THE SPECTRUM IS CONTINUOUS ON THE SET OF QUASI-n-HYPONORMAL OPERATORS

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Abstract. In this paper it is shown that the spectrum σ, a set valued function, is continuous when the function is restricted to the set of all 'quasi-n-hyponormal' operators acting on an infinite-dimensional separable Hilbert space, where a quasi-n-hyponormal operator is defined to be unitarily equivalent to an $n \times n$ upper triangular operator matrix whose diagonal entries are hyponormal operators.

1 Introduction

Throughout the paper, $\mathcal{H}$ denotes an infinite-dimensional separable Hilbert space. We write $\mathcal{B}(\mathcal{H})$ for the algebra of bounded linear operators on $\mathcal{H}$ and $K(\mathcal{H})$ for the ideal of compact operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called an *n-normal* operator if there exists a maximal abelian self-adjoint algebra $\mathcal{R}$ such that T is in the commutant of $\mathcal{R}^{(n)}$, where $\mathcal{R}^{(n)}$ denotes the direct sum of n copies of $\mathcal{R}$. The class of n-normal operators was first studied by A. Brown [Br] and have been much studied (see, for example, [Br], [Fo], [Hoo], [Pe], [RR1], [RR2]). From the definition we can see that $T \in \mathcal{B}(\mathcal{H})$ is n-normal if and only if it is unitarily equivalent to an $n \times n$ operator matrix (N_{ij}) acting on $\mathcal{K}^{(n)}$, where $\{N_{ij}\}$ is a collection of commuting normal operators on a separable Hilbert space $\mathcal{K}$ (cf. [RR2, Theorem 7.17]). In fact, the notion of n-normality was chosen as to be a generalization in operator form of the $n \times n$ complex-valued matrices in a way parallel to the way in which a normal operator is a generalization of a complex number (cf.[Br]). Moreover it was well known ([Fo], [RR2, Theorem 7.2]) that each n-normal operator has an upper triangular form: i.e., if T is n-normal then T is unitarily equivalent to

$$\begin{pmatrix} N_{11} & N_{12} & \dots & N_{1n} \\ 0 & N_{22} & \dots & N_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & N_{nn} \end{pmatrix}, \tag{1,1}$$

where $\{N_{ij}\}_{1 \le i \le j \le n}$ consists of mutually commuting normal operators on a separable Hilbert space $\mathcal{K}$. Evidently, the classes of normal and 1-normal operators coincide.

We now introduce a class of operators which contains the class of hyponormal operators as well as n-normal operators.

Definition 1. Let $\mathcal{K}$ be a separable complex Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a *quasi-n-hyponormal operator* (for $n \in \mathbb{N}$) if it is unitarily equivalent to an $n \times n$ upper triangular operator matrix (N_{ij}) acting on $\mathcal{K}^{(n)}$, where the diagonal entries N_{jj} $(j = 1, 2, \cdots, n)$ are hyponormal operators in $\mathcal{B}(\mathcal{K})$.

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Clearly, the classes of hyponormal and quasi-1-hyponormal operators coincide. The term "n-hyponormal" operators is reserved for $n \times n$ upper triangular operator matrices (N_{ij}) whose all entries are commuting hyponormal operators. So evidently, n-normal $\Rightarrow$ n-hyponormal $\Rightarrow$ quasi-n-hyponormal. For example, every algebraic operator (i.e., an operator T for which $p(T) = 0$ for a non-zero polynomial p) is quasi-n-hyponormal (see [Pe, Theorem 6.11]).

Let $\mathbf{K}$ denote the set, equipped with the Hausdorff metric, of all compact subsets of the complex plane $\mathbf{C}$. Then the spectrum σ can be viewed as a function $\sigma : \mathcal{B}(\mathcal{H}) \to \mathbf{K}$, mapping each operator T to its spectrum $\sigma(T)$. It is well-known that the function σ is upper semicontinuous, and that σ does have points of discontinuity. J. Newburgh [Ne] gave the fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel [CoM] have undertaken a detailed study of spectral continuity in the case where the Banach algebra is the C^*-algebra of all operators acting on a complex separable Hilbert space. Of interest is the identification of classes $\mathfrak{C}$ of operators for which σ becomes continuous when restricted to $\mathfrak{C}$. In [Ne] it was shown that σ is continuous on the set of normal operators (also see [Hal, Solution 105]). This argument can be easily extended to the set of hyponormal operators. In [FaL], the continuity of σ was considered when the function is restricted to certain subsets of Toeplitz operators on the Hardy space of the unit circle. Also it was shown in [BGS] that σ is discontinuous on the entire manifold of Toeplitz operators. Recently it was shown in [HwL] that σ is continuous on the set of p-hyponormal operators (i.e., $(T^*T)^p \geq (TT^*)^p$ for some $0 < p \leq 1$). The purpose of the present paper is to show that the function σ is continuous when restricted to the set of all quasi-n-hyponormal operators.

Theorem 1. *The spectrum σ is continuous on the set of all quasi-n-hyponormal operators.*

In Section 2 we provide auxiliary lemmas needed to prove the main theorem and Section 3 devotes a proof of the main theorem.

2 Auxiliary lemmas

If $T \in \mathcal{B}(\mathcal{H})$, we write $\rho(T)$ for the resolvent of T; $\sigma(T)$ for the spectrum of T; $\pi_0(T)$ for the eigenvalues of T; $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. An operator $T \in \mathcal{B}(\mathcal{H})$ is called left-Fredholm if it has closed range with finite-dimensional null space and right-Fredholm if it has closed range with its range of finite co-dimension. If T is either left- or right-Fredholm we call it semi-Fredholm and Fredholm if it is both. The index of a semi-Fredholm operator T, denoted by ind(T), is given by the integer $\text{ind}\,(T) := \dim T^{-1}(0) - \dim T(\mathcal{H})^{\perp}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called Weyl if it is Fredholm of index zero. The essential spectrum, $\sigma_e(T)$, and the Weyl spectrum, $\omega(T)$, of $T \in \mathcal{B}(\mathcal{H})$ are defined by

$$\sigma_e(T) := \{\lambda \in \mathbf{C} : \ T - \lambda \text{ is not Fredholm}\};$$
$$\omega(T) := \{\lambda \in \mathbf{C} : \ T - \lambda \text{ is not Weyl}\}.$$

H. Weyl [We] has shown that every hermitian operator $T \in \mathcal{B}(\mathcal{H})$ satisfies the equality

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T). \tag{2.1}$$

Today we say that *Weyl's theorem holds for T* if T satisfies the equality (2.1). Weyl's theorem has been extended from hermitian operator to hyponormal operators, to Toeplitz operators by L. Coburn [Co] and to several classes of operators including hyponormal operators by many authors.

If $T \in \mathcal{B}(\mathcal{H})$, a *hole* in $\sigma_e(T)$ is a bounded component of $\mathbb{C} \setminus \sigma_e(T)$ and a *pseudohole* in $\sigma_e(T)$ is a component of $\sigma_e(T) \setminus \sigma_{le}(T)$ or $\sigma_e(T) \setminus \sigma_{re}(T)$. The *spectral picture*, $\mathcal{SP}(T)$, of $T \in \mathcal{B}(\mathcal{H})$ is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes. Recall ([Pe, Definition 4.8]) that an operator $T \in \mathcal{B}(\mathcal{H})$ is called *quasitriangular* if there exists a sequence $\{P_n\}_{n=1}^{\infty}$ of projections of finite rank in $\mathcal{B}(\mathcal{H})$ that converges strongly to 1 and satisfies $||P_nTP_n - TP_n|| \to 0$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *coquasitriangular* if T^* is quasitriangular. By Apostol, Foias, and Voiculescu [AFV], T is quasitriangular [coquasitriangular] if and only if for $\lambda \in \mathbf{C}$, $\mathcal{SP}(T)$ contains no hole or pseudohole associated with a negative [positive] number.

Lemma 1. *If $T \in \mathcal{B}(\mathcal{H})$ is quasi-n-hyponormal then it is coquasitriangular.*

Proof. If $n = 1$, this statement is clear. Assume that this is true for $n = k$. Suppose that T is quasi-$(k+1)$-hyponormal. So we can write $T = \left(\begin{smallmatrix} S & A \\ 0 & N \end{smallmatrix}\right)$, where S is quasi-k-hyponormal and N is hyponormal. We first show that $\mathcal{SP}(T)$ contains no pseudohole associated with a positive number, that is, for $\lambda \in \mathbb{C}$,

$$T - \lambda \text{ is right-Fredholm } \Longrightarrow T - \lambda \text{ is Fredholm.} \tag{2.2}$$

Towards (2.2) suppose that $T-\lambda$ is right-Fredholm. We may assume $\lambda = 0$. Then N is right-Fredholm. Since N is hyponormal, N must be Fredholm. On the other hand, by the Atkinson's theorem, there exist operators X, Y, Z and W such that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} S & A \\ 0 & N \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in \mathcal{K}(\mathcal{H}).$$

Thus $1-(SX+AZ)$ and NZ are both compact. But since N is Fredholm it follows that Z is compact. Therefore $1 - SX$ is compact, and hence S is right-Fredholm. By the inductive hypothesis on S we can see that S is Fredholm. Therefore T is Fredholm, which proves (2.2). Furthermore if $\lambda \notin \sigma_e(T)$ then

$$\text{ind}\,(T - \lambda) = \text{ind}\,(S - \lambda) + \text{ind}\,(N - \lambda) \leq 0,$$

which shows that $\mathcal{SP}(T)$ has no hole associated with a positive number. Hence T is coquasitriangular. □

Lemma 2. *Weyl's theorem holds for quasi-n-hyponormal operators.*

Proof. We use an induction. If $n = 1$, this is true. Assume that Weyl's theorem holds for quasi-k-hyponormal operators. Suppose T is quasi-$(k+1)$-hyponormal. Then we can write $T = \left(\begin{smallmatrix} S & A \\ 0 & N \end{smallmatrix}\right)$, where S is quasi-k-hyponormal and N is hyponormal. Thus we can see that (cf. [HLL,Corollary 11]) $\sigma(S \oplus N) = \sigma(S) \cup \sigma(N) = \sigma(T)$. On the other hand, remember ([Le, Corollary 5]) that if T is weyl then (i) S is left-Fredholm; (ii) N is right-Fredholm; and (iii) $S^{-1}(0) \oplus N^{-1}(0) \cong (\text{ran}\, S)^{\perp} \oplus (\text{ran}\, N)^{\perp}$. But since N is hyponormal, we have that N is Fredholm, and hence by (iii), S is also Fredholm. Since by Lemma 1, S and N are coquasitriangular, and $0 = \text{ind}(T) = \text{ind}(S) + \text{ind}(N)$, we can see that $\text{ind}(S) = \text{ind}(N) = 0$, i.e., S and N are both Weyl. Applying this for $T - \lambda$ in place of T gives that $\omega(S \oplus N) = \omega(S) \cup \omega(N) = \omega(T)$. On the other hand, we argue that every quasi-n-hyponormal operator is *isoloid*, in the sense that all isolated points of the spectrum are eigenvalues. To see this, suppose R is quasi-n-hyponormal. Then R is unitarily equivalent to

$$\begin{pmatrix} N_1 & * & \dots & * \\ 0 & N_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & N_n \end{pmatrix}, \qquad (N_j \text{ is hyponormal for } 1 \leq j \leq n).$$

Then $\sigma(R) = \bigcup_{j=1}^{n} \sigma(N_j)$ since the N_j are all hyponormal (see [HLL, Corollary 11]). Let $\lambda \in \text{iso}\,\sigma(T)$. We assume, without loss of generality, that $\lambda = 0$. Then $0 \in \text{iso} \bigcup_{j=1}^{n} \sigma(N_j)$, so that $0 \in \text{iso}\,\sigma(N_j) \cup \rho(N_j)$ for j $(1 \le j \le n)$. Suppose k is the first integer such that $0 \in \text{iso}\,\sigma(N_k)$ (there exists such an integer since $0 \in \sigma(R)$). But since N_k is isoloid, N_k is not one-one. So there exists a vector $\mathbf{x} := (x_1, \cdots, x_k, 0 \cdots, 0)^T$ belongs to $R^{-1}(0)$, so 0 is an eigenvalue of R. This shows that every quasi-n-hyponormal operator is isoloid. Thus since S and N are isoloid and Weyl's theorem holds for S and N by the inductive hypothesis, we have that

$$\begin{aligned}\pi_{00}(S \oplus N) &= \big(\pi_{00}(S) \cap \rho(N)\big) \cup \big(\rho(S) \cap \pi_{00}(N)\big) \cup \big(\pi_{00}(S) \cap \pi_{00}(N)\big)\\ &= \big(\sigma(S) \cup \sigma(N)\big) \setminus \big(\omega(S) \cup \omega(N)\big)\\ &= \sigma(S \oplus N) \setminus \omega(S \oplus N),\end{aligned}$$

which says that Weyl's theorem holds for $S \oplus N$. Thus we have that $\sigma(T) \setminus \omega(T) = \sigma(S \oplus N) \setminus \omega(S \oplus N) = \pi_{00}(S \oplus N)$. But since $\text{iso}\,\sigma(T) = \text{iso}\,\sigma(S \oplus N)$, it follows at once that $\sigma(T) \setminus \omega(T) \subset \pi_{00}(T)$. For the reverse inclusion, suppose $\lambda \in \pi_{00}(T)$. We must show that $\lambda \in \pi_{00}(S \oplus N)$. Since S and N are isoloid, it will suffice to show that $(S-\lambda)^{-1}(0) \oplus (N-\lambda)^{-1}(0)$ is finite-dimensional. Evidently, $(S-\lambda)^{-1}(0)$ is finite-dimensional because $(S-\lambda)^{-1}(0) \oplus \{0\} \subset (T-\lambda)^{-1}(0)$. Thus $\lambda \in \pi_{00}(S)$. We now assume to the contrary that $(N-\lambda)^{-1}(0)$ is infinite-dimensional. Thus $A\big((N-\lambda)^{-1}(0)\big)$ is also infinite-dimensional; if it were not so then $\big(A|_{(N-\lambda)^{-1}(0)}\big)^{-1}(0)$ is infinite-dimensional and hence, so is $(T-\lambda)^{-1}(0)$, a contradiction. On the other hand, since $\lambda \in \pi_{00}(S)$ and Weyl's theorem holds for S, we have that $S-\lambda$ is Weyl, so that $(S-\lambda)(\mathcal{H})^{\perp}$ is finite-dimensional. Therefore $A\big((N-\lambda)^{-1}(0)\big) \cap (S-\lambda)(\mathcal{H})$ is infinite-dimensional. Then there exist an orthonormal sequence $\{y_j\}$ in $(N-\lambda)^{-1}(0)$ and an orthonormal sequence $\{x_j\}$ in $\mathcal{H}$ such that $Ay_j = (S-\lambda)x_j$. Thus we have

$$\begin{pmatrix} S-\lambda & A \\ 0 & N-\lambda \end{pmatrix} \begin{pmatrix} x_j \\ -y_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for each } j = 1, 2, \cdots,$$

which implies that $(T-\lambda)^{-1}(0)$ is infinite-dimensional, a contradiction. Thus $(N-\lambda)^{-1}(0)$ is finite-dimensional, which completes the proof. ☐

If $T \in \mathcal{B}(\mathcal{H})$ then the *reduced minimum modulus*, denoted $\gamma(T)$, of T is defined by

$$\gamma(T) := \inf_{x \in \mathcal{H}} \frac{||Tx||}{\text{dist}\,(x,\, T^{-1}(0))},$$

where $\frac{0}{0}$ is defined to be ∞. If $T \in \mathcal{B}(\mathcal{H})$ is a non-zero operator then we can see that $\gamma(T) = \inf\,(\sigma(|T|) \setminus \{0\})$. In particular if T is invertible then $\gamma(T) = ||T^{-1}||^{-1}$. On the other hand, if we write $r(A)$ for the spectral radius of an operator A then

$$\text{dist}\,(\lambda,\ \sigma(T)) = \text{dist}\,(0,\ \sigma(T-\lambda)) = \min_{\mu \in \sigma(T-\lambda)} |\mu| = \frac{1}{\max_{\frac{1}{\nu} \in \sigma(T-\lambda)} |\nu|} = \frac{1}{r((T-\lambda)^{-1})}.$$

Thus if T is hyponormal and $\lambda \notin \sigma(T)$ then since $(T-\lambda)^{-1}$ is normaloid, i.e., norm equals radius, it follows that

$$\text{dist}\,(\lambda,\ \sigma(T)) = \frac{1}{r((T-\lambda)^{-1})} = \frac{1}{||(T-\lambda)^{-1}||} = \gamma(T-\lambda). \tag{2.3}$$

By comparison we have:

Lemma 3. *If T is quasi-n-hyponormal and $\lambda \notin \sigma(T)$ then*

$$||(T-\lambda)^{-1}|| \le \frac{(1+||T||)^{n-1}}{\min\Big\{1,\ \big[\text{dist}\,(\lambda,\ \sigma(T))\big]^n\Big\}}. \tag{2.4}$$

Proof. Note that if $n=1$, then (2.4) is obvious from (2.3). We also use an induction. It is easy to see that if $A, B \in \mathcal{B}(\mathcal{H})$ and A is invertible then $\gamma(AB) \ge \gamma(A)\gamma(B)$. We suppose that T is quasi-2-hyponormal. So we can write $T = \left(\begin{smallmatrix} N_1 & N_3 \\ 0 & N_2 \end{smallmatrix}\right)$, where N_1 and N_2 are hyponormal operators. If $\lambda \notin \sigma(T)$ then $N_2 - \lambda$ is right invertible. But since N_2 is hyponormal it follows that $N_2 - \lambda$ is invertible. We thus have that

$$\begin{aligned}\gamma(T-\lambda) = \gamma\begin{pmatrix} N_1-\lambda & N_3 \\ 0 & N_2-\lambda \end{pmatrix} &\ge \gamma\begin{pmatrix} 1 & 0 \\ 0 & N_2-\lambda \end{pmatrix}\gamma\begin{pmatrix} 1 & N_3 \\ 0 & 1 \end{pmatrix}\gamma\begin{pmatrix} N_1-\lambda & 0 \\ 0 & 1 \end{pmatrix} \\ &= \gamma\begin{pmatrix} 1 & N_3 \\ 0 & 1 \end{pmatrix}\min\Big\{1,\ \gamma(N_2-\lambda)\Big\}\min\Big\{1,\ \gamma(N_1-\lambda)\Big\}.\end{aligned}$$

But since

$$\gamma\begin{pmatrix} 1 & N_3 \\ 0 & 1 \end{pmatrix} = \frac{1}{||\left(\begin{smallmatrix} 1 & N_3 \\ 0 & 1 \end{smallmatrix}\right)^{-1}||} = \frac{1}{||\left(\begin{smallmatrix} 1 & -N_3 \\ 0 & 1 \end{smallmatrix}\right)||} \ge \frac{1}{1+||N_3||} \ge \frac{1}{1+||T||},$$

it follows that

$$\gamma(T-\lambda) \ge (1+||T||)^{-1}\min\Big\{1,\ \gamma(N_1-\lambda),\ \gamma(N_2-\lambda),\ \gamma(N_1-\lambda)\gamma(N_2-\lambda)\Big\}. \tag{2.5}$$

By (2.3) and the fact that $\sigma(T) = \sigma(N_1) \cup \sigma(N_2)$, we have that $\gamma(N_j - \lambda) = \text{dist}\,(\lambda,\ \sigma(N_j)) \ge \text{dist}\,(\lambda,\ \sigma(T))$ for each $j = 1, 2$, and hence, $\gamma(T-\lambda) \ge (1+||T||)^{-1}\min\Big\{1,\ \big[\text{dist}\,(\lambda,\ \sigma(T))\big]^2\Big\}$. So if $\lambda \notin \sigma(T)$ then

$$||(T-\lambda)^{-1}|| = \frac{1}{\gamma(T-\lambda)} \le \frac{1+||T||}{\min\Big\{1,\ \big[(\text{dist}\,(\lambda,\ \sigma(T))\big]^2\Big\}}.$$

Thus (2.4) holds for $n=2$. We assume that (2.4) holds for $n=k$. Suppose T is quasi-$(k+1)$-hyponormal. Then we can write $T = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$, where A is quasi-k-hyponormal and B is hyponormal. If $\lambda \notin \sigma(T)$, write

$$T-\lambda = \begin{pmatrix} 1 & 0 \\ 0 & B-\lambda \end{pmatrix}\begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A-\lambda & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $B-\lambda$ is right invertible. But since B is hyponormal, it follows that $B-\lambda$ is invertible. Thus by the same argument as (2.5) we have that

$$\gamma(T-\lambda) \ge (1+||T||)^{-1}\min\Big\{1,\ \gamma(A-\lambda),\ \gamma(B-\lambda),\ \gamma(A-\lambda)\gamma(B-\lambda)\Big\}. \tag{2.6}$$

By the inductive hypothesis on A, we have

$$\gamma(A-\lambda) \ge \frac{\min\Big\{1,\ \big[\text{dist}\,(\lambda,\ \sigma(A))\big]^k\Big\}}{(1+||T||)^{k-1}}.$$

But since $\gamma(B-\lambda)=\text{dist}\,(\lambda,\ \sigma(B))\geq \text{dist}\,(\lambda,\ \sigma(T))$, it follows from (2.6) that

$$\gamma(T-\lambda)\geq \frac{\min\left\{1,\ \left[\text{dist}\,(\lambda,\ \sigma(T))\right]^{k+1}\right\}}{(1+||T||)^k},$$

which implies that (2.4) holds for $n=k+1$. This completes the proof. □

3 Proof of Theorem 1

We are ready for:

Proof of Theorem 1. We write $(Q_n)(\mathcal{H})$ for the set of all quasi-n-hyponormal operators. If $T\in\mathcal{B}(\mathcal{H})$, define $m_e(T)$ for the essential minimum modulus of T (cf.[Bo]): i.e., $m_e(T):=\inf\sigma_e(|T|)$. Obviously,

$$m_e(T)>0 \iff T \text{ is left-Fredholm.} \tag{3.1}$$

On the other hand, m_e can be viewed as a function from $\mathcal{B}(\mathcal{H})$ to $\mathbf{R}$, mapping each operator T to its essential minimum modulus $m_e(T)$. We claim that m_e is a continuous function: indeed, if $T, T_n\in\mathcal{B}(\mathcal{H})$ $(n\in\mathbf{Z}_+)$ are such that T_n converges to T in norm then $|T_n|$ converges to $|T|$ in norm (cf.[HwL, Lemma 1]) and $\lim\sigma_e(|T_n|)=\sigma_e(|T|)$ because σ_e is continuous on the set of normal elements in a unital C^*-algebra (cf. [Ne, Corollary 2]), which implies that $\lim m_e(T_n)=m_e(T)$. We also claim that there exists a constant $c>0$ such that if $T\in(Q_n)(\mathcal{H})$ then

$$m_e(T-\lambda)\geq c\,\min\left\{1,\ \left[\text{dist}\,(\lambda,\ \sigma_e(T))\right]^n\right\}\quad \text{for } \lambda\notin\sigma_e(T). \tag{3.2}$$

To prove (3.2) suppose $T\in(Q_n)(\mathcal{H})$ and $0\notin\sigma_e(T)$. If $\pi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})/K(\mathcal{H})$ is the Calkin homomorphism then we have that $m_e(T)=\inf\sigma(|\pi(T)|)$. We thus argue that if $\mathcal{B}(\mathcal{H})/K(\mathcal{H})$ is regarded as a C^*-subalgebra of $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ then since $\pi(T)$ is quasi-n-hyponormal, we have that by Lemma 3,

$$\begin{aligned}
m_e(T)&=\inf\sigma(|\pi(T)|)\\
&=\inf\left\{||\pi(T)x||:\ ||x||=1,\ x\in\mathcal{K}\right\}\\
&=\frac{1}{||\pi(T)^{-1}||}\\
&\geq c\,\min\left\{1,\ \left[\text{dist}\,(0,\ \sigma(\pi(T))\right]^n\right\}\quad \text{with } c:=\frac{1}{(1+||\pi(T)||)^{n-1}}\\
&=c\,\min\left\{1,\ \left[\text{dist}\,(0,\ \sigma_e(T))\right]^n\right\}.
\end{aligned}$$

Applying this result with $T-\lambda$ in place of T proves (3.2). Now suppose that $T_n, T\in(Q_n)(\mathcal{H})$, for $n\in\mathbf{Z}_+$, are such that T_n converges to T in norm. Since σ is upper semicontinuous and $\liminf_n\sigma(T_n)\subset\sigma(T)$, it suffices to show that $\sigma(T)\subset\liminf_n\sigma(T_n)$. We first claim that $\text{iso}\,\sigma(T)\subset\liminf\sigma(T_n)$: indeed this follows at once from an argument of Newburgh [Ne. Lemma 3]: if $\lambda\in\text{iso}\,\sigma(T)$ then for every neighborhood $N(\lambda)$ of λ there exists an $N\in\mathbf{Z}_+$ such that $n>N$ implies $\sigma(T_n)\cap N(\lambda)\neq\emptyset$. This shows that $\lambda\in\liminf_n\sigma(T_n)$. So it suffices to show that $\text{acc}\,\sigma(T)\subset\liminf\sigma(T_n)$. To show this let $\lambda\in\text{acc}\,\sigma(T)$ and assume to the contrary that $\lambda\notin\liminf_n\sigma(T_n)$. Then there exists a neighborhood $N(\lambda)$ of λ which does not intersect infinitely many $\sigma(T_n)$. Thus we can choose a subsequence $\{T_{n_k}\}_k$ of $\{T_n\}_n$ such that for some $\epsilon > 0$, $\operatorname{dist}(\lambda, \sigma(T_{n_k})) > \epsilon$ for all $k \in \mathbf{Z}_+$. Since $\operatorname{dist}(\lambda, \sigma(T_{n_k})) \le \operatorname{dist}(\lambda, \sigma_e(T_{n_k}))$, it follows that $m_e(T_{n_k} - \lambda) > \delta$ for some $\delta > 0$ and all $k \in \mathbf{Z}_+$. Since m_e is continuous, we have that $m_e(T-\lambda) \ge \delta$, which by (3.1), implies that $T-\lambda$ is left-Fredholm. By the continuity of the semi-Fredholm index, $\operatorname{ind}(T-\lambda) = \lim_{k\to\infty} \operatorname{ind}(T_{n_k} - \lambda) = 0$, which implies that $T - \lambda$ is Weyl. Since by Lemma 2, Weyl's theorem holds for every quasi-n-hyponormal operator, it follows $\lambda \in \pi_{00}(T)$, which implies $\lambda \in \operatorname{iso}\sigma(T)$, a contradiction. Therefore $\lambda \in \liminf_n \sigma(T_n)$ and this completes the proof. $\square$

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