THE SPECTRUM IS CONTINUOUS ON THE
SET OF QUASI-$n$-HYPONORMAL OPERATORS

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Abstract. In this paper it is shown that the spectrum $\sigma$, a set valued function, is continuous when the function is restricted to the set of all ‘quasi-$n$-hyponormal’ operators acting on an infinite-dimensional separable Hilbert space, where a quasi-$n$-hyponormal operator is defined to be unitarily equivalent to an $n \times n$ upper triangular operator matrix whose diagonal entries are hyponormal operators.

1 Introduction

Throughout the paper, $\mathcal{H}$ denotes an infinite-dimensional separable Hilbert space. We write $\mathcal{B}(\mathcal{H})$ for the algebra of bounded linear operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ for the ideal of compact operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called an $n$-normal operator if there exists a maximal abelian self-adjoint algebra $\mathcal{R}$ such that $T$ is in the commutant of $\mathcal{R}^{(n)}$, where $\mathcal{R}^{(n)}$ denotes the direct sum of $n$ copies of $\mathcal{R}$. The class of $n$-normal operators was first studied by A. Brown [Br] and have been much studied (see, for example, [Br], [Fo], [Hoo], [Pe], [RR1], [RR2]). From the definition we can see that $T \in \mathcal{B}(\mathcal{H})$ is $n$-normal if and only if it is unitarily equivalent to an $n \times n$ operator matrix $(N_{ij})$ acting on $\mathcal{K}^{(n)}$, where $\{N_{ij}\}$ is a collection of commuting normal operators on a separable Hilbert space $\mathcal{K}$ (cf. [RR2, Theorem 7.17]). In fact, the notion of $n$-normality was chosen as to be a generalization in operator form of the $n \times n$ complex-valued matrices in a way parallel to the way in which a normal operator is a generalization of a complex number (cf. [Br]). Moreover it was well known ([Fo], [RR2, Theorem 7.2]) that each $n$-normal operator has an upper triangular form: i.e., if $T$ is $n$-normal then $T$ is unitarily equivalent to

$$
\begin{pmatrix}
N_{11} & N_{12} & \cdots & N_{1n} \\
0 & N_{22} & \cdots & N_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & N_{nn}
\end{pmatrix}
$$

where $\{N_{ij}\}_{1 \leq i \leq j \leq n}$ consists of mutually commuting normal operators on a separable Hilbert space $\mathcal{K}$. Evidently, the classes of normal and 1-normal operators coincide.

We now introduce a class of operators which contains the class of hyponormal operators as well as $n$-normal operators.

Definition 1. Let $\mathcal{K}$ be a separable complex Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a quasi-$n$-hyponormal operator (for $n \in \mathbb{N}$) if it is unitarily equivalent to an $n \times n$ upper triangular operator matrix $(N_{ij})$ acting on $\mathcal{K}^{(n)}$, where the diagonal entries $N_{jj}$ ($j = 1, 2, \cdots, n$) are hyponormal operators in $\mathcal{B}(\mathcal{K})$.

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Clearly, the classes of hyponormal and quasi-1-hyponormal operators coincide. The term “n-hyponormal” operators is reserved for \(n \times n\) upper triangular operator matrices \((N_{ij})\) whose all entries are commuting hyponormal operators. So evidently, \(n\)-normal \(\Rightarrow n\)-hyponormal \(\Rightarrow\) quasi-\(n\)-hyponormal. For example, every algebraic operator (i.e., an operator \(T\) for which \(p(T) = 0\) for a non-zero polynomial \(p\)) is quasi-\(n\)-hyponormal (see [Pe, Theorem 6.11]).

Let \(K\) denote the set, equipped with the Hausdorff metric, of all compact subsets of the complex plane \(C\). Then the spectrum \(\sigma\) can be viewed as a function \(\sigma: \mathcal{B}(\mathcal{H}) \to K\), mapping each operator \(T\) to its spectrum \(\sigma(T)\). It is well-known that the function \(\sigma\) is upper semicontinuous, and that \(\sigma\) does have points of discontinuity. J. Newburgh [Ne] gave the fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel [CoM] have undertaken a detailed study of spectral continuity in the case where the Banach algebra is the \(C^*\)-algebra of all operators acting on a complex separable Hilbert space. Of interest is the identification of classes \(E\) of operators for which \(\sigma\) becomes continuous when restricted to \(E\). In [Ne] it was shown that \(\sigma\) is continuous on the set of normal operators (also see [Hal, Solution 105]). This argument can be easily extended to the set of hyponormal operators. In [FaL], the continuity of \(\sigma\) was considered when the function is restricted to certain subsets of Toeplitz operators on the Hardy space of the unit circle. Also it was shown in [BGS] that \(\sigma\) is discontinuous on the entire manifold of Toeplitz operators. Recently it was shown in [HwL] that \(\sigma\) is continuous on the set of \(p\)-hyponormal operators (i.e., \((T^*T)^p \geq (TT^*)^p\) for some \(0 < p \leq 1\)). The purpose of the present paper is to show that the function \(\sigma\) is continuous when restricted to the set of all quasi-\(n\)-hyponormal operators.

**Theorem 1.** The spectrum \(\sigma\) is continuous on the set of all quasi-\(n\)-hyponormal operators.

In Section 2 we provide auxiliary lemmas needed to prove the main theorem and Section 3 devotes a proof of the main theorem.

### 2 Auxiliary lemmas

If \(T \in \mathcal{B}(\mathcal{H})\), we write \(\rho(T)\) for the resolvent of \(T\); \(\sigma(T)\) for the spectrum of \(T\); \(\pi_0(T)\) for the eigenvalues of \(T\); \(\pi_{00}(T)\) for the isolated points of \(\sigma(T)\) which are eigenvalues of finite multiplicity. An operator \(T \in \mathcal{B}(\mathcal{H})\) is called left-Fredholm if it has closed range with finite-dimensional null space and right-Fredholm if it has closed range with its range of finite co-dimension. If \(T\) is either left- or right-Fredholm we call it semi-Fredholm and Fredholm if it is both. The index of a semi-Fredholm operator \(T\), denoted by \(\text{ind}(T)\), is given by the integer \(\text{ind}(T) := \dim T^{-1}(0) - \dim T(\mathcal{H})\)\(^{-1}\). An operator \(T \in \mathcal{B}(\mathcal{H})\) is called Weyl if it is Fredholm of index zero. The essential spectrum, \(\sigma_e(T)\), and the Weyl spectrum, \(\omega(T)\), of \(T \in \mathcal{B}(\mathcal{H})\) are defined by

\[
\sigma_e(T) := \{\lambda \in C : T - \lambda \text{ is not Fredholm}\};
\]

\[
\omega(T) := \{\lambda \in C : T - \lambda \text{ is not Weyl}\}.
\]

H. Weyl [We] has shown that every hermitian operator \(T \in \mathcal{B}(\mathcal{H})\) satisfies the equality

\[
\sigma(T) \setminus \omega(T) = \pi_{00}(T).
\]

Today we say that *Weyl’s theorem holds for T* if \(T\) satisfies the equality (2.1). Weyl’s theorem has been extended from hermitian operator to hyponormal operators, to Toeplitz operators by L. Coburn [Co] and to several classes of operators including hyponormal operators by many authors.

If \(T \in \mathcal{B}(\mathcal{H})\), a hole in \(\sigma_e(T)\) is a bounded component of \(C \setminus \sigma_e(T)\) and a pseudohole in \(\sigma_e(T)\) is a component of \(\sigma_e(T) \setminus \sigma_{le}(T)\) or \(\sigma_e(T) \setminus \sigma_{re}(T)\). The spectral picture, \(\mathcal{SP}(T)\), of \(T \in \mathcal{B}(\mathcal{H})\) is
the structure consisting of the set \( \sigma_e(T) \), the collection of holes and pseudoholes in \( \sigma_e(T) \), and the indices associated with these holes and pseudoholes. Recall (see [Pe, Definition 4.8]) that an operator \( T \in \mathcal{B}(\mathcal{H}) \) is called *quasitriangular* if there exists a sequence \( \{P_n\}_{n=1}^\infty \) of projections of finite rank in \( \mathcal{B}(\mathcal{H}) \) that converges strongly to 1 and satisfies \( \|P_nTP_n - TP_n\| \to 0 \). An operator \( T \in \mathcal{B}(\mathcal{H}) \) is called *coquasitriangular* if \( T^* \) is quasitriangular. By Apostol, Foias, and Voiculescu [AFV], \( T \) is quasitriangular [coquasitriangular] if and only if for \( \lambda \in \mathbb{C} \), \( \mathcal{SP}(T) \) contains no hole or pseudohole associated with a positive number.

**Lemma 1.** If \( T \in \mathcal{B}(\mathcal{H}) \) is quasi-\( n \)-hyponormal then it is coquasitriangular.

**Proof.** If \( n = 1 \), this statement is clear. Assume that this is true for \( n = k \). Suppose that \( T \) is quasi-\((k + 1)\)-hyponormal. So we can write \( T = \begin{pmatrix} S & A \\ 0 & N \end{pmatrix} \), where \( S \) is quasi-\( k \)-hyponormal and \( N \) is hyponormal. We first show that \( \mathcal{SP}(T) \) contains no pseudohole associated with a positive number, that is, for \( \lambda \in \mathbb{C} \),

\[
T - \lambda \text{ is right-Fredholm } \Rightarrow T - \lambda \text{ is Fredholm.}
\]

Towards (2.2) suppose that \( T - \lambda \) is right-Fredholm. We may assume \( \lambda = 0 \). Then \( N \) is right-Fredholm. Since \( N \) is hyponormal, \( N \) must be Fredholm. On the other hand, by the Atkinson’s theorem, there exist operators \( X, Y, Z \) and \( W \) such that

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} S & A \\ 0 & N \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in \mathcal{K}(\mathcal{H}).
\]

Thus \( 1 - (SX + AZ) \) and \( NZ \) are both compact. But since \( N \) is Fredholm it follows that \( Z \) is compact. Therefore \( 1 - SX \) is compact, and hence \( S \) is right-Fredholm. By the inductive hypothesis on \( S \) we can see that \( S \) is Fredholm. Therefore \( T \) is Fredholm, which proves (2.2). Furthermore if \( \lambda \notin \sigma_e(T) \) then

\[
\text{ind}(T - \lambda) = \text{ind}(S - \lambda) + \text{ind}(N - \lambda) \leq 0,
\]

which shows that \( \mathcal{SP}(T) \) has no hole associated with a positive number. Hence \( T \) is coquasitriangular.

**Lemma 2.** Weyl’s theorem holds for quasi-\( n \)-hyponormal operators.

**Proof.** We use an induction. If \( n = 1 \), this is true. Assume that Weyl’s theorem holds for quasi-\( k \)-hyponormal operators. Suppose \( T \) is quasi-\((k + 1)\)-hyponormal. Then we can write \( T = \begin{pmatrix} S & A \\ 0 & N \end{pmatrix} \), where \( S \) is quasi-\( k \)-hyponormal and \( N \) is hyponormal. Thus we can see that (cf. [HLL, Corollary 11]) \( \sigma(S \oplus N) = \sigma(S) \cup \sigma(N) = \sigma(T) \). On the other hand, remember ([Le, Corollary 5]) that if \( T \) is weyl then (i) \( S \) is left-Fredholm; (ii) \( N \) is right-Fredholm; and (iii) \( S^{-1}(0) \oplus N^{-1}(0) \cong (\text{ran } S)^\perp \oplus (\text{ran } N)^\perp \). But since \( N \) is hyponormal, we have that \( N \) is Fredholm, and hence by (iii), \( S \) is also Fredholm. Since by Lemma 1, \( S \) and \( N \) are coquasitriangular, and \( 0 = \text{ind}(T) = \text{ind}(S) + \text{ind}(N) \), we can see that \( \text{ind}(S) = \text{ind}(N) = 0 \), i.e., \( S \) and \( N \) are both Weyl. Applying this for \( T - \lambda \) in place of \( T \) gives that \( \omega(S \oplus N) = \omega(S) \cup \omega(N) = \omega(T) \). On the other hand, we argue that every quasi-\( n \)-hyponormal operator is isolated, in the sense that all isolated points of the spectrum are eigenvalues. To see this, suppose \( R \) is quasi-\( n \)-hyponormal. Then \( R \) is unitarily equivalent to

\[
\begin{pmatrix} N_1 & * & \cdots & * \\ 0 & N_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & N_n \end{pmatrix}, \quad (N_j \text{ is hyponormal for } 1 \leq j \leq n).
\]
Then $\sigma(R) = \bigcup_{j=1}^{n} \sigma(N_j)$ since the $N_j$ are all hyponormal (see [HLL, Corollary 11]). Let $\lambda \in \sigma(T)$. We assume, without loss of generality, that $\lambda = 0$. Then $0 \in \sigma(N_j)$ for $j (1 \leq j \leq n)$. Suppose $k$ is the first integer such that $0 \in \sigma(N_k)$ (there exists such an integer since $0 \in \sigma(R)$). But since $N_k$ is isoid, $N_k$ is not one-one. So there exists a vector $x := (x_1, \ldots, x_k, 0 \cdots, 0)^T$ belongs to $R^{-1}(0)$, so $0$ is an eigenvalue of $R$. This shows that every quasi-$n$-hyponormal operator is isoid. Thus since $S$ and $N$ are isoid and Weyl’s theorem holds for $S$ and $N$ by the inductive hypothesis, we have that

$$
\pi_{00}(S \oplus N) = (\pi_{00}(S) \cap \rho(N)) \cup (\rho(S) \cap \pi_{00}(N)) \cup (\pi_{00}(S) \cap \pi_{00}(N)) = (\sigma(S) \cup \sigma(N)) \setminus \omega(S) \cup \omega(N)) = \sigma(S \oplus N) \setminus \omega(S \oplus N),
$$

which says that Weyl’s theorem holds for $S \oplus N$. Thus we have that $\sigma(T) \setminus \omega(T) = \sigma(S \oplus N) \setminus \omega(S \oplus N) = \pi_{00}(S \oplus N)$. But since $\sigma(T) = \sigma(S \oplus N)$, it follows at once that $\sigma(T) \setminus \omega(T) \subset \pi_{00}(T)$. For the reverse inclusion, suppose $\lambda \in \pi_{00}(T)$. We must show that $\lambda \in \pi_{00}(S \oplus N)$. Since $S$ and $N$ are isoid, it will suffice to show that $(S - \lambda)^{-1}(0) \oplus (N - \lambda)^{-1}(0)$ is finite-dimensional. Evidently, $(S - \lambda)^{-1}(0)$ is finite-dimensional because $(S - \lambda)^{-1}(0) \oplus \{0\} \subset (T - \lambda)^{-1}(0)$. Thus $\lambda \in \pi_{00}(S)$. We now assume to the contrary that $(N - \lambda)^{-1}(0)$ is infinite-dimensional. Thus $A((N - \lambda)^{-1}(0))$ is also infinite-dimensional; if it were not so then $(A(N - \lambda)^{-1}(0))^{-1}(0)$ is infinite-dimensional and hence, so is $(T - \lambda)^{-1}(0)$, a contradiction. On the other hand, since $\lambda \in \pi_{00}(S)$ and Weyl’s theorem holds for $S$, we have that $S - \lambda$ is Weyl, so that $(S - \lambda)(\mathcal{H})$ is finite-dimensional. Therefore $A((N - \lambda)^{-1}(0)) \cap (S - \lambda)(\mathcal{H})$ is infinite-dimensional. Then there exist an orthonormal sequence $\{y_j\}$ in $(N - \lambda)^{-1}(0)$ and an orthonormal sequence $\{x_j\}$ in $\mathcal{H}$ such that $Ay_j = (S - \lambda)x_j$. Thus we have

$$
\begin{pmatrix}
S - \lambda & A \\
0 & N - \lambda
\end{pmatrix}
\begin{pmatrix}
x_j \\
y_j
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

for each $j = 1, 2, \cdots$,

which implies that $(T - \lambda)^{-1}(0)$ is infinite-dimensional, a contradiction. Thus $(N - \lambda)^{-1}(0)$ is finite-dimensional, which completes the proof. \hfill \Box

If $T \in \mathcal{B}(\mathcal{H})$ then the reduced minimum modulus, denoted $\gamma(T)$, of $T$ is defined by

$$
\gamma(T) := \inf_{x \in \mathcal{H}} \frac{\|Tx\|}{\text{dist}(x, T^{-1}(0))},
$$

where $\frac{0}{0}$ is defined to be $\infty$. If $T \in \mathcal{B}(\mathcal{H})$ is a non-zero operator then we can see that $\gamma(T) = \inf (\sigma(|T|) \setminus \{0\})$. In particular if $T$ is invertible then $\gamma(T) = \|T^{-1}\|^{-1}$. On the other hand, if we write $r(A)$ for the spectral radius of an operator $A$ then

$$
\text{dist}(\lambda, \sigma(T)) = \text{dist}(0, \sigma(T - \lambda)) = \min_{\mu \in \sigma(T - \lambda)} |\mu| = \frac{1}{\max_{\mu \in \sigma(T - \lambda)} |\mu|} = \frac{1}{r((T - \lambda)^{-1})}.
$$

Thus if $T$ is hyponormal and $\lambda \notin \sigma(T)$ then since $(T - \lambda)^{-1}$ is normaloid, i.e., norm equals radius, it follows that

$$
\text{dist}(\lambda, \sigma(T)) = \frac{1}{r((T - \lambda)^{-1})} = \frac{1}{\|T - \lambda\|^{-1}} = \gamma(T - \lambda).
$$

By comparison we have:
Lemma 3. If $T$ is quasi-$n$-hyponormal and $\lambda \notin \sigma(T)$ then

\[(2.4) \quad ||(T - \lambda)^{-1}|| \leq \frac{(1 + ||T||)^{n-1}}{\min\{1, \text{dist}(\lambda, \sigma(T)) \}^n}.\]

Proof. Note that if $n = 1$, then (2.4) is obvious from (2.3). We also use an induction. It is easy to see that if $A, B \in B(H)$ and $A$ is invertible then $\gamma(AB) \geq \gamma(A)\gamma(B)$. We suppose that $T$ is quasi-$2$-hyponormal. So we can write $T = \begin{pmatrix} N_1 & N_3 \\ 0 & N_2 \end{pmatrix}$, where $N_1$ and $N_2$ are hyponormal operators. If $\lambda \notin \sigma(T)$ then $N_2 - \lambda$ is right invertible. But since $N_2$ is hyponormal it follows that $N_2 - \lambda$ is invertible. We thus have that

\[
\gamma(T - \lambda) = \gamma \left( \begin{pmatrix} N_1 - \lambda & N_3 \\ 0 & N_2 - \lambda \end{pmatrix} \right) \geq \gamma \left( \begin{pmatrix} 1 & 0 \\ 0 & N_2 - \lambda \end{pmatrix} \right) \gamma \left( \begin{pmatrix} 1 & N_3 \\ 0 & 1 \end{pmatrix} \right) = \gamma \left( \begin{pmatrix} 1 & N_3 \\ 0 & 1 \end{pmatrix} \right) \min\{1, \gamma(N_2 - \lambda)\} \min\{1, \gamma(N_1 - \lambda)\}.
\]

But since

\[
\gamma \left( \begin{pmatrix} 1 & N_3 \\ 0 & 1 \end{pmatrix} \right) = \frac{1}{|| \begin{pmatrix} 1 & N_3 \\ 0 & 1 \end{pmatrix}^{-1} ||} = \frac{1}{|| \begin{pmatrix} 1 & -N_3 \\ 0 & 1 \end{pmatrix} ||} \geq \frac{1}{1 + ||N_3||} \geq \frac{1}{1 + ||T||},
\]

it follows that

\[(2.5) \quad \gamma(T - \lambda) \geq (1 + ||T||)^{-1} \min\{1, \gamma(N_1 - \lambda), \gamma(N_2 - \lambda), \gamma(N_1 - \lambda)\gamma(N_2 - \lambda)\}.
\]

By (2.3) and the fact that $\sigma(T) = \sigma(N_1) \cup \sigma(N_2)$, we have that $\gamma(N_j - \lambda) = \text{dist}(\lambda, \sigma(N_j)) \geq \text{dist}(\lambda, \sigma(T))$ for each $j = 1, 2$, and hence, $\gamma(T - \lambda) \geq (1 + ||T||)^{-1} \min\{1, \text{dist}(\lambda, \sigma(T))^2\}$. So if $\lambda \notin \sigma(T)$ then

\[
||(T - \lambda)^{-1}|| = \frac{1}{\gamma(T - \lambda)} \leq \frac{1 + ||T||}{\min\{1, \text{dist}(\lambda, \sigma(T))^2\}}.
\]

Thus (2.4) holds for $n = 2$. We assume that (2.4) holds for $n = k$. Suppose $T$ is quasi-$(k + 1)$-hyponormal. Then we can write $T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where $A$ is quasi-$k$-hyponormal and $B$ is hyponormal. If $\lambda \notin \sigma(T)$, write

\[
T - \lambda = \begin{pmatrix} 1 & 0 \\ 0 & B - \lambda \end{pmatrix} \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - \lambda & 0 \\ 0 & 1 \end{pmatrix}.
\]

Then $B - \lambda$ is right invertible. But since $B$ is hyponormal, it follows that $B - \lambda$ is invertible. Thus by the same argument as (2.5) we have that

\[(2.6) \quad \gamma(T - \lambda) \geq (1 + ||T||)^{-1} \min\{1, \gamma(A - \lambda), \gamma(B - \lambda), \gamma(A - \lambda)\gamma(B - \lambda)\}.
\]

By the inductive hypothesis on $A$, we have

\[
\gamma(A - \lambda) \geq \frac{\min\{1, \text{dist}(\lambda, \sigma(A))^k\}}{(1 + ||T||)^{k-1}}.
\]
But since $\gamma(B - \lambda) = \text{dist}(\lambda, \sigma(B)) \geq \text{dist}(\lambda, \sigma(T))$, it follows from (2.6) that

$$
\gamma(T - \lambda) \geq \frac{\min\left\{1, \left[\text{dist}(\lambda, \sigma(T))\right]^{k+1}\right\}}{(1 + ||T||)^k},
$$

which implies that (2.4) holds for $n = k + 1$. This completes the proof. \hfill \Box

3 Proof of Theorem 1

We are ready for:

Proof of Theorem 1. We write $(Q_n)(\mathcal{H})$ for the set of all quasi-$n$-hyponormal operators. If $T \in \mathcal{B}(\mathcal{H})$, define $m_e(T)$ for the essential minimum modulus of $T$ (cf.[Bo]): i.e., $m_e(T) := \inf \sigma_e(|T|)$. Obviously,

$$(3.1) \quad m_e(T) > 0 \iff T \text{ is left-Fredholm.}$$

On the other hand, $m_e$ can be viewed as a function from $\mathcal{B}(\mathcal{H})$ to $\mathbb{R}$, mapping each operator $T$ to its essential minimum modulus $m_e(T)$. We claim that $m_e$ is a continuous function: indeed, if $T, T_n \in \mathcal{B}(\mathcal{H})$ ($n \in \mathbb{Z}_+$) are such that $T_n$ converges to $T$ in norm then $|T_n|$ converges to $|T|$ in norm (cf.[HwL, Lemma 1]) and $\lim \sigma_e(|T_n|) = \sigma_e(|T|)$ because $\sigma_e$ is continuous on the set of normal elements in a unital C*-algebra (cf. [Ne, Corollary 2]), which implies that $\lim m_e(T_n) = m_e(T)$. We also claim that there exists a constant $c > 0$ such that if $T \in (Q_n)(\mathcal{H})$ then

$$(3.2) \quad m_e(T - \lambda) \geq c \min\left\{1, \left[\text{dist}(\lambda, \sigma_e(T))^n\right]\right\} \quad \text{for } \lambda \not\in \sigma_e(T).$$

To prove (3.2) suppose $T \in (Q_n)(\mathcal{H})$ and $0 \not\in \sigma_e(T)$. If $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/K(\mathcal{H})$ is the Calkin homomorphism then we have that $m_e(T) = \inf \sigma(|\pi(T)|)$. We thus argue that if $\mathcal{B}(\mathcal{H})/K(\mathcal{H})$ is regarded as a $C^*$-subalgebra of $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ then since $\pi(T)$ is quasi-$n$-hyponormal, we have that by Lemma 3,

$$
m_e(T) = \inf \sigma(|\pi(T)|)
= \inf\left\{||\pi(T)x|| : ||x|| = 1, \ x \in \mathcal{K}\right\}
= \frac{1}{||\pi(T)^{-1}||}
\geq c \min\left\{1, \left[\text{dist}(0, \sigma(\pi(T)))^n\right]\right\} \quad \text{with } c := \frac{1}{(1 + ||\pi(T)||)^{n-1}}
= c \min\left\{1, \left[\text{dist}(0, \sigma_e(T))^n\right]\right\}.
$$

Applying this result with $T - \lambda$ in place of $T$ proves (3.2). Now suppose that $T_n, T \in (Q_n)(\mathcal{H})$, for $n \in \mathbb{Z}_+$, are such that $T_n$ converges to $T$ in norm. Since $\sigma$ is upper semicontinuous and $\lim \inf_n \sigma(T_n) \subset \sigma(T)$, it suffices to show that $\sigma(T) \subset \lim \inf_n \sigma(T_n)$. We first claim that $\sigma(T) \subset \lim \inf \sigma(T_n)$: indeed this follows at once from an argument of Newburgh [Ne, Lemma 3]: if $\lambda \in \sigma(T)$ then for every neighborhood $N(\lambda)$ of $\lambda$ there exists an $N \in \mathbb{Z}_+$ such that $n > N$ implies $\sigma(T_n) \cap N(\lambda) \neq \emptyset$. This shows that $\lambda \in \lim \inf_n \sigma(T_n)$. So it suffices to show that $\sigma(T) \subset \lim \inf \sigma(T_n)$. To show this let $\lambda \in \text{acc } \sigma(T)$ and assume to the contrary that $\lambda \not\in \lim \inf_n \sigma(T_n)$. Then there exists a
neighborhood \( N(\lambda) \) of \( \lambda \) which does not intersect infinitely many \( \sigma(T_n) \). Thus we can choose a subsequence \( \{T_{n_k}\}_k \) of \( \{T_n\}_n \) such that for some \( \epsilon > 0 \), \( \text{dist} (\lambda, \sigma(T_{n_k})) > \epsilon \) for all \( k \in \mathbb{Z}_+ \). Since \( \text{dist} (\lambda, \sigma(T_{n_k})) \leq \text{dist}(\lambda, \sigma(T_{n_k} - \lambda)) \), it follows that \( m_e(T_{n_k} - \lambda) > \delta \) for some \( \delta > 0 \) and all \( k \in \mathbb{Z}_+ \). Since \( m_e \) is continuous, we have that \( m_e(T - \lambda) \geq \delta \), which by (3.1), implies that \( T - \lambda \) is left-Fredholm. By the continuity of the semi-Fredholm index, \( \text{ind}(T - \lambda) = \lim_{k \to \infty} \text{ind}(T_{n_k} - \lambda) = 0 \), which implies that \( T - \lambda \) is Weyl. Since by Lemma 2, Weyl’s theorem holds for every quasi-n-hyponormal operator, it follows \( \lambda \in \pi_0(T) \), which implies \( \lambda \in \sigma(T) \), a contradiction. Therefore \( \lambda \in \liminf_n \sigma(T_n) \) and this completes the proof. \( \square \)

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