HYPONORMAL OPERATORS WITH RANK-TWO SELF-COMMUTATORS

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Abstract. In this paper it is shown that if $T \in \mathcal{L}(\mathcal{H})$ satisfies
(i) T is a pure hyponormal operator;
(ii) $[T^*, T]$ is of rank-two; and
(iii) $\ker [T^*, T]$ is invariant for T,
then T is either a subnormal operator or the Putinar's matricial model of rank two. More precisely, if $T|_{\ker [T^*,T]}$ has the rank-one self-commutator then T is subnormal and if instead $T|_{\ker [T^*,T]}$ has the rank-two self-commutator then T is either a subnormal operator or the k-th minimal partially normal extension, $\widehat{T_k}^{(k)}$, of a $(k+1)$-hyponormal operator T_k which has rank-two self-commutator for any $k \in \mathbb{Z}_+$. Hence, in particular, every weakly subnormal (or 2-hyponormal) operator with rank-two self-commutator is either a subnormal operator or a finite rank-perturbation of a k-hyponormal operator for any $k \in \mathbb{Z}_+$.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, quasinormal if $T^*T^2 = TT^*T$, hyponormal if $T^*T \geq TT^*$, and subnormal if it has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K}$ containing $\mathcal{H}$. In general it is quite difficult to determine the subnormality of an operator by definition. An alternative description of subnormality is given by the Bram-Halmos criterion, which states that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([Bra], [Con, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1\ ). \tag{1.1}$$

Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.1) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{1.2}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (1.2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (1.1); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([CMX]). The classes of k-hyponormal operators have been studied in an attempt to bridge the

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gap between subnormality and hyponormality (cf. [Cu1], [Cu2], [CF1], [CF2], [CF3], [CL1], [CL2], [CL3], [CMX], [DPY], [McCP]).

The Bram-Halmos characterization of subnormality indicates that 2-hyponormality is generally far from subnormality. There are special classes of operators, however, for which these two notions are equivalent. A trivial example is given by the classes of operators whose square is compact. There are many nontrivial examples: for example, every 2-hyponormal Toeplitz operator with polynomial symbol is subnormal (see [CL1]). So it seems to be interesting to consider the following problem:

$$\text{Which 2-hyponormal operators are subnormal?} \tag{1.3}$$

The first inquiry involves the self-commutator. The self-commutator of an operator plays an important role in the study of subnormality. Subnormal operators with finite rank self-commutators have been extensively studied ([Ale], [McCY], [OTT], [StX], [Xi1], [Xi2], [Ya1], [Ya2]). Particular attention has been paid to hyponormal operators with rank-one or rank-two self-commutators ([GuP], [Mor], [Pu1], [Pu2], [Pu3], [StX], [Xi3], [Ya3]). In particular, B. Morrel [Mor] showed that a pure subnormal operator with rank-one self-commutator (pure means having no normal summand) is unitarily equivalent to a linear function of the unilateral shift. Morrel's theorem can be essentially stated (also see [Con, p.162]) that if

$$\begin{cases} \text{(i) } T \text{ is hyponormal;} \\ \text{(ii) } [T^*, T] \text{ is of rank-one; and} \\ \text{(iii) } \ker [T^*, T] \text{ is invariant for } T, \end{cases} \tag{1.4}$$

then $T-\beta$ is quasinormal for some $\beta \in \mathbb{C}$. Now remember that every pure quasinormal operator is unitarily equivalent to $U \otimes P$, where U is the unilateral shift and P is a positive operator with trivial kernel. Thus if $[T^*, T]$ is of rank-one (and hence so is $[(T-\beta)^*, (T-\beta)]$), we must have $P \cong \alpha\ (\neq 0) \in \mathbb{C}$, so that $T-\beta \cong \alpha U$, or $T \cong \alpha U + \beta$. It would be interesting (in the sense of giving a simple sufficiency for the subnormality) to note that Morrel's theorem gives that

$$\text{if } T \text{ satisfies the condition (1.4) then } T \text{ is subnormal.} \tag{1.5}$$

On the other hand, it was shown ([CL2, Lemma 2.2]) that if T is 2-hyponormal then

$$T(\ker [T^*, T]) \subseteq \ker [T^*, T].$$

Therefore by Morrel's theorem, we can see that

$$\text{every 2-hyponormal operator with rank-one self-commutator is subnormal.} \tag{1.6}$$

On the other hand, M. Putinar [Pu4] gave a matricial model for the hyponormal operator $T \in \mathcal{L}(\mathcal{H})$ with finite rank self-commutator, in the cases where

$$\mathcal{H}_0 := \bigvee_{k=0}^{\infty} T^{*k}\big(\operatorname{ran} [T^*, T]\big) \text{ has finite dimension } d \quad \text{and} \quad \mathcal{H} = \bigvee_{n=0}^{\infty} T^n \mathcal{H}_0.$$

In this case, if we write

$$\mathcal{H}_n := G_n \ominus G_{n-1} \quad (n \geq 1) \quad \text{and} \quad G_n := \bigvee_{k=0}^{n} T^k \mathcal{H}_0 \quad (n \geq 0),$$

then T has the following two-diagonal structure relative to the decomposition $H = H_0 \oplus H_1 \oplus \cdots$:

$$T = \begin{pmatrix} B_0 & 0 & 0 & 0 & \cdots \\ A_0 & B_1 & 0 & 0 & \cdots \\ 0 & A_1 & B_2 & 0 & \cdots \\ 0 & 0 & A_2 & B_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.7}$$

where

$$\begin{cases} \dim(\mathcal{H}_n) = \dim(\mathcal{H}_{n+1}) = d \quad (n \geq 0); \\ [T^*, T] = ([B_0^*, B_0] + A_0^* A_0) \oplus 0_\infty; \\ [B_{n+1}^*, B_{n+1}] + A_{n+1}^* A_{n+1} = A_n A_n^* \quad (n \geq 0); \\ A_n^* B_{n+1} = B_n A_n^* \quad (n \geq 0). \end{cases} \tag{1.8}$$

We will refer the operator (1.7) to the *Putinar's matricial model of rank d*. This model was also introduced in [GuP], [Pu1], [Xi3], [Ya1], and etc.

The purpose of the present paper is to obtain an extension of Morrel's theorem to the cases of *rank-two* self-commutators via the Putinar's matricial model. The main idea proving this result comes from the notion of "weak subnormality", which was first introduced in [CL2], with an aim at providing a model for 2-hyponormal operators.

2. The Main Result

We review first a few essential facts concerning weak subnormality that we will need to begin with. Note that the operator T is subnormal if and only if there exist operators A and B such that $\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ is normal, i.e.,

$$\begin{cases} [T^*, T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*, B] + A^*A = 0. \end{cases} \tag{2.1}$$

The operator $\widehat{T}$ is called a *normal extension* of T. We also say that $\widehat{T}$ in $\mathcal{L}(\mathcal{K})$ is a *minimal normal extension* (briefly, m.n.e.) of T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a normal extension of T. It is known that

$$\widehat{T} = \text{ m.n.e.}(T) \iff \mathcal{K} = \bigvee \{\widehat{T}^{*n} h : \ h \in \mathcal{H}, \ n \geq 0\},$$

and the m.n.e.(T) is unique.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly subnormal* if there exist operators $A \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}')$ such that the first two conditions in (2.1) hold:

$$[T^*, T] = AA^* \quad \text{and} \quad A^*T = BA^*, \tag{2.2}$$

or equivalently, there is an extension $\widehat{T}$ of T such that

$$\widehat{T}^* \widehat{T} f = \widehat{T} \widehat{T}^* f \quad \text{for all } f \in \mathcal{H}.$$

The operator $\widehat{T}$ is called a *partially normal extension* (briefly, p.n.e.) of T. We also say that $\widehat{T}$ in $\mathcal{L}(\mathcal{K})$ is a *minimal partially normal extension* (briefly, m.p.n.e.) of T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a partially normal extension of T. It is known ([CL2, Lemma 2.5 and Corollary 2.7]) that

$$\widehat{T} = \text{ m.p.n.e.}(T) \iff \mathcal{K} = \bigvee \{\widehat{T}^{*n} h : \ h \in \mathcal{H}, \ n = 0, 1\},$$

and the m.p.n.e.(T) is unique. For convenience, if $\widehat{T} = $ m.p.n.e.(T) is also weakly subnormal then we write $\widehat{T}^{(2)} := \widehat{\widehat{T}}$ and more generally,

$$\widehat{T}^{(n)} := \widehat{\widehat{T}^{(n-1)}},$$

which will be called the *n-th minimal partially normal extension of* T. It was ([CL2], [CJP]) shown that

$$\text{2-hyponormal} \implies \text{weakly subnormal} \implies \text{hyponormal} \tag{2.3}$$

and the converses of both implications in (2.3) are not true in general. It was ([CL2]) known that

$$T \text{ is weakly subnormal} \Longrightarrow T(\ker[T^*, T]) \subseteq \ker[T^*, T] \tag{2.4}$$

and it was ([CJP]) kown that if $\widehat{T} := \text{m.p.n.e.}(T)$ then for any $k \geq 1$,

$$T \text{ is } (k+1)\text{-hyponormal} \Longleftrightarrow T \text{ is weakly subnormal and } \widehat{T} \text{ is } k\text{-hyponormal.} \tag{2.5}$$

So, in particular, one can see that

$$\text{if } T \text{ is subnormal then } \widehat{T} \text{ is subnormal.} \tag{2.6}$$

It is worth to noticing that in view of (2.3) and (2.4), Morrel's theorem gives that

$$\text{every weakly subnormal operator with rank-one self-commutator is subnormal.} \tag{2.7}$$

We are ready for stating the main theorem.

Theorem 1. *Let $T \in \mathcal{L}(\mathcal{H})$. If*

(i) *T is a pure hyponormal operator;*
(ii) *$[T^*, T]$ is of rank-two; and*
(iii) *$\ker [T^*, T]$ is invariant for T,*

then the following hold:

1. *If $T|_{\ker [T^*,T]}$ has the rank-one self-commutator then T is subnormal;*
2. *If $T|_{\ker [T^*,T]}$ has the rank-two self-commutator then T is either a subnormal operator or the Putinar's matricial model (1.7) of rank two.*

We would like to remark that in the latter case of the case (2), if T is the Putinar's matricial model (1.7) of the form

$$T = \begin{pmatrix} B_0 & 0 & 0 & 0 & \cdots \\ A_0 & B_1 & 0 & 0 & \cdots \\ 0 & A_1 & B_2 & 0 & \cdots \\ 0 & 0 & A_2 & B_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{on } H = H_0 \oplus H_1 \oplus \cdots , \tag{2.8}$$

where $H_n = \operatorname{ran} [T_n^*, T_n]$ and $\dim H_n = 2$ $(n \geq 0)$, and if T_n denotes the compression of T to the space $H_n \oplus H_{n+1} \oplus \cdots$, $n \geq 0$ then $T_n = \text{m.p.n.e.}\,(T_{n+1})$ $(n \geq 0)$. Consequently, by (2.5), T is the k-th minimal partially normal extension, $\widehat{T_k}^{(k)}$, of a $(k+1)$-hyponormal operator T_k which has rank-two self-commutator for any $k \in \mathbb{Z}_+$.

The essence of our approach is a comparison of two operations. The first one associates with a hyponormal operator T of the type considered to the hyponormal operator $T_1 = T|_{\ker [T^*,T]}$. The second one starts with a k-hyponormal operator T and associates with it a $(k-1)$-hyponormal operator m.p.n.e(T) $(n \geq 2)$. These two operations not always are inverse to each other. The main point of part (2) of Theorem 1 is that if T is not subnormal then one has $T = \widehat{T_k}^{(k)}$ for any k, that is, the above two operations are inverse to each other in this case.

The following corollary follows at once from Theorem 1 and (2.4).

Corollary. *If T is a weakly subnormal (or 2-hyponormal) operator with rank-two self-commutator then T is either a subnormal operator or a finite rank perturbation of a k-hyponormal operator for any $k \in \mathbb{Z}_+$.*

Since the operator (2.8) can be constructed from the pair of matrices $\{A_0, B_0\}$, we know that the pair $\{A_0, B_0\}$ is a complete set of unitary invariants for the operator (2.8). Many authors used the following Xia's unitary invariants $\{\Lambda, C\}$ to describe pure subnormal operators with finite rank self-commutators:

$$\Lambda := \left(T^*|_{\operatorname{ran} [T^*,T]}\right)^* \quad \text{and} \quad C := [T^*, T]|_{\operatorname{ran} [T^*,T]}.$$

Consequently,

$$\Lambda = B_0 \quad \text{and} \quad C = [B_0^*, B_0] + A_0^2.$$

We know that given Λ and C (or equivalently, A_0 and B_0) corresponding to a pure subnormal operator we can reconstruct T. Now the following question naturally arises: "what are the restrictions on matrices A_0 and B_0 such that they represent a subnormal operator ?" In the cases where A_0 and B_0 operate on a finite dimensional Hilbert space, D. Yakubovich [Ya1] showed that such a description can be given in terms of a topological property of a certain algebraic curve, associated with A_0 and B_0. However there is a subtle difference between Yakubovich's criterion and the Putinar's model operator (2.8). In fact, in some sense, Yakubovich gave conditions on A_0 and B_0 such that the operator (2.8) can be constructed so that the condition (1.8) is satisfied. By comparison, the Putinar's model operator (2.8) was already constructed so that it satisfies the condition (1.8). Thus we would guess that if the operator (2.8) can be constructed so that the condition (1.8) is satisfied then two matrices $\{A_0, B_0\}$ in (2.8) must satisfy the Yakubovich's criterion. In this viewpoint, we have the following:

Conjecture. *The Putinar's matricial model (2.8) of rank two is subnormal.*

If the operator T in (2.8) is rationally cyclic, then the spectrum of T is the closure of an order two quadrature domain by [Pu4, Proposition 3.1]. Since there are only three types of order two quadrature domains(cf. [GuP]): a couple of disjoint disks, a lemniscate or a cardioid, the first case is the direct sum of hyponormal operators with rank one self-commutators and hence it is subnormal by Morrel's theorem. If the spectrum of T is either a lemniscate or a cardioid, then the essential spectrum of T should be the boundary union finitely many points(cf. [Pu4, The remarks after Theorem 3.5]). Thus the principal function is known. This may support the conjecture. An affirmative answer to the conjecture would show that if T is a hyponormal operator with rank-two self-commutator and satisfying that ker $[T^*, T]$ is invariant for T then T is subnormal. Hence, in particular, one could obtain:

Every weakly subnormal operator with rank-two self-commutator is subnormal.

In the sequel we will provide an affirmative evidence towards the above conjecture.

Theorem 2. *The operator T in (2.8) is subnormal if B_n is normal for some $n \geq 0$.*

One may ask whether the operators described by Theorem 2 really exist. The following example shows that such operators exist: this is basically due to S.Campbell and R. Gellar ([CaG]).

Example 3. Let $A_j := \begin{pmatrix} p_j & 0 \\ 0 & q_j \end{pmatrix}$ $(p_j, q_j \in \mathbb{R}_+)$ and $B_j = \begin{pmatrix} 0 & a_j \\ b_j & 0 \end{pmatrix}$ $(a_j, b_j \in \mathbb{R})$ for $j = 0, 1, \cdots$. Then $[T^*, T] = \mathrm{diag}\{C, 0, 0, \cdots\}$ if and only if

$$[B_0^*, B_0] + A_0^* A_0 = C, \quad A_{n+1}^* A_{n+1} = A_n A_n^* - [B_{n+1}^*, B_{n+1}] \quad \text{and} \quad A_n^* B_{n+1} = B_n A_n^* \quad (n \geq 0). \tag{2.10}$$

Let $C := \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ $(0 < \alpha < 1)$. Then by the first equality of (2.10), we have $p_0^2 + b_0^2 - a_0^2 = 1$ and $q_0^2 + a_0^2 - b_0^2 = \alpha$. Set $a_0 := \frac{\alpha}{\sqrt{2+2\alpha}}$ and $b_0 := \frac{1}{\sqrt{2+2\alpha}}$. Then $p_0 = \sqrt{\frac{1+\alpha}{2}} = q_0$. Thus the second equation of (2.10) becomes

$$p_{n+1}^2 = p_n^2 + a_{n+1}^2 - b_{n+1}^2, \quad q_{n+1}^2 = q_n^2 + b_{n+1}^2 - a_{n+1}^2 \quad (n \geq 0), \tag{2.11}$$

while the third equation of (2.10) is

$$a_{n+1} = \frac{a_n q_n}{p_n}, \quad b_{n+1} = \frac{b_n p_n}{q_n} \quad (n \geq 0). \tag{2.12}$$

Note that a_{n+1}, b_{n+1} can successively be defined by (2.12) and p_{n+1}, q_{n+1} can successively be defined by (2.11). A straightforward calculation shows that $A_{n+6} = A_n$ and $B_{n+6} = B_n$. More explicitly,

$$A_0 = \begin{pmatrix} \sqrt{\frac{1+\alpha}{2}} & 0 \\ 0 & \sqrt{\frac{1+\alpha}{2}} \end{pmatrix}, \; A_1 = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & 1 \end{pmatrix}, \; A_2 = A_1, \; A_3 = A_0, \; A_4 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix}, \; A_5 = A_4$$

and

$$B_0 = \begin{pmatrix} 0 & \frac{\alpha}{\sqrt{2+2\alpha}} \\ \frac{1}{\sqrt{2+2\alpha}} & 0 \end{pmatrix}, \; B_1 = B_0, \; B_2 = \begin{pmatrix} 0 & \sqrt{\frac{\alpha}{2+2\alpha}} \\ \sqrt{\frac{\alpha}{2+2\alpha}} & 0 \end{pmatrix}, \; B_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2+2\alpha}} \\ \frac{\alpha}{\sqrt{2+2\alpha}} & 0 \end{pmatrix}, \; B_4 = B_3, \; B_5 = B_2.$$

Since B_2 is normal it follows from Theorem 2 that T is subnormal.

Remark. We need not expect that every weakly subnormal operators with *finite rank* self-commutator is subnormal. For example, if W_α is the weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$, where

$$\alpha_0 = \frac{1}{3}, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_n = 1 \ \ (n \geq 2),$$

then W_α is weakly subnormal (see [CL2, Theorem 5.4]) and $\operatorname{rank}[W_\alpha^*, W_\alpha] = 3$, but W_α is not subnormal. In particular, W_α is a partially normal extension of the unilateral shift U: indeed, look at

$$W_\alpha \cong \left(\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & U & & 0 & 0 & 0 \\ & & & \vdots & \vdots & \vdots \\ \hline & & & 0 & \frac{1}{2} & 0 \\ & 0 & & 0 & 0 & \frac{1}{3} \\ & & & 0 & 0 & 0 \end{array}\right) = \text{p.n.e.}(U).$$

So we need not expect that every partially normal extension of a subnormal operator T is subnormal even though p.n.e.(T) is weakly subnormal.

In sections 3 and 4 we provide proofs of Theorems 1 and 2.

3. Proof of Theorem 1

Proof of Theorem 1. (1) Suppose that T is a pure hyponormal operator with $\operatorname{rank}[T^*, T] = 2$ and $\ker[T^*, T]$ is invariant for T. Then T has the following representation relative to the direct sum $\ker[T^*, T] \oplus \operatorname{ran}[T^*, T]$:

$$T = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix}.$$

Since

$$[T^*, T] = \begin{pmatrix} [S^*, S] - AA^* & S^*A - AB^* \\ A^*S - BA^* & [B^*, B] + A^*A \end{pmatrix} : \begin{pmatrix} \ker[T^*, T] \\ \operatorname{ran}[T^*, T] \end{pmatrix} \to \begin{pmatrix} \ker[T^*, T] \\ \operatorname{ran}[T^*, T] \end{pmatrix}$$

we have that

$$[S^*, S] - AA^* = 0 \quad \text{and} \quad A^*S - BA^* = 0,$$

which shows that the condition (2.2) holds with S in place of T, and hence S is weakly subnormal and $T = \text{p}.n.e.(S)$. Note that S is pure because every restriction of a pure hyponormal operator is also pure: indeed if $T = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix}$ is a pure hyponormal operator on $\mathfrak{M} \oplus \mathfrak{M}^\perp$ and if S has a normal summand N acting on $\mathfrak{N}$ then we write $A := \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} : \mathfrak{M}^\perp \to \mathfrak{N} \oplus (\mathfrak{M} \ominus \mathfrak{N})$ and so $[T^*, T] = \begin{pmatrix} -A_1A_1^* & * \\ * & * \end{pmatrix} \geq 0$, which implies $A_1 = 0$, so that N is a normal summand of T, a contradiction. Now suppose $T|_{\ker[T^*, T]}$ has the rank-one self-commutator. Thus $[S^*, S] = AA^*$ is of rank one. Since by (2.3) S is hyponormal, $[S^*, S]$ is of rank-one, and by (2.4), $\ker[S^*, S]$ is invariant for S, it follows from the Morrel's theorem that $S \cong \alpha U + \beta$, where U is the unilateral shift and $\alpha, \beta \in \mathbb{C}$. Therefore

$$T \cong \begin{pmatrix} \alpha U + \beta & A \\ 0 & B \end{pmatrix}.$$

For subnormality for T we may reformulate

$$T = \begin{pmatrix} U & A \\ 0 & B \end{pmatrix}.$$

Note that $T =$ p.n.e.(U) because if R is weakly subnormal then for any $\lambda \in \mathbb{C}$, $R - \lambda$ is also weakly subnormal and p.n.e.$(R - \lambda) =$ p.n.e.$(R) - \lambda$. Thus $[U^*, U] = AA^*$. Since $\operatorname{ran}[T^*, T]$ is 2-dimensional,

A is of the form

$$A=\begin{pmatrix} x & y\\ 0 & 0\\ 0 & 0\\ \vdots & \vdots \end{pmatrix}\qquad \text{with } |x|^2+|y|^2=1.$$

If we write $B=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ then since $AB^*=U^*A=0$, it follows that

$$0=\begin{pmatrix} x & y\\ 0 & 0\\ 0 & 0\\ \vdots & \vdots \end{pmatrix}\begin{pmatrix} \bar a & \bar c\\ \bar b & \bar d\end{pmatrix}\Longrightarrow\begin{cases} x\bar a+y\bar b=0\\ x\bar c+y\bar d=0\end{cases}\Longrightarrow \operatorname{rank} B=1,$$

which implies that B should be of the form $B=\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}$. A direct calculation shows that

$$[T^*,T]=\begin{pmatrix} 0 & 0\\ 0 & [B^*,B]+A^*A\end{pmatrix}=0_\infty\bigoplus\begin{pmatrix} |x|^2-|b|^2 & \overline{a}b+\overline{x}y\\ a\bar b+x\overline{y} & |b|^2+|y|^2\end{pmatrix}.$$

But since $[T^*,T]\geq 0$ and $\operatorname{rank}[T^*,T]=2$, we must have that $|x|>|b|$, and in turn $|y|>|a|$ since $x\overline{a}+y\overline{b}=0$. So we can write

$$T=\left(\begin{array}{cccc|cc} 0 & & & & x & y\\ 1 & 0 & & & 0 & 0\\ & 1 & \ddots & & 0 & 0\\ & & \ddots & \ddots & \vdots & \vdots\\ \hline 0 & 0 & \cdots & \cdots & a & b\\ 0 & 0 & \cdots & \cdots & 0 & 0\end{array}\right)\cong\begin{pmatrix} 0 & 0 & 0 & & \cdots & \cdots & \cdots\\ b & a & 0 & & \cdots & \cdots & \cdots\\ y & x & 0 & & & & \\ 0 & 0 & 1 & 0 & & & \\ \vdots & \vdots & & 1 & 0 & & \\ \vdots & \vdots & & & 1 & 0 & \\ \vdots & \vdots & & & & \ddots & \ddots\end{pmatrix},$$

satisfying

$$\begin{cases} |x|^2+|y|^2=1\\ |b|<|x|,\ |a|<|y|\\ \overline{a}x+\overline{b}y=0\end{cases}.$$

We claim that $||T||=1$: indeed, since $|a|^2+|b|^2<1$, and hence B is a finite dimensional contraction, it follows from an argument of [HLL, Corollary 8] - if B is a compact operator then $\sigma\begin{pmatrix} A & C\\ 0 & B\end{pmatrix}=\sigma(A)\cup\sigma(B)$ for every bounded operator C - that $\sigma(T)=\sigma(U)\cup\sigma(B)=$ the closed unit disk, where $\sigma(\cdot)$ denotes the spectrum, so that $||T||=1$ since T is hyponormal and hence normaloid (i.e., norm equals spectral radius). We will prove that T is subnormal using the Agler's criterion which states that T is subnormal if and only if $\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}T^{*k}T^k\geq 0$ for all $n\geq 0$. A straightforward calculation shows that

$$T^{*k}T^k=\begin{pmatrix} p_k & q_k\\ \overline{q_k} & r_k\end{pmatrix}\bigoplus 1_\infty,$$

where

$$p_k := |b|^2|a|^{2(k-1)} + |b|^2|x|^2 \sum_{j=0}^{k-2} |a|^{2j} + |y|^2$$

$$q_k := a\overline{b}\,|a|^{2(k-1)} + a\overline{b}|x|^2 \sum_{j=0}^{k-2} |a|^{2j} + x\overline{y}$$

$$r_k := |a|^{2k} + |x|^2 \sum_{j=0}^{k-1} |a|^{2j}.$$

For notational convenience we let $p_0 := 1 =: r_0$ and $q_0 := 0$. Since $\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k} = 0$ for every $n \ge 0$, it suffices to show that

$$\sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} \begin{pmatrix} p_k & q_k \\ \overline{q_k} & r_k \end{pmatrix} \ge 0 \quad \text{for all } n \ge 0. \tag{3.1}$$

If $n = 0$ then

$$\begin{aligned}
\sum_{k=0}^{1}(-1)^k \binom{1}{k} \begin{pmatrix} p_k & q_k \\ \overline{q_k} & r_k \end{pmatrix} &= \begin{pmatrix} p_0 & q_0 \\ \overline{q_0} & r_0 \end{pmatrix} - \begin{pmatrix} p_1 & q_1 \\ \overline{q_1} & r_1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} |b|^2 + |y|^2 & a\overline{b} + x\overline{y} \\ \overline{a}b + \overline{x}y & |a|^2 + |x|^2 \end{pmatrix} \\
&= \begin{pmatrix} |x|^2 - |b|^2 & -(a\overline{b} + x\overline{y}) \\ -(\overline{a}b + \overline{x}y) & |y|^2 - |a|^2 \end{pmatrix} =: Q.
\end{aligned}$$

A straightforward calculation shows that $\det Q = 0$. But since $|x| > |b|$ it follows that $Q \ge 0$. If $n \ge 1$, then

$$\begin{aligned}
&\sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} \begin{pmatrix} p_k & q_k \\ \overline{q_k} & r_k \end{pmatrix} \\
&= \sum_{k=1}^{n}(-1)^k \binom{n+1}{k} \begin{pmatrix} p_k & q_k \\ \overline{q_k} & r_k \end{pmatrix} + \begin{pmatrix} p_0 & q_0 \\ \overline{q_0} & r_0 \end{pmatrix} + (-1)^{n+1} \begin{pmatrix} p_{n+1} & q_{n+1} \\ \overline{q_{n+1}} & r_{n+1} \end{pmatrix} \\
&= \sum_{k=1}^{n}(-1)^k \binom{n}{k} \begin{pmatrix} p_k & q_k \\ \overline{q_k} & r_k \end{pmatrix} + \sum_{k=1}^{n}(-1)^k \binom{n}{k-1} \begin{pmatrix} p_k & q_k \\ \overline{q_k} & r_k \end{pmatrix} + \begin{pmatrix} p_0 & q_0 \\ \overline{q_0} & p_0 \end{pmatrix} + (-1)^{n+1} \begin{pmatrix} p_{n+1} & q_{n+1} \\ \overline{q_{n+1}} & r_{n+1} \end{pmatrix} \\
&= \sum_{k=0}^{n}(-1)^k \binom{n}{k} \begin{pmatrix} p_k & q_k \\ \overline{q_k} & r_k \end{pmatrix} + \sum_{k=1}^{n+1}(-1)^k \binom{n}{k-1} \begin{pmatrix} p_k & q_k \\ \overline{q_k} & r_k \end{pmatrix} \\
&= \sum_{k=0}^{n}(-1)^k \binom{n}{k} \begin{pmatrix} p_k & q_k \\ \overline{q_k} & r_k \end{pmatrix} - \sum_{k=0}^{n}(-1)^k \binom{n}{k} \begin{pmatrix} p_{k+1} & q_{k+1} \\ \overline{q_{k+1}} & r_{k+1} \end{pmatrix} \\
&= \sum_{k=0}^{n}(-1)^k \binom{n}{k} \begin{pmatrix} p_k - p_{k+1} & q_k - q_{k+1} \\ \overline{q_k} - \overline{q_{k+1}} & r_k - r_{k+1} \end{pmatrix} \\
&= \sum_{k=0}^{n}(-1)^k \binom{n}{k} \begin{pmatrix} |a|^{2(k-1)}(1 - |a|^2 - |x|^2)|b|^2 & |a|^{2(k-1)}(1 - |a|^2 - |x|^2)a\overline{b} \\ |a|^{2(k-1)}(1 - |a|^2 - |x|^2)\overline{a}b & |a|^{2k}(1 - |a|^2 - |x|^2) \end{pmatrix} \\
&= \frac{(1 - |a|^2 - |x|^2)}{|a|^2} \begin{pmatrix} |b|^2 & a\overline{b} \\ \overline{a}b & |a|^2 \end{pmatrix} \sum_{k=0}^{n}(-1)^k \binom{n}{k} |a|^{2k} \\
&= \frac{(|y|^2 - |a|^2)}{|a|^2} \begin{pmatrix} |b|^2 & a\overline{b} \\ \overline{a}b & |a|^2 \end{pmatrix} \sum_{k=0}^{n}(-1)^k \binom{n}{k} |a|^{2k}.
\end{aligned}$$

If n is even, then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}|a|^{2k}=\sum_{k=0}^{n}\binom{n}{k}|a|^{2k}(-1)^{n-k}=(|a|^2-1)^n\ge 0.$$

If instead n is odd, then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}|a|^{2k}=-\sum_{k=0}^{n}\binom{n}{k}|a|^{2k}(-1)^{n-k}=-(|a|^2-1)^n\ge 0.$$

This proves (3.1) and therefore T is subnormal. This proves the first statement.

(2) Towards the second statement, we suppose that $T|_{\ker[T^*,T]}$ has the rank-two self-commutator, that is, $[S^*,S]$ is of rank-two. So, A is of rank-two, and hence A^* is also of rank-two. Since for each $h\in\ \ker[T^*,T]$,

$$T^*h=\begin{pmatrix}S^*&0\\A^*&B^*\end{pmatrix}\begin{pmatrix}h\\0\end{pmatrix}=\begin{pmatrix}S^*h\\A^*h\end{pmatrix},$$

it follows

$$\mathcal{H}=\bigvee\{T^{*n}h:\ h\in\ \ker[T^*,T],\ n=0,1\}.$$

So $T=\ $m.p.n.e.$\,(S)=:\widehat{S}$. By (2.5), S is 2-hyponormal. Since again $\ker[S^*,S]$ is invariant for S and $\operatorname{rank}[S^*,S]=2$, we can repeat the preceding argument for S instead of T. Write $T_1:=S$ and

$$T_1=\begin{pmatrix}T_2&A_1\\0&B_1\end{pmatrix}:\ \begin{pmatrix}\ker[T_1^*,T_1]\\\operatorname{ran}[T_1^*,T_1]\end{pmatrix}\to\begin{pmatrix}\ker[T_1^*,T_1]\\\operatorname{ran}[T_1^*,T_1]\end{pmatrix}.$$

If A_1 is of rank-one then again $T_2\cong\alpha_1U+\beta_1$, so by the first statement we can see that T_1 is subnormal. Since by (2.6), the minimal partially normal extension of a subnormal operator is also subnormal, we can conclude that $T=\widehat{T_1}$ is subnormal. If instead A_1 is of rank-two then again we have

$$T_1=\ \text{m.p.n.e.}\,(T_2)=:\widehat{T_2}.$$

Since T_1 is 2-hyponormal it follows from (2.5) that T_2 is 3-hyponormal and $T=\widehat{T_2}^{(2)}$. If this process stops after finite steps, then T is subnormal. If instead this process does not stop after finite steps, then we can obtain a sequence $\{T_n\}$ such that

(i) $\operatorname{rank}[T_n^*,T_n]=2$;
(ii) $T_n=\ $m.p.n.e.$(T_{n+1})=:\widehat{T_{n+1}}$;
(iii) T_n is $(n+1)$-hyponormal.

So we have that for each $n\ge 1$, $T=\widehat{T_n}^{(n)}$ and T_n is $(n+1)$-hyponormal. Consequently, T is a finite rank perturbation of T_n which is an $(n+1)$-hyponormal operator. Since n is arbitrary, the first assertion of the statement (2) follows. Note that in this case T has the following two-diagonal structure, with respect to the orthogonal decomposition $H=H_0\oplus H_1\oplus\cdots$:

$$T=\begin{pmatrix}B_0&0&0&0&\dots\\A_0&B_1&0&0&\dots\\0&A_1&B_2&0&\dots\\0&0&A_2&B_3&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix},\tag{3.2}$$

where

(i)

$$T_n:=\begin{pmatrix}B_n&0&0&0&\dots\\A_n&B_{n+1}&0&0&\dots\\0&A_{n+1}&B_{n+2}&0&\dots\\0&0&A_{n+2}&B_{n+3}&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\tag{3.3}$$

is the minimal partially normal extension of T_{n+1};

(ii) $H_n = \operatorname{ran}[T_{n+1}^*, T_{n+1}]$;
(iii) the A_j and B_j are all 2×2 matrices;
(iv) the A_j are invertible.

Since $T_n = $ m.p.n.e.(T_{n+1}) we have that

$$\begin{cases} [T^*, T] = ([B_0^*, B_0] + A_0^* A_0) \oplus 0_\infty; \\ [B_{n+1}^*, B_{n+1}] + A_{n+1}^* A_{n+1} = A_n A_n^* \quad (n \geq 0); \\ A_n^* B_{n+1} = B_n A_n^* \quad (n \geq 0), \end{cases} \tag{3.4}$$

which gives that the operator T in (3.2) is exactly the Putinar's matricial model (1.7). This proves the second statement. □

4. Proof of Theorem 2

Proof of Theorem 2. We split the proof into two cases.
(**Case1**: B_n is normal for some $n \geq 1$): The program is to show that if B_n is normal for some $n \geq 1$ then

$$T_n := \begin{pmatrix} B_n & 0 & 0 & 0 & \dots \\ A_n & B_{n+1} & 0 & 0 & \dots \\ 0 & A_{n+1} & B_{n+2} & 0 & \dots \\ 0 & 0 & A_{n+2} & B_{n+3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is subnormal, and hence by (2.6), T is subnormal since $T = \widehat{T}^n$. So, we may assume, without loss of generality, that B_1 is normal. By the fourth equality of (2.9), we have $A_0 = A_1$. Since A_0 is diagonalizable, we can write $A_0 = A_1 := \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)$. We also write $B_n := \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$ $(n = 0, 1)$. By the third equality of (2.9), we have

$$\begin{cases} a_0 = a_1 =: a; \\ d_0 = d_1 =: d; \\ p b_1 = b_0 q; \\ c_0 p = q c_1 \,. \end{cases}$$

There are two cases to consider.
(**Case1-1**: $a \neq d$): By translation, write $B_n := \left(\begin{smallmatrix} a & b_n \\ c_n & 0 \end{smallmatrix}\right)$ $(a \neq 0; n = 0, 1)$. We may assume without loss of generality that a is a real number (In fact, if we multiply T by $e^{i\theta}$, then A_n is not positive. But we can proceed in the same way with the notations of (3.4). So we hold the notations of (2.9).) So, $c_1 = \overline{b_1}$. By the third equality of (2.9), we have

$$B_2 = A_1^{-1} B_1 A_1 = \begin{pmatrix} a & \frac{q}{p} b_1 \\ \frac{p}{q} \overline{b_1} & 0 \end{pmatrix}$$

and

$$B_0 = A_0 B_1 A_0^{-1} = \begin{pmatrix} a & \frac{p}{q} b_1 \\ \frac{q}{p} \overline{b_1} & 0 \end{pmatrix} = B_2^*.$$

Note that $A_0^2 + [B_0^*, B_0] = A_1^2 - [B_2^*, B_2] = A_2^2$. Now if we define $A_{-1} := \left([B_0^*, B_0] + A_0^2\right)^{\frac{1}{2}} = A_2$ and in turn, $B_{-1} := A_{-1} B_0 A_{-1}^{-1} = A_2 B_0 A_2^{-1} = A_2 B_2^* A_2^{-1} = B_3^*$, then $T_1 := \left(\begin{smallmatrix} B_{-1} & 0 \\ A_{-1} & T \end{smallmatrix}\right) = \left(\begin{smallmatrix} B_3^* & 0 \\ A_2 & T \end{smallmatrix}\right)$ is the minimal partially normal extension of T. Moreover, we have

$$[T_1^*, T_1] = \left([B_3, B_3^*] + A_2^2\right) \oplus\ 0_\infty = \left(A_2^2 - [B_3^*, B_3]\right) \oplus\ 0_\infty = A_3^2 \oplus\ 0_\infty \geq 0.$$

Since $T_1 = $ m.p.n.e.(T) it follows from (2.5) that T is 2-hyponormal. Similarly, if we define

$$T_2 := \begin{pmatrix} B_4^* & 0 & 0 \\ A_3 & B_3^* & 0 \\ 0 & A_2 & T \end{pmatrix}$$

then $T_2 =$ m.p.n.e. (T_1) and $[T_2^*, T_2] = A_4^2 \oplus 0_\infty \geq 0$, so that T_1 is 2-hyponormal, and hence T is 3-hyponormal. Continuing this process gives that if we define, for each $n = 0, 1, \cdots$,

$$T_n := \begin{pmatrix} B_{n+2}^* & 0 & & & \\ A_{n+1} & B_{n+1}^* & \ddots & & \\ 0 & A_n & \ddots & \ddots & \\ & \ddots & \ddots & B_3^* & 0 \\ & & 0 & A_2 & T \end{pmatrix}$$

then $T_n =$ m.n.p.e. (T_{n-1}) and T_n is hyponormal, so that T is $(n+1)$-hyponormal for every n. Therefore T is subnormal.

(**Case1-2**: $a = d$): By translation we can write $B_n = \left(\begin{smallmatrix} 0 & b_n \\ c_n & 0 \end{smallmatrix}\right)$ $(n = 0, 1)$. So by the third equality of (2.9), B_2 is skew diagonal and in turn, by the fourth equality of (2.9), A_2 is diagonal. Repeating this argument with a telescoping method shows that B_n is skew diagonal and A_n is diagonal for each $n = 0, 1, \cdots$. Thus T is of the form:

$$T = \begin{pmatrix} 0 & b_0 & 0 & 0 & \cdots \\ c_0 & 0 & 0 & 0 & \cdots \\ p_0 & 0 & 0 & b_1 & \cdots \\ 0 & q_0 & c_1 & 0 & \cdots \\ 0 & 0 & p_1 & 0 & \cdots \\ 0 & 0 & 0 & q_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.1}$$

Since B_1 is normal, we have $|b_1| = |c_1|$. Write

$$T_1 := \begin{pmatrix} B_1 & 0 & 0 & 0 & \cdots \\ A_1 & B_2 & 0 & 0 & \cdots \\ 0 & A_2 & B_2 & 0 & \cdots \\ 0 & 0 & A_3 & B_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We claim that T_1 is subnormal. For notational convenience we assume that B_0 is normal and hence $|b_0| = |c_0|$. We must show that T is subnormal. By the third and fourth equalities of (2.9) we have

$$\begin{cases} p_{n+1}^2 = p_n^2 + |b_{n+1}|^2 - |c_{n+1}|^2; \\ q_{n+1}^2 = q_n^2 - |b_{n+1}|^2 + |c_{n+1}|^2; \\ p_n b_{n+1} = b_n q_n; \\ c_n p_n = q_n c_{n+1}. \end{cases} \tag{4.2}$$

We want to define p_j, q_j, b_j and c_j for $j = -1, -2, \cdots$. To do so, from (4.2), we need to confirm that p_j and q_j are all positive for $j = -1, -2, \cdots$. But since B_0 is normal, A_{-1} is defined by A_0 and then B_{-1} is automatically defined by (4.2). Now the remaining is to show that $p_{-1}^2 - |b_{-1}|^2 + |c_{-1}|^2 > 0$ and $q_{-1}^2 + |b_{-1}|^2 - |c_{-1}|^2 > 0$. Indeed a straightforward calculation shows that

$$p_{-1}^2 - |b_{-1}|^2 + |c_{-1}|^2 = p_0^2 - |c_1|^2 + |b_1|^2 = p_1^2 > 0$$

and

$$q_{-1}^2 + |b_{-1}|^2 - |c_{-1}|^2 = q_0^2 + |c_1|^2 - |b_1|^2 = q_1^2 > 0.$$

Repeating this process, we obtain p_n, q_n, b_n and c_n satisfying the condition (4.2) for all $n \in \mathbb{Z}$. Therefore T is a subnormal operator whose minimal normal extension is given by

$$N = \begin{pmatrix} \ddots & \ddots & & \vdots & \vdots & \\ & A_{-1} & B_{-1} & 0 & 0 & \\ \dots & 0 & A_{-1} & B_0 & 0 & \dots \\ \dots & \dots & 0 & A_0 & B_1 & \dots \\ \dots & \dots & 0 & 0 & A_1 & \ddots \\ & & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(**Case2**: B_0 is normal): Since A_0 is diagonalizable, we can write, by some translation,

$$A_0 := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \quad \text{and} \quad B_0 := \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \quad (|b| = |c|).$$

Then by the fourth equality of (2.9), we can define $A_{-1} := A_0 = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ and

$$B_{-1} := A_{-1} B_0 A_{-1}^{-1} = A_0 B_0 A_0^{-1} = \begin{pmatrix} a & \frac{p}{q} b \\ \frac{q}{p} c & 0 \end{pmatrix}.$$

On the other hand, from the third equality of (2.9), we have

$$B_1 = A_0^{-1} B_0 A_0 = \begin{pmatrix} a & \frac{q}{p} b \\ \frac{p}{q} c & 0 \end{pmatrix}.$$

A straightforward calculation shows that $[B_1^*, B_1] = -[B_{-1}^*, B_{-1}]$. So, we have

$$A_{-1}^2 + [B_{-1}^*, B_{-1}] = A_0^2 - [B_1^*, B_1] = A_1^2 > 0.$$

If we let $\widehat{T} := \begin{pmatrix} B_{-1} & 0 \\ A_{-1} & T \end{pmatrix}$, then $\widehat{T}$ is the minimal partially normal extension of T. Moreover, since

$$[\widehat{T}^*, \widehat{T}] = \left(A_{-1}^2 + [B_{-1}^*, B_{-1}]\right) \oplus 0_\infty = A_1^2 \oplus 0_\infty \geq 0,$$

it follows from (2.5) that T is 2-hyponormal. By the previous argument of the Case1, we can conclude that T is subnormal. □

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