}{\overline{a_N}}; \\ \Phi_{N-m+1} = \frac{c_{N-m+1}}{1 - |c_{N-m}|^2} = \frac{\frac{a_{-m}}{\overline{a_N}} \sum_{j=m+1}^{N} \overline{\zeta_j}}{1 - \ \frac{a_{-m}}{\overline{a_N}}^{\,2}}. \end{cases} \tag{6.3}$$

Therefore if T_φ is hyponormal then by Lemma 1, $|\Phi_{N-m+1}| \leq 1$, i.e.,

$$\left| \frac{a_{-m}}{a_N} \right| \left| \sum_{j=m+1}^{N} \zeta_j \right| \leq 1 - \left| \frac{a_{-m}}{a_N} \right|^2$$

or equivalently, since $\psi = \frac{a_N}{\overline{a_{-m}}} \prod_{j=m+1}^{N} (z - \zeta_j)$,

$$\left| \sum_{\zeta \in \mathcal{Z}(\psi)} \zeta \right| \leq \left| \frac{a_N}{a_{-m}} \right| - \left| \frac{a_{-m}}{a_N} \right| = |\mu| - \frac{1}{|\mu|}. \tag{6.4}$$

which proves (6.1). The second assertion is straightforward from (6.1). □

We are ready for:

Theorem 7. *Let $\varphi = \bar{g} + f$, where f and g are analytic polynomials of degrees N and m $(m \geq 3)$, respectively. Suppose that g divides f and the leading coefficient of $\psi := \frac{f}{g}$ is μ with $|\mu| > 1$. If $\left|\sum_{\zeta \in \mathcal{Z}(\psi)} \zeta\right| = |\mu| - \frac{1}{|\mu|}$ then we have:*

- (i) *If $N < 2m - 1$ then T_φ is never hyponormal;*
- (ii) *If $N \geq 2m - 1$ then T_φ is hyponormal if and only if the Fourier coefficients $\hat{\psi}(j)$ $(N - 2m + 1 \leq j \leq N - m)$ of ψ satisfy the following equation:*

$$\begin{pmatrix} \hat{\psi}(N-2m+2) \\ \hat{\psi}(N-2m+3) \\ \vdots \\ \vdots \\ \hat{\psi}(N-m-1) \end{pmatrix} = \frac{\mu - \frac{1}{\bar{\mu}}}{\hat{\psi}(N-m-1)} \begin{pmatrix} \hat{\psi}(N-2m+1) \\ \hat{\psi}(N-2m+2) \\ \vdots \\ \vdots \\ \hat{\psi}(N-m-2) \end{pmatrix}. \tag{7.1}$$

In particular, the hyponormality of T_φ is independent of the particular values of the Fourier coefficients $\hat{\psi}(0), \hat{\psi}(1), \cdots, \hat{\psi}(N-2m)$ of ψ.

Proof. We first claim that if T_φ is hyponormal then

$$\operatorname{rank}\,[T_\varphi^*, T_\varphi] = N - m + 1 \iff \left|\sum_{\zeta \in \mathcal{Z}(\psi)} \zeta\right| = |\mu| - \frac{1}{|\mu|}. \tag{7.2}$$

Indeed if $\operatorname{rank}\,[T_\varphi^*, T_\varphi] = N - m + 1$, then in view of Lemma 3 (iv), the finite Blaschke product $k \in \mathcal{E}(\varphi)$ is of the form

$$k(z) = e^{i\theta} z^{N-m} \frac{z - \xi}{1 - \bar{\xi} z} \quad (0 < |\xi| < 1,\ \theta \in [0, 2\pi)).$$

But by (1.1), $k_p(z) = c_{N-m} z^{N-m} + c_{N-m+1} z^{N-m+1}$ is the unique analytic polynomial of degree less that $N - m + 2$ satisfying $\varphi - k_p \bar{\varphi} \in H^\infty$. A straightforward calculation shows that $k(z)$ should be of the form

$$k(z) = e^{i\theta}(-\xi) z^{N-m} + e^{i\theta}(1 - |\xi|^2) z^{N-m+1} + \sum_{j=N-m+2}^{\infty} c_j z^j.$$

By the uniqueness of k_p, we have that $c_{N-m} = -e^{i\theta}\xi$ and $c_{N-m+1} = e^{i\theta}(1 - |\xi|^2)$, which implies that $|c_{N-m+1}| = 1 - |c_{N-m}|^2$. Thus by (6.3) we have that $|\Phi_{N-m+1}| = 1$, or equivalently, $\left|\sum_{\zeta \in \mathcal{Z}(\psi)} \zeta\right| = |\mu| - \frac{1}{|\mu|}$. Conversely, suppose $\left|\sum_{\zeta \in \mathcal{Z}(\psi)} \zeta\right| = |\mu| - \frac{1}{|\mu|}$, i.e., $|\Phi_{N-m+1}| = 1$. Then by Lemma 5, $\mathcal{E}(\varphi)$ consists of only one element as the following finite Blaschke product:

$$k(z) = \frac{c_{N-m+1}}{1 - |c_{N-m}|^2} z^{N-m} \frac{z - \alpha}{1 - \overline{\alpha}\, z} \quad \text{with } \alpha = -\frac{c_{N-m}\overline{c_{N-m+1}}}{|c_{N-m+1}|}.$$

Since $\deg(k) = N - m + 1$, it follows that $\operatorname{rank}[T_\varphi^*, T_\varphi] = N - m + 1$. This proves (7.2). Write

$$\psi(z) := \mu \prod_{j=1}^{N-m} (z - \gamma_j).$$

Suppose T_φ is hyponormal. By our assumption and (7.2), $[T_\varphi^*, T_\varphi]$ is of rank $N - m + 1$. Thus by Lemma 3 (iv), the finite Blaschke product $k \in \mathcal{E}(\varphi)$ should be of the form

$$k(z) = e^{i\omega} z^{N-m} \frac{z - \xi}{1 - \bar{\xi} z} \quad (0 < |\xi| < 1,\ \omega \in [0, 2\pi)). \tag{7.3}$$

Thus we have

$$\begin{aligned} T_\varphi \text{ hyponormal} &\Longleftrightarrow \varphi - k\,\bar{\varphi} \in H^\infty \quad \text{with } k \text{ in (7.3)} \\ &\Longleftrightarrow \bar{g} - e^{i\omega} z^{N-m} \frac{z - \xi}{1 - \bar{\xi} z} \bar{f} \in H^\infty \\ &\Longleftrightarrow \bar{g} - e^{i\omega} z^{N-m} \frac{z - \xi}{1 - \bar{\xi} z} \bar{g}\, \bar{\mu} \prod_{j=1}^{N-m} (\bar{z} - \overline{\gamma_j}) \in H^\infty \quad \text{(because } f = g\psi) \\ &\Longleftrightarrow \bar{g} \left(1 - e^{i\omega} \frac{z - \xi}{1 - \bar{\xi} z} \bar{\mu} \prod_{j=1}^{N-m} (1 - \overline{\gamma_j} z) \right) \in H^\infty \\ &\Longleftrightarrow 1 - e^{i\omega} \frac{z - \xi}{1 - \bar{\xi} z} \bar{\mu} \prod_{j=1}^{N-m} (1 - \overline{\gamma_j} z) \in z^m H^\infty \qquad (7.4) \end{aligned}$$

Substituting $z = 0$ into (7.4) gives that $1 - e^{i\omega}(-\xi)\bar{\mu} = 0$, and hence $e^{i\omega} = -(\bar{\mu}\xi)^{-1}$. Thus T_φ is hyponormal if and only if

$$\frac{z - \xi}{1 - \bar{\xi} z} \bar{\mu} \prod_{j=1}^{N-m} (1 - \overline{\gamma_j} z) = -\bar{\mu}\xi + \sum_{j=m}^{\infty} c_j z^j \quad \text{for some } c_j\ (j = m, m+1, \cdots)$$

or equivalently,

$$(z - \xi)\, \bar{\mu} \prod_{j=1}^{N-m} (1 - \overline{\gamma_j} z) = (1 - \bar{\xi} z) \left(-\bar{\mu}\xi + \sum_{j=m}^{\infty} c_j z^j \right) \quad \text{for some } c_j\ (j = m, m+1, \cdots). \tag{7.5}$$

Note that

$$\bar{\mu} \prod_{j=1}^{N-m} (1 - \overline{\gamma_j} z) = z^{N-m} \overline{\psi(z)} = \overline{\hat{\psi}(N-m)} + \overline{\hat{\psi}(N-m-1)}\, z + \cdots + \overline{\hat{\psi}(0)}\, z^{N-m}.$$

Thus solving (7.5) gives
(7.6)

$$\begin{cases} \bar{\mu}\,|\xi|^2 = \bar{\mu} - \xi\,\overline{\hat{\psi}(N-m-1)} \\ 0 = \overline{\hat{\psi}(j)} - \xi\,\overline{\hat{\psi}(j-1)} \quad (j = N-m-1, \cdots, N-2m+2) \\ c_m = \overline{\hat{\psi}(N-2m+1)} - \xi\,\overline{\hat{\psi}(N-2m)} \\ c_{m+j} - \bar{\xi}\,c_{m+j-1} = \overline{\hat{\psi}(N-2m+1-j)} - \xi\,\overline{\hat{\psi}(N-2m-j)} \quad (j = 1, \cdots, N-2m+1) \\ c_{m+j} - \bar{\xi}\,c_{m+j-1} = 0 \quad (j = N-2m+2, N-2m+3, \cdots), \end{cases}$$

where for notational convenience, we let $\hat{\psi}(j) := 0$ for $j < 0$. If $N < 2m-1$ then a telescoping argument with the second equation of (7.6) gives that $\hat{\psi}(j) = 0$ for all $j = N-2m+2, \cdots, N-m-1$, so that from the first equation of (7.6) we have that $\bar{\mu}\,|\xi|^2 = \bar{\mu}$ and hence $|\xi| = 1$, a contradiction. Therefore, if $N < 2m-1$ then T_φ is never hyponormal. Now suppose that $N \geq 2m-1$. Since $\left|\frac{c_{m+j}}{c_{m+j-1}}\right| = |\xi| < 1$ for all $j = N-2m+2, N-2m+3, \cdots$, we must have that $\sum_{j=m}^{\infty} |c_j|^2 < \infty$, i.e., $\sum_{j=m}^{\infty} c_j z^j \in H^\infty$. Therefore the solutions of (7.6) equal those of the first two equations of (7.6), i.e.,

$$\begin{cases} \frac{\hat{\psi}(N-m-1)}{\hat{\psi}(N-m-2)} = \frac{\hat{\psi}(N-m-2)}{\hat{\psi}(N-m-3)} = \cdots = \frac{\hat{\psi}(N-2m+2)}{\hat{\psi}(N-2m+1)} = \bar{\xi} = -\frac{e^{i\omega}}{\mu} \\ \mu\,\left|\frac{\hat{\psi}(N-m-1)}{\hat{\psi}(N-m-2)}\right|^2 = \mu - \frac{\hat{\psi}(N-m-1)^2}{\hat{\psi}(N-m-2)}. \end{cases} \tag{7.7}$$

But the second equality in (7.7) is equivalent to

$$\frac{1}{\bar{\mu}} = \mu + \frac{e^{i\omega}}{\mu}\hat{\psi}(N-m-1), \tag{7.8}$$

which implies

$$-\frac{e^{i\omega}}{\mu} = \frac{1}{\hat{\psi}(N-m-1)}(\mu - \frac{1}{\bar{\mu}}). \tag{7.9}$$

Therefore by the first equation of (7.7) and (7.9), T_φ is hyponormal if and only if

$$\frac{\hat{\psi}(N-m-1)}{\hat{\psi}(N-m-2)} = \frac{\hat{\psi}(N-m-2)}{\hat{\psi}(N-m-3)} = \cdots = \frac{\hat{\psi}(N-2m+2)}{\hat{\psi}(N-2m+1)} = \frac{1}{\hat{\psi}(N-m-1)}(\mu - \frac{1}{\bar{\mu}}).$$

This completes the proof. □

The following is an immediate result of Theorem 7.

Corollary 8. *Suppose that* $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$*, where* $3 \le m \le N$ *and* $|a_{-m}| < |a_N|$*. If* $|a_N|^2 - |a_{-m}|^2 = \left| det \begin{pmatrix} \overline{a_{-m}} & \overline{a_{-m+1}} \\ a_N & a_{N-1} \end{pmatrix} \right|$*, then*

$$T_\varphi \text{ hyponormal} \iff d_{j+1} = \left[\frac{|a_N|^2 - |a_{-m}|^2}{det \begin{pmatrix} \overline{a_{-m}} & \overline{a_{-m+1}} \\ a_N & a_{N-1} \end{pmatrix}} \overline{\left(\frac{a_{-m}}{a_N} \right)} \right] \cdot d_j \quad (j = 1, \cdots, m-2),$$

where the d_j *are given by*

$$\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_m \end{pmatrix} := \begin{pmatrix} \overline{a_{-m}} & \overline{a_{-m+1}} & \cdots & \overline{a_{-2}} & \overline{a_{-1}} \\ & \overline{a_{-m}} & \overline{a_{-m+1}} & \cdots & \overline{a_{-2}} \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & \overline{a_{-m+1}} \\ & & & & \overline{a_{-m}} \end{pmatrix}^{-1} \begin{pmatrix} a_{N-m+1} \\ a_{N-m+2} \\ \vdots \\ \vdots \\ a_N \end{pmatrix}. \tag{8.1}$$

Proof. Write $\varphi = \bar{g} + f$, where $f = P\varphi$. Applying Lemma 4 shows that there exists a trigonometric polynomial $\widetilde{\varphi}$ of the form $\widetilde{\varphi} := \bar{g} + \widetilde{f}$, where $\widetilde{f}$ is an analytic polynomial of degree $2m$ such that g divides $\widetilde{f}$ and T_φ is hyponormal if and only if $T_{\widetilde{\varphi}}$ is. Note that if $\psi := \frac{\widetilde{f}}{g}$ then from (4.1) we can see that the Fourier coefficients $\hat{\psi}(j)$ are given by the values d_j in (8.1). A straightforward calculation shows that

$$\hat{\psi}(m) = \frac{a_N}{\overline{a_{-m}}} \quad \text{and} \quad \hat{\psi}(m-1) = \frac{a_{N-1}\overline{a_{-m}} - \overline{a_{-m+1}}a_N}{\overline{a_{-m}}^2},$$

so

$$\left| \sum_{\zeta \in \mathcal{Z}(\psi)} \zeta \right| = \left| \frac{\hat{\psi}(m-1)}{\hat{\psi}(m)} \right| = \left| \frac{a_{N-1}\overline{a_{-m}} - \overline{a_{-m+1}}a_N}{\overline{a_{-m}}a_N} \right|.$$

Therefore we have

$$|a_N|^2 - |a_{-m}|^2 = \left| \det \begin{pmatrix} \overline{a_{-m}} & \overline{a_{-m+1}} \\ a_N & a_{N-1} \end{pmatrix} \right| \iff \left| \sum_{\zeta \in \mathcal{Z}(\psi)} \zeta \right| = |\hat{\psi}(m)| - \frac{1}{|\hat{\psi}(m)|}.$$

Now applying Theorem 7 with $N = 2m$ and $\hat{\psi}(n) = d_n$ $(n = 1, \cdots, m)$ gives the result. $\square$

Example 9. Consider the trigonometric polynomial

$$\varphi(z) = z^{-3} + 2z^{-2} + \alpha z^{-1} + \beta z + z^2 + 2z^3 \quad (\alpha, \beta \in \mathbb{C}). \tag{9.1}$$

Then φ satisfies the assumptions of Corollary 8. By (8.1),

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & \bar{\alpha} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \beta \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \beta - 2\bar{\alpha} + 6 \\ -3 \\ 2 \end{pmatrix}.$$

Thus by Corollary 8, T_φ is hyponormal if and only if

$$d_2 = -\frac{1}{2} d_1, \quad \text{i.e.,} \quad 2\bar{\alpha} - \beta = 0.$$

Therefore we have

$$\{(\alpha, \beta) \in \mathbb{C}^2 : T_\varphi \text{ is hyponormal}\} = \{(\alpha, \beta) \in \mathbb{C}^2 : \beta = 2\bar{\alpha}\}. \tag{9.2}$$

Example 10. Consider the trigonometric polynomial

$$\varphi(z) = z^{-4} + 2z^{-3} + \alpha z^{-2} + \beta z^2 + z^3 + 2z^4 \quad (\alpha, \beta \in \mathbb{C}).$$

We are tempted to guess that (9.2) is still true. But this is not the case. To see this observe that φ satisfies the assumptions of Corollary 8. By (8.1),

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & \bar{\alpha} & 0 \\ 0 & 1 & 2 & \bar{\alpha} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \beta \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2\beta+7\bar{\alpha}-12 \\ \beta-2\bar{\alpha}+6 \\ -3 \\ 2 \end{pmatrix}.$$

Thus by Corollary 8, T_φ is hyponormal if and only if $d_{j+1} = -\frac{1}{2}\, d_j$ $(j = 1, 2)$. Therefore T_φ is hyponormal if and only if $\alpha = \beta = 0$.

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Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
E-mail address: `ishwang@math.skku.ac.kr`

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
E-mail address: `ihkim@math.skku.ac.kr`

Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
E-mail address: `wylee@yurim.skku.ac.kr`