A Gap between Hyponormality and Subnormality for Block Toeplitz Operators

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Abstract. This paper concerns a gap between hyponormality and subnormality for block Toeplitz operators. We show that there is no gap between 2-hyponormality and subnormality for a certain class of trigonometric block Toeplitz operators (e.g., its co-analytic outer coefficient is invertible). In addition we consider the extremal cases for the hyponormality of trigonometric block Toeplitz operators: in this case, hyponormality and normality coincide.

Keywords. Block Toeplitz operators, matrix-valued trigonometric polynomials, hyponormal, 2-hyponormal, subnormal.

1. Introduction

The Bram-Halmos criterion of subnormality ([Br]) states that an operator T on a Hilbert space $\mathcal{H}$ is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$ for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$. It is easy to see that this is equivalent to the following positivity test:

$$\begin{bmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{bmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{1.1}$$

Condition (1.1) provides a measure of the gap between hyponormality and subnormality. In fact the positivity condition (1.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.1) for all k. For $k \geq 1$, an operator T is said to be *k-hyponormal* if T satisfies the positivity condition (1.1) for a fixed k. Thus the Bram-Halmos criterion can be stated as: T is subnormal if and only if T is k-hyponormal for all $k \geq 1$. The k-hyponormality has been considered by many authors with an aim at understanding the gap between hyponormality and subnormality. For instance, the Bram-Halmos criterion on subnormality indicates that 2-hyponormality is generally far from subnormality. There are special classes of operators, however, for which these two notions are equivalent. For example, in [CL1, Example 3.1], it was shown that there is no gap between 2-hyponormality and subnormality for a back-step extension of the recursively generated subnormal weighted shift. The purpose of this paper is to consider a gap between hyponormality and subnormality (or normality) for Toeplitz operators with matrix-valued symbols. We establish that there is no gap between 2-hyponormality and normality for a certain class of block Toeplitz operators with matrix-valued trigonometric polynomial symbols and in the extremal cases, hyponormality and normality coincide.

2. Preliminaries

Throughout this paper, let $\mathcal{H}$ denote a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators acting on $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, T^* denotes the adjoint of

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2000 *Mathematics Subject Classification.* Primary 47B20, 47B35
The work of the first author was supported by National Research Foundation of Korea (NRF) grant funded by the Korea government(MEST)(2010-0016369). The work of the third author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST)(2010-0001983) .

T. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if its self-commutator $[T^*, T] \equiv T^*T - TT^*$ is positive semi-definite, and *subnormal* if T has a normal extension N, i.e., there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subseteq \mathcal{H}$ and $T = N|_{\mathcal{H}}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, we write $\ker T$ for the kernel of T. For a set $\mathcal{M}$, $\mathcal{M}^\perp$ denotes the orthogonal complement of $\mathcal{M}$.

We review a few essential facts for (block) Toeplitz operators and (block) Hankel operators that we will need to begin with, using [Do1], [Do2], and [Ni]. Let $L^2 \equiv L^2(\mathbb{T})$ be the set of square-integrable measurable functions on the unit circle $\mathbb{T} \equiv \partial\mathbb{D}$ in the complex plane and $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Let $L^\infty \equiv L^\infty(\mathbb{T})$ be the set of bounded measurable functions on $\mathbb{T}$ and let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$. For $\mathcal{X}$ a Hilbert space, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy space. We observe that $L^2_{\mathbb{C}^n} = L^2(\mathbb{T}) \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2(\mathbb{T}) \otimes \mathbb{C}^n$. Let M_n denote the set of $n \times n$ complex matrices. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ $(= L^\infty(\mathbb{T}) \otimes M_n)$ then the block Toeplitz operator T_Φ and the block Hankel operator H_Φ on $H^2_{\mathbb{C}^n}$ are defined by

$$T_\Phi f = P(\Phi f) \quad \text{and} \quad H_\Phi f = JP^\perp(\Phi f) \quad (f \in H^2_{\mathbb{C}^n}), \tag{2.1}$$

where P and $P^\perp$ denote the orthogonal projections that map from $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$ and $\left(H^2_{\mathbb{C}^n}\right)^\perp$, respectively and J denotes the unitary operator from $L^2_{\mathbb{C}^n}$ to $L^2_{\mathbb{C}^n}$ given by $J(g)(z) = \overline{z} I_n g(\overline{z})$ for $g \in L^2_{\mathbb{C}^n}$ (I_n :=the $n \times n$ identity matrix). If $n = 1$, T_Φ and H_Φ are called the (scalar) Toeplitz operator and the (scalar) Hankel operator, respectively. For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}). \tag{2.2}$$

An inner (matrix) function $\Theta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is one satisfying $\Theta^*\Theta = I_m$ for almost all $z \in \mathbb{T}$, where $M_{n\times m}$ denotes the set of $n \times m$ complex matrices. The following basic relations can be easily derived from the definition:

(2.3) $T^*_\Phi = T_{\Phi^*}$, $H^*_\Phi = H_{\widetilde{\Phi}}$ $\quad (\Phi \in L^\infty_{M_n})$;
(2.4) $T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi$ $\quad (\Phi, \Psi \in L^\infty_{M_n})$;
(2.5) $H_\Phi T_\Psi = H_{\Phi\Psi}$, $H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi$ $\quad (\Phi \in L^\infty_{M_n},\ \Psi \in H^\infty_{M_n})$.

A matrix-valued trigonometric polynomial $\Phi \in L^\infty_{M_n}$ is of the form

$$\Phi(z) = \sum_{j=-m}^{N} A_j z^j \ (A_j \in M_n),$$

where A_N and A_{-m} are called the *outer* coefficients of Φ. For a matrix-valued function $A(z) = \sum_{j=-\infty}^{\infty} A_j z^j \in L^2_{M_n}$, we define

$$||A||_2^2 := \int_{\mathbb{T}} \operatorname{tr}(A^*A)\, d\mu = \sum_{j=-\infty}^{\infty} \operatorname{tr}(A_j^* A_j),$$

where $\operatorname{tr}(\cdot)$ means the trace of the matrix and if $A \in L^\infty_{M_n}$, we define

$$||A||_\infty := \sup_{t\in\mathbb{T}} ||A(t)|| \quad (||\cdot|| \text{ means the spectral norm of the matrix}).$$

The hyponormality of the scalar Toeplitz operators T_φ was completely characterized by a property of their symbols by C. Cowen [Co] in 1988.

Cowen's Theorem ([Co], [NT]) *For $\varphi \in L^\infty$, write*

$$\mathcal{E}(\varphi) := \Big\{k \in H^\infty : \ ||k||_\infty \le 1 \ \text{and}\ \varphi - k\overline{\varphi} \in H^\infty\Big\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

In 2006, Gu, Hendricks and Rutherford [GHR] considered the hyponormality of block Toeplitz operators and characterized the hyponormality of block Toeplitz operators in terms of their symbols. In particular they showed that if T_Φ is a hyponormal block Toeplitz operator on $H^2_{\mathbb{C}^n}$, then Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$. Their characterization for hyponormality of block Toeplitz operators resembles the Cowen's theorem except for an additional condition – the normality of the symbol.

Lemma 2.1. (Hyponormality of Block Toeplitz Operators) [GHR] *For each $\Phi \in L^\infty_{M_n}$, let*

$$\mathcal{E}(\Phi) := \Big\{K \in H^\infty_{M_n} : \ ||K||_\infty \le 1 \ \ \textit{and} \ \ \Phi - K\Phi^* \in H^\infty_{M_n}\Big\}.$$

Then a block Toeplitz operator T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ $(m \le n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime.

Remark 2.2. If $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$.

Proof. Assume to the contrary that $\Phi = \Delta A$ with $\Delta \in H^2_{M_{n\times r}}$ $(r < n)$. Then for almost all $z \in \mathbb{T}$, $\operatorname{rank}\Phi(z) \le \operatorname{rank}\Delta(z) \le r < n$, so that $\det \Phi(z) = 0$ for almost all $z \in \mathbb{T}$. This shows that any left inner divisor Δ of Φ is square. $\square$

If $\Phi \in H^2_{M_n}$ is such that $\det\Phi$ is not identically zero then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$.

For brevity we write I for the identity matrix and

$$I_\zeta := \zeta I \quad (\zeta \in L^\infty).$$

For $\Phi \in L^\infty_{M_n}$ we write

$$\Phi_+ := P_n\Phi \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \big(P_n^\perp \Phi\big)^* \in H^2_{M_n},$$

where P_n denotes the orthogonal projection from $L^2_{M_n}$ onto $H^2_{M_n}$. Thus we can write $\Phi = \Phi_-^* + \Phi_+$. If Ψ is a matrix-valued analytic polynomial then we can write

$$\Psi = \Theta A^* \quad (A \in H^2_{M_n} \text{ and } \Theta = I_{z^N} \text{ for some nonnegative integer } N). \tag{2.6}$$

If Ω is the greatest common right inner divisor of A and Θ in the representation (2.6), then $\Theta = \Omega_r\Omega$ and $A = A_r\Omega$ for some inner matrix Ω_r (where $\Omega_r \in H^2_{M_n}$ because $\det\Theta$ is not identically zero) and some $A_r \in H^2_{M_n}$. Therefore we can write

$$\Psi = \Omega_r A_r^*, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime:} \tag{2.7}$$

in this case, $\Omega_r A_r^*$ is called the *right coprime decomposition* of Φ.

In general, it is not easy to check the condition "Θ and A are right coprime" for the representation $\Phi = \Theta A^*$ (Θ is inner and $A \in H^2_{M_n}$) even though $\Theta = I_\theta$ for an inner function θ. But if Φ is a matrix-valued analytic polynomial then we have a more tractable criterion (cf. [CHL, Lemma 3.10]): if $A \in H^\infty_{M_n}$ is a matrix-valued analytic polynomial and $\Theta = I_{z^N}$, then

$$\Theta \text{ and } A \text{ are right coprime} \iff A(0) \text{ is invertible.} \tag{2.8}$$

If $\Phi \in L^\infty_{M_n}$ is a matrix-valued trigonometric polynomial then T_Φ will be called a *trigonometric block Toeplitz operator*. In Section 3 we show that there is no gap between 2-hyponormality and normality for a certain class of trigonometric block Toeplitz operators. In Section 4, we consider the extremal cases for the hyponormality of trigonometric block Toeplitz operators: in this case, hyponormality and normality coincide.

3. 2-hyponormality of trigonometric block Toeplitz operators

We begin with:

Lemma 3.1. *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued trigonometric polynomial of the form $\Phi(z) = \sum_{j=-m}^{N} A_j z^j$ $(m \le N)$ and write*

$$\Phi_- = \Theta F^* \text{ (right coprime decomposition).}$$

Suppose I_z is an inner divisor of Θ. If

(i) *T_Φ is hyponormal;*

(ii) *$\ker [T_\Phi^*, T_\Phi]$ is invariant for T_Φ,*

then T_Φ is normal. Hence in particular, if T_Φ is 2-hyponormal then it is normal.

Proof. By assumption we write $\Theta = I_z\Theta_1$ for some inner matrix Θ_1. Suppose T_Φ is hyponormal. Since $\Phi^*\Phi = \Phi\Phi^*$, it follows from (2.4) that $[T_\Phi^*, T_\Phi] = H^*_{\Phi^*}H_{\Phi^*} - H^*_\Phi H_\Phi$. Note that by (2.8), $F_0 := F(0)$ is an invertible matrix since F and I_z are right coprime. Since Φ^* and Φ are trigonometric polynomials of co-analytic degrees N and m, respectively, we can see that

$$\text{ran}\,[T_\Phi^*, T_\Phi] = \text{ran}\left(H^*_{\Phi^*}H_{\Phi^*} - H^*_\Phi H_\Phi\right) \subseteq \mathcal{H}(I_{z^N}). \tag{3.1}$$

We now suppose that N_1 is the smallest integer such that

$$\text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(I_{z^{N_1}}). \tag{3.2}$$

Assume to the contrary that $\text{ran}\,[T_\Phi^*, T_\Phi] \neq \{0\}$. We choose an element $B \in \text{ran}\,[T_\Phi^*, T_\Phi]$ of the greatest analytic degree. Write

$$B := \sum_{j=0}^{N_1-1} B_j z^j \quad (B_{N_1-1} \neq 0).$$

We thus have

$$\begin{aligned}
T_{\Theta_1^*}T_{I_{z^{-N_1}}}T_{\Phi^*}B &= T_{\Theta_1^* I_{z^{-N_1}}\Phi^*}B \\
&= P\Big(\Theta_1^* I_{z^{-N_1}}\left(\Phi_+^* + I_z\Theta_1 F^*\right)\sum_{j=0}^{N_1-1} B_j z^j\Big) \\
&= P\Big(\Theta_1^*\big(I_{z^{-1}}\Phi_+^* + \Theta_1 F^*\big)\sum_{j=0}^{N_1-1} B_j z^{-(N_1-1-j)}\Big) \\
&= P\Big(F^*\sum_{j=0}^{N_1-1} B_j z^{-(N_1-1-j)}\Big) \\
&= F_0^* B_{N_1-1}.
\end{aligned}$$

But since F_0 is invertible and $B_{N_1-1} \neq 0$, it follows that $T^*_{\Theta_1}\Big(T_{I_{z^{-N_1}}}T_{\Phi^*}B\Big) \neq 0$, which implies that $T_{I_{z^{-N_1}}}T_{\Phi^*}B \neq 0$ and in turn,

$$T_{\Phi^*}B \notin \mathcal{H}(I_{z^{N_1}}).$$

But if $\ker [T_\Phi^*, T_\Phi]$ is invariant for T_Φ, and hence $\text{ran}\,[T_\Phi^*, T_\Phi]$ is invariant for T_Φ^*, then by (3.2),

$$T_\Phi^* B \in \text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(I_{z^{N_1}}),$$

which leads a contradiction. Therefore we must have that ran $[T_\Phi^*, T_\Phi] = \{0\}$, i.e., T_Φ is normal. The second assertion follows from the first assertion together with the fact that every 2-hyponormal operator $T \in \mathcal{B}(\mathcal{H})$ satisfies that ker $[T^*, T]$ is invariant for T (cf. [CL2]). This completes the proof. $\square$

Write $\Phi(z) \equiv \sum_{j=-m}^{N} A_j z^j \in L^\infty_{M_n}$. Define

$$G_{0,r} := A_{-m+r} \quad (r = 0, \dots, m-1)$$

and put

$$M_0 := \ker G_{00} \ (= \ker A_{-m}).$$

We now define, recursively, $G_{s,r}$ and M_s as follows: for $r = 0, \dots, m-1$ and $s = 0, \dots, m-1$,

$$\begin{cases} G_{s+1,r} := G_{s,r} P_{M_s^\perp} + G_{s,r+1} P_{M_s} \\ M_s := \ker G_{s,0}\,, \end{cases} \tag{3.3}$$

where $P_{\mathcal{X}}$ denotes the orthogonal projection of $\mathbb{C}^n$ onto $\mathcal{X}$ and $G_{s,m}$ is defined to be the zero matrix for all s.

Remark 3.2. *The sequence* (dim M_s) *is decreasing.*

Proof. By definition we can write

$$G_{s,0} = \begin{bmatrix} C_s & 0 \\ D_s & 0 \end{bmatrix} : \begin{bmatrix} M_s^\perp \\ M_s \end{bmatrix} \to \begin{bmatrix} M_s^\perp \\ M_s \end{bmatrix}.$$

Let

$$G_{s,1} := \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix} : \begin{bmatrix} M_s^\perp \\ M_s \end{bmatrix} \to \begin{bmatrix} M_s^\perp \\ M_s \end{bmatrix}.$$

Since

$$G_{s+1,0} = G_{s,0} P_{M_s^\perp} + G_{s,1} P_{M_s} = \begin{bmatrix} C_s & 0 \\ D_s & 0 \end{bmatrix} + \begin{bmatrix} 0 & E_2 \\ 0 & E_4 \end{bmatrix} = \begin{bmatrix} C_s & E_2 \\ D_s & E_4 \end{bmatrix},$$

it follows that rank $G_{s,0} \le$ rank $G_{s+1,0}$, i.e., $\dim\ker G_{s,0} \ge \dim\ker G_{s+1,0}$, giving the result. $\square$

We note that if $G_{s_0,0}$ is invertible for some s_0, then $G_{s,r} = G_{s_0,r}$ for all $s \ge s_0$ and $0 \le r \le m-1$.

We are ready for:

Theorem 3.3. *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued trigonometric polynomial of the form $\Phi(z) = \sum_{j=-m}^{N} A_j z^j$ $(m \le N)$ and suppose some $G_{s_0,0}$ $(0 \le s_0 \le m-1)$ defined by* (3.3) *is invertible. If T_Φ is 2-hyponormal then T_Φ is normal.*

Proof. Let $G_{s,r}$ be defined by (3.3) and write

$$G_0(z) \equiv \sum_{r=0}^{m-1} G_{0,r} z^r = \sum_{r=0}^{m-1} A_{-m+r} z^r. \tag{3.4}$$

Put $M_0 := \ker G_{00}$ $(= \ker A_{-m})$ as above. Therefore we can write

$$G_{00} = \begin{bmatrix} C_0 & 0 \\ D_0 & 0 \end{bmatrix} : \begin{bmatrix} M_0^\perp \\ M_0 \end{bmatrix} \to \begin{bmatrix} M_0^\perp \\ M_0 \end{bmatrix}.$$

Observe that

$$\begin{bmatrix} C_0 & 0 \\ D_0 & 0 \end{bmatrix} = \begin{bmatrix} C_0 & 0 \\ D_0 & 0 \end{bmatrix} \begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0} \end{bmatrix},$$

so that

$$G_{00} = G_{00}\Big(P_{M_0^\perp} + P_{M_0}\Big) = G_{00}P_{M_0^\perp}\begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0}\end{bmatrix} \tag{3.5}$$

and for $1 \le r \le m-1$,

$$\begin{aligned} G_{0,r}z^r &= G_{0,r}\Big(P_{M_0^\perp} + P_{M_0}\Big)\begin{bmatrix} z^r|_{M_0^\perp} & 0 \\ 0 & z^{r-1}|_{M_0}\end{bmatrix}\begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0}\end{bmatrix} \\ &= \Big((G_{0,r}P_{M_0^\perp})z^r + (G_{0,r}P_{M_0})z^{r-1}\Big)\begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0}\end{bmatrix}. \end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we have

$$\begin{aligned} G_0(z) &= \sum_{r=0}^{m-1} G_{0,r}z^r \\ &= G_{00}P_{M_0^\perp}\begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0}\end{bmatrix} \\ &\quad + \Big((G_{0,1}P_{M_0^\perp})z^1 + (G_{0,1}P_{M_0})z^0\Big)\begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0}\end{bmatrix} \\ &\quad + \Big((G_{0,2}P_{M_0^\perp})z^2 + (G_{0,2}P_{M_0})z^1\Big)\begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0}\end{bmatrix} \\ &\quad \cdots \\ &\quad + \Big((G_{0,m-1}P_{M_0^\perp})z^{m-1} + (G_{0,m-1}P_{M_0})z^{m-2}\Big)\begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0}\end{bmatrix} \\ &= \left(\sum_{r=0}^{m-1}\Big(G_{0,r}P_{M_0^\perp} + G_{0,r+1}P_{M_0}\Big)z^r\right)\begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0}\end{bmatrix} \\ &= \left(\sum_{r=0}^{m-1} G_{1,r}z^r\right)\begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0}\end{bmatrix}, \end{aligned}$$

where the third equality follows from regrouping the terms and adding the term

$$G_{0,m}P_{M_0}z^{m-1}\begin{bmatrix} 1|_{M_0^\perp} & 0 \\ 0 & z|_{M_0}\end{bmatrix}$$

(this is equal to zero because $G_{s,m}$ is defined to be the zero matrix for all s). Repeating the above argument for $G_1(z) \equiv \sum_{r=0}^{m-1} G_{1,r}z^r$, we have

$$G_1(z) = \left(\sum_{r=0}^{m-1} G_{2,r}z^r\right)\begin{bmatrix} 1|_{M_1^\perp} & 0 \\ 0 & z|_{M_1}\end{bmatrix}.$$

By induction we obtain

$$G_0(z) = \left(\sum_{r=0}^{m-1} G_{s,r}z^r\right)\prod_{j=1}^{s}\begin{bmatrix} 1|_{M_{s-j}^\perp} & 0 \\ 0 & z|_{M_{s-j}}\end{bmatrix} \quad \text{for } s = 1, \dots, m-1.$$

We now assume that $G_{s_0,0}$ is invertible for some s_0 $(0 \le s_0 \le m-1)$. Then the invertibility of $G_{s_0,0}$ implies that $\sum_{r=0}^{m-1} G_{s_0,r} z^r$ is right coprime with I_z. We observe

$$\begin{aligned}\Phi_- &= A_{-1}^* z + \cdots + A_{-m}^* z^m = z^m G_0(z)^* \\ &= z^m \left(\left(\sum_{r=0}^{m-1} G_{s_0,r} z^r \right) \prod_{j=1}^{s_0} \begin{bmatrix} 1|_{M_{s_0-j}^\perp} & 0 \\ 0 & z|_{M_{s_0-j}} \end{bmatrix} \right)^* \\ &= z^{m-s_0} \prod_{j=1}^{s_0} \begin{bmatrix} z|_{M_{s_0-j}^\perp} & 0 \\ 0 & 1|_{M_{s_0-j}} \end{bmatrix} \left(\sum_{r=0}^{m-1} G_{s_0,r} z^r \right)^* .\end{aligned}$$

By assumption we must have that $m - s_0 \ge 1$. We claim that

$$\Theta \equiv z^{m-s_0} \prod_{j=1}^{s_0} \begin{bmatrix} z|_{M_{s_0-j}^\perp} & 0 \\ 0 & 1|_{M_{s_0-j}} \end{bmatrix} \text{ and } F \equiv \sum_{r=0}^{m-1} G_{s_0,r} z^r \text{ are right coprime.} \tag{3.7}$$

To see (3.7) we assume to the contrary that Θ and F are not right coprime. Then $\widetilde{\Theta}$ and $\widetilde{F}$ are not left coprime. Thus there exists an inner matrix function $\widetilde{\Delta} \in H^2_{M_{n\times l}}$ such that

$$\widetilde{\Theta} = \widetilde{\Delta} C_1, \quad \widetilde{F} = \widetilde{\Delta} C_2 \quad (\text{for some } C_1, C_2 \in H^2_{M_{l\times n}}),$$

where Δ is not unitary constant. Since $G_{s_0,0}$ is invertible it follows that $\det \widetilde{F}$ is not identically zero, and hence $\widetilde{\Delta} \in H^2_{M_n}$. Therefore Δ becomes a common right inner divisor of Θ and F. Put

$$\Omega := \prod_{j=0}^{s_0-1} \begin{bmatrix} 1|_{M_j^\perp} & 0 \\ 0 & z|_{M_j} \end{bmatrix} .$$

Then $I_{z^m} = \Omega\Theta = \Omega C_1 \Delta$ and $F = C_2 \Delta$ are not right coprime. But since $F(0) = G_{s_0,0}$ is invertible, it follows from (2.8) that I_{z^m} and F are right coprime, a contradiction. This proves (3.7). But since Θ contains an inner factor I_z, applying Lemma 3.1 with F and Θ gives the result. □

The following corollary shows that there is no gap between 2-hyponormality and normality for Toeplitz operators with matrix-valued trigonometric polynomial symbols whose co-analytic outer coefficient is invertible.

Corollary 3.4. *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued trigonometric polynomial whose co-analytic outer coefficient is invertible. If T_Φ is 2-hyponormal then T_Φ is normal.*

Proof. Write

$$\Phi_- = \sum_{j=1}^{m} A_{-j} z^j .$$

Under the notation of Theorem 3.3, we have that $G_{00} = A_{-m}$ (=the co-analytic outer coefficient). Thus the result follows at once from Theorem 3.3. □

In Corollary 3.4, the condition "the coanalytic outer coefficient is invertible" is essential. To see this, let

$$\Phi := \begin{bmatrix} z + \overline{z} & 0 \\ 0 & z \end{bmatrix}.$$

Then

$$T_\Phi = \begin{bmatrix} T_z + T_z^* & 0 \\ 0 & T_z \end{bmatrix}.$$

Thus T_Φ is subnormal (and hence 2-hyponormal). Clearly, T_Φ is neither normal nor analytic even though the analytic outer coefficient $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is invertible. Note that the co-analytic outer coefficient $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is singular.

Of course, the assumption of Corollary 3.4 is superfluous. For example, if $\Phi = \sum_{j=-m}^{N} A_j z^j$ is a matrix-valued trigonometric polynomial of the form

$$A_{-m} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_{-m+1} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then by Theorem 3.3, the conclusion of Corollary 3.4 is still true even though A_{-m} is not invertible.

4. Extremal cases

It was known ([FL]) that if φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ then '$|a_{-m}| \le |a_N|$' is a necessary condition for T_φ to be hyponormal. In this sense, the condition '$|a_{-m}| = |a_N|$' is an extremal case for T_φ to be hyponormal: in particular, in this case, T_φ is hyponormal if and only if the Fourier coefficients of φ have a symmetric relation, i.e., there exists $\theta \in [0, 2\pi)$ such that (cf. [FL, Theorem 1.4])

$$\begin{bmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ a_{-m} \end{bmatrix} = e^{i\theta} \begin{bmatrix} \overline{a}_{N-m+1} \\ \overline{a}_{N-m+2} \\ \vdots \\ \overline{a}_N \end{bmatrix} \quad \text{for some } \theta \in [0, 2\pi).$$

We now consider the extremal cases for hyponormal Toeplitz operators with matrix-valued trigonometric polynomial symbols. What is a matrix version of the extremal condition '$|a_{-m}| = |a_N|$' for a matrix-valued trigonometric polynomial $\Phi(z) = \sum_{j=-m}^{N} A_j z^j$ (where each A_j is an $n \times n$ matrix and A_N is invertible) ? We may suggest the following conditions as the corresponding matrix version of the extremal case:

(4.1) $A_{-m}^* A_{-m} = A_N A_N^*$;
(4.2) $|\det A_{-m}| = |\det A_N|$;
(4.3) $||A_{-m}||_2 = ||A_N||_2$.

Evidently, (4.1) $\Rightarrow$ (4.2) and (4.3). However (4.2) is independent of (4.3). In [GHR], the authors established the hyponormality of T_Φ with symbol Φ satisfying the condition (4.1): indeed, there is a symmetric relation such as

$$A_{-m+j} = U\, A_{N-j}^* \;\text{ with a constant unitary matrix } U \quad (j = 0, 1, \dots, m-1).$$

In this section, we consider the cases (4.2) and (4.3): in fact, we get to the same conclusion.

Theorem 4.1. *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued trigonometric polynomial of the form $\Phi(z) = \sum_{j=-m}^{N} A_j z^j$ (A_N is invertible). If T_Φ is hyponormal then*

$$|\det A_{-m}| \le |\det A_N|. \tag{4.4}$$

Moreover if $|\det A_{-m}| = |\det A_N|$, then T_Φ is hyponormal if and only if $\Phi^*\Phi = \Phi\Phi^*$ and there exists a constant unitary matrix U such that

$$A_{-m+j} = U\, A_{N-j}^* \quad \textit{for each } j = 0, 1, \dots, m-1. \tag{4.5}$$

Proof. Suppose T_Φ is hyponormal. Then by Lemma 2.1, there exists a matrix function $K \in H^\infty_{M_n}$ such that $||K||_\infty \le 1$ and $\Phi_-^* - K\Phi_+^* \in H^\infty_{M_n}$, i.e.,

$$\sum_{j=-m}^{-1} A_j z^j - K \sum_{j=1}^{N} A_j^* z^{-j} \in H^\infty_{M_n}. \tag{4.6}$$

Since A_N is invertible, we can write $K = z^{N-m}\sum_{j=0}^{\infty} K_j z^j$ and $A_{-m} = K_0 A_N^*$. On the other hand, since $||K_0|| \le 1$ (because $||K||_\infty \le 1$) and

$$||K_0|| = \max\Big\{\sqrt{\lambda_j} : \lambda_j \text{ is an eigenvalue of } K_0^* K_0\Big\},$$

we have $0 \le \lambda_j \le ||K_0||^2 \le 1$ for each j. Thus

$$|\det K_0|^2 = \det K_0^* K_0 = \lambda_1\lambda_2\cdots\lambda_n \le 1\,, \tag{4.7}$$

which implies $|\det K_0| \le 1$. Thus we have

$$|\det A_{-m}| = |\det K_0|\,|\det A_N| \le |\det A_N|\,,$$

giving (4.4). For the second assertion, we assume that

$$|\det A_{-m}| = |\det A_N| \ne 0,$$

so that $\lambda_1\lambda_2\cdots\lambda_n = |\det K_0|^2 = 1$. Since $0 \le \lambda_j \le 1$ for each j, it follows that $\lambda_j = 1$ for all $j = 1, \dots, n$. Thus $K_0^* K_0$ is unitarily equivalent to I, so that K_0 is unitary. On the other hand,

$$1 = \frac{1}{n}||K_0||_2^2 \le \frac{1}{n}\sum_{j=0}^{\infty}||K_j||_2^2 = \frac{1}{n}||K||_2^2 \le ||K||_\infty^2 \le 1,$$

which implies that $K_1 = K_2 = \ldots = 0$. Hence $U \equiv K_0 = \sum_{j=0}^{\infty} K_j z^j$ is unitary. In particular, from (4.6),

$$\sum_{j=-m}^{-1} A_j z^j - U\sum_{j=N-m+1}^{N} A_j^* z^{N-m-j} \in H^\infty_{M_n},$$

giving (4.5). The converse is similar. □

Theorem 4.2. *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued trigonometric polynomial of the form $\Phi(z) = \sum_{j=-m}^{N} A_j z^j$ (A_N is invertible). If T_Φ is hyponormal then*

$$||A_{-m}||_2 \le ||A_N||_2. \tag{4.8}$$

Moreover if $||A_{-m}||_2 = ||A_N||_2$, then T_Φ is hyponormal if and only if $\Phi^*\Phi = \Phi\Phi^*$ and there exists a constant unitary matrix U such that

$$A_{-m+j} = U\,A_{N-j} ^* \quad \textit{for each } j = 0, 1, \dots, m-1. \tag{4.9}$$

Proof. Suppose T_Φ is hyponormal. Thus by Lemma 2.1, there exists a matrix function $K \in H^\infty_{M_n}$ such that $||K||_\infty \le 1$ and $\Phi_-^* - K\Phi_+^* \in H^\infty_{M_n}$, i.e.,

$$\sum_{j=-m}^{-1} A_j z^j - K\sum_{j=1}^{N} A_j^* z^{-j} \in H^\infty_{M_n}.$$

Thus we can write $K = z^{N-m}\sum_{j=0}^{\infty} K_j z^j$ and $A_{-m} = K_0\,A_N^*$. Observe that

$$||A_N||_2^2 - ||A_{-m}||_2^2 = \text{tr}\,(A_N A_N^*) - \text{tr}\,\big(A_{-m}^* A_{-m}\big) = \text{tr}\,(A_N(I - K_0^* K_0)A_N^*) \ge 0 \tag{4.10}$$

because K_0 is a contraction. This gives (4.8). For the second assertion we assume that $||A_{-m}||_2 = ||A_N||_2$. By (4.10), we have tr $(A_N(I - K_0^*K_0)A_N^*) = 0$, so that $A_N(I - K_0^*K_0)^{\frac{1}{2}} = 0$. But since A_N is invertible it follows that K_0 is unitary. Now the same argument as the proof of Theorem 4.1 gives the result. □

We conclude with the following observation which shows that hyponormality and normality coincide for the extremal cases.

Corollary 4.3. *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued trigonometric polynomial of the form* $\Phi(z) = \sum_{j=-N}^{N} A_j z^j$ *(A_N is invertible) satisfying*

$$\text{either} \quad |\det A_{-N}| = |\det A_N| \quad \text{or} \quad ||A_{-N}||_2 = ||A_N||_2 \,,$$

then T_Φ is hyponormal if and only if T_Φ is normal.

Proof. In this case, Theorems 4.1 and 4.2 give that $\Phi_+ - \Phi(0) = \Phi_- U$ for some constant unitary matrix U. Further since A_N is invertible, $\det(\Phi_+ - \Phi(0))$ is not identically zero. Thus the result follows at once from Theorem 4.3 of [GHR]. $\square$

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