COWEN SETS FOR TOEPLITZ OPERATORS WITH FINITE RANK SELFCOMMUTATORS

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Abstract. Cowen's theorem states that if $\varphi \in L^\infty(\mathbb{T})$ then the Toeplitz operator T_φ is hyponormal if and only if the following 'Cowen' set $\mathcal{E}(\varphi)$ is nonempty:

$$\mathcal{E}(\varphi) = \{k \in H^\infty(\mathbb{T}) : ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\}.$$

In this paper, we give a complete description on the Cowen set $\mathcal{E}(\varphi)$ if the selfcommutator $[T_\varphi^*, T_\varphi]$ is of finite rank. In particular, it is shown that the solution for the cases where φ is of bounded type has a connection with a H^∞ optimization problem.

1. Introduction

A bounded linear operator A on a Hilbert space $\mathfrak{H}$ is said to be hyponormal if its selfcommutator $[A^*, A] = A^*A - AA^*$ is positive semidefinite. Recall that given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator with symbol φ is the operator T_φ on the Hardy space $H^2(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$ in the complex plane $\mathbb{C}$ defined by

$$T_\varphi f = P(\varphi \cdot f),$$

where $f \in H^2(\mathbb{T})$ and P denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. Relationships between hyponormal operators and Toeplitz-like operators were discovered in papers [NF] and [Cla]. More recently, the problem of determining which symbols induce hyponormal Toeplitz operators was completely solved by C. Cowen [Co] in 1988. Here we shall employ an equivalent variant of Cowen's theorem that was proposed by T. Nakazi and K. Takahashi in [NT].

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Cowen's theorem. [Co], [NT] *Suppose that* $\varphi \in L^\infty(\mathbb{T})$ *is arbitrary and put*

$$\mathcal{E}(\varphi) := \{k \in H^\infty(\mathbb{T}) : ||k||_\infty \leq 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty(\mathbb{T})\}.$$

Then T_φ *is hyponormal if and only if the set* $\mathcal{E}(\varphi)$ *is nonempty.*

Cowen's method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution of a certain functional equation involving its symbol. This approach has been put to use in the works [CL1], [CL2], [FL], [GS], [HKL], [HL], [NT], [Zhu] to study Toeplitz operators on the Hardy space of the unit circle.

Now the set $\mathcal{E}(\varphi)$ will be called the *Cowen set* for the function $\varphi \in L^\infty(\mathbb{T})$. The question about the Cowen set $\mathcal{E}(\varphi)$ is of great interest. Indeed, $\mathcal{E}(\varphi)$ has been studied intensively in recent literature because when φ is of bounded type (i.e., quotient of two bounded analytic functions), it has a connection with the following H^∞ optimization problem which naturally arise in robust control theory (cf. [FF]):

H^∞ **optimization problem.** *Let* $k_0 \in L^\infty(\mathbb{T})$ *and* θ *a fixed inner function in* $H^\infty(\mathbb{T})$*. Find* μ *where*

$$\mu = \text{dist}\,(k_0, \theta H^\infty) \equiv \inf_{h \in H^\infty} ||k_0 - \theta h||_\infty.$$

In this paper it is shown that via Nehari's theorem and Adamyan-Arov-Krein theorem, a solution of a H^∞ optimization problem provides information on $\mathcal{E}(\varphi)$ when φ is of bounded type and T_φ has finite rank selfcommutator.

2. Main Results

We begin with the connection between Hankel and Toeplitz operators. For φ in $L^\infty(\mathbb{T})$, the *Hankel operator* $H_\varphi : H^2 \to H^2$ is defined by

$$H_\varphi f = J(I - P)(\varphi f),$$

where $J : (H^2)^\perp \to H^2$ is given by $Jz^{-n} = z^{n-1}$ for $n \geq 1$. For $\zeta \in L^\infty(\mathbb{T})$, we define

$$\widetilde{\zeta} = \overline{\zeta(\overline{z})}.$$

The following is a basic connection between Hankel and Toeplitz operators:

$$T_{\varphi\psi} - T_\varphi T_\psi = H^*_{\overline{\varphi}} H_\psi \quad (\varphi, \psi \in L^\infty) \quad \text{and} \quad H_\varphi T_h = H_{\varphi h} = T^*_{\widetilde{h}} H_\varphi \quad (h \in H^\infty).$$

From this we can see that if $k \in \mathcal{E}(\varphi)$ then

(1) $$[T_\varphi^*, T_\varphi] = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_\varphi H_\varphi = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_{k\,\overline{\varphi}} H_{k\,\overline{\varphi}} = H^*_{\overline{\varphi}} (1 - T_{\widetilde{k}} T^*_{\widetilde{k}}) H_{\overline{\varphi}}.$$

For an inner function θ, we write

$$\mathcal{H}(\theta) \equiv H^2 \ominus \theta H^2.$$

If $\varphi \in L^\infty$, write

$$\varphi_+ \equiv P(\varphi) \in H^2 \quad \text{and} \quad \varphi_- \equiv \overline{(I-P)(\varphi)} \in zH^2.$$

Thus we can write $\varphi = \varphi_+ + \overline{\varphi_-}$. Assume that φ is of bounded type, i.e., there are functions ψ_1, ψ_2 in $H^\infty(\mathbb{D})$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$$

for almost all $z \in \mathbb{T}$. Since $T_{\overline{z}} H_\varphi = H_\varphi T_z$ it follows from Beurling's theorem that $\ker H_{\overline{\varphi_-}} = \theta H^2$ and $\ker H_{\overline{\varphi_+}} = \theta_+ H^2$ for some inner functions θ, θ_+. If T_φ is hyponormal then by (1), $||H_{\overline{\varphi_+}} f|| \geq ||H_{\overline{\varphi_-}} f||$ for all $f \in H^2$, so that

$$\theta_+ H^2 = \ker H_{\overline{\varphi_+}} \subseteq \ker H_{\overline{\varphi_-}} = \theta H^2,$$

which implies that θ divides θ_+, i.e., $\theta_+ = \theta_0 \theta$ for some inner function θ_0. Thus if $\varphi = \varphi_+ + \overline{\varphi_-}$ is of bounded type and T_φ is hyponormal then we can write (cf. [GS])

$$\varphi_+ = \theta_0 \theta \bar{a} \quad \text{and} \quad \varphi_- = \theta \bar{b},$$

where $a \in \mathcal{H}(\theta_0 \theta)$ and $b \in \mathcal{H}(\theta)$. If $k_0 \in H^\infty$ is a solution of equation

(2) $$b - k_0 a = \theta h \quad \text{for some } h \in H^2$$

then $\mathcal{E}(\varphi)$ can be written as

$$\mathcal{E}(\varphi) = \{\theta_0(k_0 + \theta f) : \ f \in H^\infty \text{ and } ||k_0 + \theta f||_\infty \leq 1\}.$$

By Nehari's Theorem [Ne], we have

(3) $$\text{dist}\,(k_0,\ \theta H^\infty) = \inf_{f \in H^\infty} ||\bar{\theta} k_0 + f||_\infty = ||H_{\bar{\theta} k_0}||.$$

Thus we have (see [GS, Theorem 8])

(4) $$T_\varphi \text{ is hyponormal} \iff ||H_{\bar{\theta} k_0}|| \leq 1.$$

The following theorem is our main result, which gives a description on the Cowen set $\mathcal{E}(\varphi)$ when the selfcommutator $[T_\varphi^*, T_\varphi]$ is of finite rank. In fact we can prove more:

Theorem 1. *If φ is of bounded type then we have that:*

(a) *If* $\ker H_{\overline{\varphi}} \nsubseteq \ker\,[T_\varphi^*, T_\varphi]$ *then* $\mathcal{E}(\varphi)$ *is empty;*

(b) *If* $\ker H_{\overline{\varphi}} = \ker\,[T_\varphi^*, T_\varphi]$ *and* $\operatorname{rank}\,[T_\varphi^*, T_\varphi] < \infty$ *then* $\mathcal{E}(\varphi)$ *contains infinitely many inner functions;*

(c) *If* $\ker H_{\overline{\varphi}} \subsetneq \ker\,[T_\varphi^*, T_\varphi]$ *then* $\mathcal{E}(\varphi)$ *contains a unique function which is inner.*

If instead φ is not of bounded type such that T_φ is hyponormal then $\mathcal{E}(\varphi)$ contains a unique function.

To prove Theorem 1 we need auxiliary lemmas.

The following lemma is another version of Cowen's theorem.

Lemma 2. [CL1], [CL2, Lemma 1] *If $\varphi \equiv \varphi_+ + \overline{\varphi_-} \in L^\infty$, then $\mathcal{E}(\varphi) \neq \emptyset$ if and only if the equation $H_{\overline{\varphi_+}} k = \overline{z}\widetilde{\varphi_-}$ admits a solution k satisfying $||k||_\infty \le 1$.*

T. Nakazi and K. Takahashi [NT] noticed that if T_φ is a hyponormal operator such that its selfcommutator is of finite rank then $\mathcal{E}(\varphi)$ contains a finite Blaschke product.

Lemma 3. (Nakazi-Takahashi Theorem) [NT] *A Toeplitz operator T_φ is hyponormal and the rank of the selfcommutator $[T_\varphi^*, T_\varphi]$ is finite if and only if there exists a finite Blaschke product k in $\mathcal{E}(\varphi)$ of the form*

$$k(z) = e^{i\theta} \prod_{j=1}^{n} \frac{z-\beta_j}{1-\overline{\beta_j}z} \qquad (|\beta_j| < 1 \textit{ for } j = 1, \cdots, n).$$

such that $\deg(k) = \operatorname{rank}\,[T_\varphi^*, T_\varphi]$*, where* $\deg(k)$ *denotes the degree of k – meaning the number of zeros of k in the open unit disk $\mathbb{D}$.*

The following lemma is a solution of a H^∞ optimization problem.

Lemma 4. *If b and q are finite Blaschke products then*

$$\deg(b) \ge \deg(q) \iff \operatorname{dist}(b, qH^\infty) < 1. \tag{5}$$

Proof. In general, for a continuous function u on $\mathbb{T}$ with $|u| \equiv 1$,

$$\operatorname{dist}(u,\ H^\infty) < 1 \iff \operatorname{wind}(u) \ge 0, \tag{6}$$

where wind$(\cdot)$ denotes the winding number with respect to the origin: indeed, this follows from the fact that (see [Ni, Appendix 4, Theorem 41])

$$\text{dist}\,(u,\ H^\infty) < 1 \iff T_u \text{ is left invertible} \iff \text{wind}\,(u) \geq 0,$$

where the second implication comes from the observation that T_u is Fredholm and hence, by Coburn's theorem T_u is left or right invertible and the Fredholm index of T_u is equal to $-\text{wind}\,(u)$. Applying (6) to $u = \frac{b}{q}$ gives that

$$\text{dist}\,(b,\ q\,H^\infty) < 1 \iff \text{wind}\left(\frac{b}{q}\right) \geq 0 \iff \deg\,(b) \geq \deg\,(q).$$

$\square$

We are ready for:

Proof of Theorem 1. From (1) we can see that if T_φ is hyponormal then

$$\ker H_{\overline{\varphi}} \subseteq \ker\,[T_\varphi^*, T_\varphi],$$

which proves statement (a).

Towards statement (b), suppose φ is of bounded type. So we can write $\varphi = \theta_0\theta\bar{a} + \bar{\theta}b$ for $a \in \mathcal{H}(\theta_0\theta)$ and $b \in \mathcal{H}(\theta)$. Now suppose $\ker H_{\overline{\varphi}} = \ker\,[T_\varphi^*, T_\varphi]$ and $\text{rank}\,[T_\varphi^*, T_\varphi] < \infty$. Since $\ker H_{\overline{\varphi}} = \theta_0\theta H^2$ it follows that

$$\text{ran}\,[T_\varphi^*, T_\varphi] = \big(\ker\,[T_\varphi^*, T_\varphi]\big)^\perp = (\ker H_{\overline{\varphi}})^\perp = H^2 \ominus \theta_0\theta H^2,$$

which implies that $\theta_0\theta$ is a finite Blaschke product since $\text{ran}\,[T_\varphi^*, T_\varphi]$ is finite dimensional. Also, by Lemma 3 there exists a finite Blaschke product $\theta_0 k_0$ in $\mathcal{E}(\varphi)$ such that $\deg\,(\theta_0 k_0) = \text{rank}\,[T_\varphi^*, T_\varphi]$. Thus k_0 is a finite Blaschke product such that $\deg(\theta_0 k_0) = \text{rank}\, H_{\overline{\varphi}} = \deg(\theta_0\theta)$, and hence $\deg(k_0) = \deg(\theta)$. So by Lemma 4, we have that $\text{dist}\,(k_0,\ \theta H^\infty) < 1$, and hence by (3), $||H_{\bar{\theta}k_0}|| < 1$. Remembering Adamyan-Arov-Krein theorem which states that if $f \in L^\infty$ and $\text{dist}\,(f,\ H^\infty) < 1$ then $f + H^\infty$ contains a unimodular function, we can see that if $||H_{\overline{\theta}k_0}|| < 1$, then $k_0 + \theta H^\infty$ contains an inner function. Thus $\theta_0 k_0 + \theta_0\theta H^\infty$ contains an inner function, and in turn, $\mathcal{E}(\varphi)$ contains an inner function. Since

$$\begin{aligned} 1 > \text{dist}\,(\bar{\theta}\,k_0,\ H^\infty) &= \text{dist}\,(\bar{z}\bar{\theta}k_0,\ \bar{z}\,H^\infty) \\ &= \text{dist}\,(\bar{z}\bar{\theta}k_0 + \bar{z}c,\ H^\infty) \quad \text{for a suitable } c \\ &= ||H_{\bar{z}\bar{\theta}(k_0+\theta c)}||, \end{aligned}$$

we can choose different constants α_n $(n \in \mathbb{Z}_+)$ such that $||H_{\bar{z}\bar{\theta}(k_0+\theta\alpha_n)}|| < 1$. Applying again Adamyan-Arov-Krein theorem to $H_{\bar{z}\bar{\theta}(k_0+\theta\alpha_n)}$, there exists $q_n \in H^\infty$ such that $k_0 + \theta\alpha_n + z\theta q_n$ are inner functions. Evidently, $\theta_0 k_0 + \theta_0\theta(\alpha_n + z\theta q_n) \in \mathcal{E}(\varphi)$ and are different. This proves statement (b).

Towards statement (c), suppose $\ker H_{\overline{\varphi}} \subsetneq \ker\,[T_\varphi^*, T_\varphi]$. If $\mathcal{E}(\varphi)$ contains a function k which is not inner then $\ker\,(1 - T_{\widetilde{k}}T_{\widetilde{k}}^*) = \{0\}$: indeed if $g = T_{\widetilde{k}}T_{\widetilde{k}}^* g$ then $||g||^2 = ||T_{\widetilde{k}}^* g||^2$, and hence

$$\int |g|^2 d\mu = ||g||^2 = ||T_{\widetilde{k}}^* g||^2 \le ||\overline{\widetilde{k}} g||^2 = \int |\widetilde{k}|^2 |g|^2 d\mu,$$

which implies that $g = 0$ a.e. if $\widetilde{k}$ is not inner. Thus by (1) we have that $\ker\,[T_\varphi^*, T_\varphi] \subseteq \ker H_{\overline{\varphi}}$, which forces that $\ker H_{\overline{\varphi}} = \ker\,[T_\varphi^*, T_\varphi]$, a contradiction. If instead $\mathcal{E}(\varphi)$ contains two different inner functions then $\mathcal{E}(\varphi)$ has a function which is not inner: for if k_1 and k_2 $(k_1 \ne k_2)$ are inner functions in $\mathcal{E}(\varphi)$ then we can easily see that $\frac{k_1+k_2}{2} \in \mathcal{E}(\varphi)$ and $\frac{k_1+k_2}{2}$ is not an inner function since every inner function is an extreme point of the unit ball of H^∞. Thus $\mathcal{E}(\varphi)$ contains a unique inner function. This proves statement (c).

For the second assertion write $\varphi = \varphi_+ + \overline{\varphi_-}$. If φ is not of bounded type then by an argument of Abrahamse [Ab, Lemma 3], we have that $\ker H_{\overline{\varphi_+}} = \ker H_{\overline{\varphi}} = \{0\}$. Thus the solution k of the equation $H_{\overline{\varphi_+}} k = \overline{z}\widetilde{\varphi_-}$ should be unique. Thus the second assertion follows at once from Lemma 2. □

We would like to remark that if $H_{\overline{\theta}k_0}$ attains its norm (e.g., it is of finite rank) then $\text{dist}\,(k_0, \theta H^\infty) = 1$ implies that $\mathcal{E}(\varphi)$ contains a unique inner function. To see this, recall (cf. [Ni, p.202]) that if $f \in L^\infty$ and H_f attains its norm then $f + H^\infty$ contains a unique element of least norm which is of the form $\lambda\frac{\overline{h}}{h\nu}$, where $\lambda \in \mathbb{C}$, h is an outer function and ν is an inner function. So if $||H_{\overline{\theta}k_0}|| = 1$ and $H_{\overline{\theta}k_0}$ attains its norm then by (3), $\overline{\theta}k_0 + H^\infty$ contains a unique unimodular function. Thus $\mathcal{E}(\varphi)$ contains a unique inner function.

We now turn our attention to the cases of Toelplitz operators with symbols that are trigonometric polynomials. If φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^N a_n z^n$, where a_{-m} and a_N are nonzero, then the rank of the selfcommutator $[T_\varphi^*, T_\varphi]$ is finite. Thus if T_φ is hyponormal then by Lemma 3, $\mathcal{E}(\varphi)$ contains a finite Blaschke product.

We now have:

Corollary 5. *Let* $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ *be such that* T_φ *is a hyponormal operator.*

(a) *If* $\text{rank}\,[T_\varphi^*, T_\varphi] < N$ *then* $\mathcal{E}(\varphi)$ *contains a unique finite Blaschke product;*

(b) *If* $\text{rank}\,[T_\varphi^*, T_\varphi] = N$ *then* $\mathcal{E}(\varphi)$ *contains infinitely many inner functions. Furthermore if* $b \in \mathcal{E}(\varphi)$ *is a finite Blaschke product then* $deg\,(b) \geq N$*.*

Proof. Since $\ker H_{\overline{\varphi}} = z^N H^2$, Part (a) corresponds to the case where $\ker H_{\overline{\varphi}} \subsetneq \ker\,[T_\varphi^*, T_\varphi]$ and Part (b) corresponds to the case where $\ker H_{\overline{\varphi}} = \ker\,[T_\varphi^*, T_\varphi]$. Thus the statment (a) and the first assertion of statement (b) follow at once from Theorem 1 together with Lemma 3.

For the second assertion of statement (b), assume to the contrary that $b \in \mathcal{E}(\varphi)$ is a finite Blaschke product of degree less than N. By Lemma 3, there exists a finite Blaschke product $k \in \mathcal{E}(\varphi)$ of degree N. Then we have

$$\widehat{k}(j) = \widehat{b}(j) \quad \text{for } j = 1, \cdots, N-1,$$

where $\widehat{f}(j)$ means the j-th Fourier coefficients of $f \in H^\infty$. Thus by the uniqueness argument of [HL, Lemma 1] we should have that $b = k$, a contradiction. □

Corollary 5(a) is an extended result of [HKL, Corollary 4].

The following is an immediate result from Corollary 5.

Corollary 6. *Suppose that* $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ *and that* k *is a finite Blaschke product in* $\mathcal{E}(\varphi)$*.*

(a) *If* $\deg\,(k) < N$ *then* $\text{rank}\,[T_\varphi^*, T_\varphi] = \deg\,(k)$*;*

(b) *If* $\deg\,(k) \geq N$ *then* $\text{rank}\,[T_\varphi^*, T_\varphi] = N$*.*

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