A Subnormal Toeplitz Completion Problem

Raúl E. Curto, In Sung Hwang and Woo Young Lee

Abstract. We give a brief survey of subnormality and hyponormality of Toeplitz operators on the vector-valued Hardy space of the unit circle. We also solve the following subnormal Toeplitz completion problem: Complete the unspecified *rational* Toeplitz operators (i.e., the unknown entries are rational Toeplitz operators) of the partial block Toeplitz matrix

$$G := \begin{bmatrix} T_{\overline{\omega_1}} & ? \\ ? & T_{\overline{\omega_2}} \end{bmatrix} \quad (\omega_1 \text{ and } \omega_2 \text{ are finite Blaschke products})$$

to make G subnormal.

Keywords. (Block) Toeplitz operators, bounded type functions, matrix-valued rational functions, Halmos' Problem 5, Abrahamse's Theorem, hyponormal, subnormal, completion.

1. Hyponormality and subnormality of Toeplitz operators: A brief survey

§1.1. Which operators are subnormal ?

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *hyponormal* if its self-commutator $[T^*, T] := T^*T - TT^*$ is positive (semi-definite), and *subnormal* if there exists a normal operator N on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant under N and $N|_{\mathcal{H}} = T$. The notion of subnormality was introduced by P.R. Halmos in 1950 and the study of subnormal operators has been highly successful and fruitful: we refer to [Con] for details. Indeed, the theory of subnormal operators has made significant contributions to a number of problems in functional analysis, operator theory, mathematical physics, and other fields. Oddly however, the question "Which operators are subnormal ?" is difficult to answer. In general, it is quite intricate to examine whether a normal extension exists for an operator. Of course, there are a couple of constructive methods for determining subnormality; one of them is the Bram-Halmos criterion of subnormality ([Br]), which states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$ for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$. It is easy to see that this is equivalent to the following positivity test:

$$\begin{bmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{bmatrix} \geq 0 \tag{1}$$

for all $k \geq 1$. Condition (1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1) for all k. If we denote by $[A, B] := AB - BA$ the

---

2000 *Mathematics Subject Classification.* Primary 47B20, 47B35
The work of the first named author was partially supported by NSF Grant DMS-0801168. The work of the second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology(*2011-0022577*). The work of the third author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) grant funded by the Korea government(MEST)(*2012-0000939*).

commutator of two operators A and B, and if we define T to be *k–hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k$$

is positive, or equivalently, the $(k+1) \times (k+1)$ operator matrix in (1) is positive (via the operator version of the Cholesky algorithm), then the Bram–Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k–hyponormal for every $k \geq 1$ ([CMX]). But it still may not be possible to test the positivity condition (1) for *every* positive integer k, in general. Hence the following question is interesting and challenging:

$$\text{Are there feasible tests for the subnormality of an operator?} \tag{2}$$

Recall ([At], [CMX], [CoS]) that $T \in \mathcal{B}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators. If $k = 2$ then T is called *quadratically hyponormal*, and if $k = 3$ then T is said to be *cubically hyponormal*. Similarly, $T \in \mathcal{B}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general. k-hyponormality and weak k-hyponormality have been considered by many authors with an aim at understanding the gap between hyponormality and subnormality ([Cu1], [Cu2], [CuF1], [CuF2], [CuF3], [CLL], [CL1], [CL2], [CL3], [CMX], [DPY], [McCP]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle. For weighted shifts, positive results appear in [Cu1] and [CuF3], although no concrete example of a weighted shift which is polynomially hyponormal but not subnormal has yet been found (the existence of such weighted shifts was established in [CP1] and [CP2]). The Bram-Halmos criterion on subnormality indicates that 2-hyponormality is generally far from subnormality. There are special classes of operators, however, for which these two notions are equivalent. For example, in [CL1, Theorem 3.2], it was shown that 2-hyponormality and subnormality coincide for Toeplitz operators T_φ with trigonometric polynomial symbols $\varphi \in L^\infty$. On the other hand, 2-hyponormality and subnormality enjoy some common properties. One of them is the following fact ([CL2]):

$$\text{If } T \in \mathcal{B}(\mathcal{H}) \text{ is 2-hyponormal then } \ker\,[T^*, T] \text{ is invariant for } T. \tag{3}$$

In fact, since the invariance of $\ker\,[T^*, T]$ for T is one of the most important properties for subnormal operators, we may, in view of (3), expect that 2-hyponormality and subnormality coincide for special classes of operators. Indeed, in Section 2, we shall see this phenomenon for a Toeplitz completion problem.

§1.2. (Block) Toeplitz operators and bounded type functions

Toeplitz and Hankel operators arise in a variety of problems in several fields of mathematics and physics, and nowadays the theory of Toeplitz and Hankel operators has become a very wide area. Let $\mathbb{T} \equiv \partial\,\mathbb{D}$ be the unit circle in the complex plane $\mathbb{C}$. Let $L^2 \equiv L^2(\mathbb{T})$ be the set of all square-integrable measurable functions on $\mathbb{T}$ and let $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty \cap H^2$, that is, H^∞ is the set of bounded analytic functions on the unit disk $\mathbb{D}$. Given $\varphi \in L^\infty$, the Toeplitz operator T_φ and the Hankel operator H_φ are defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi g := JP^\perp(\varphi g) \qquad (g \in H^2),$$

where P and $P^\perp$ denote the orthogonal projections that map from L^2 onto H^2 and $(H^2)^\perp$, respectively, and where J denotes the unitary operator on L^2 defined by $J(f)(z) = \overline{z} f(\overline{z})$.

We recall that a function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are analytic functions $\psi_1, \psi_2 \in H^\infty$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

It is well known [Ab, Lemma 3] that if $\varphi \in L^\infty$ then

$$\varphi \text{ is of bounded type} \iff \ker H_\varphi \neq \{0\}\,.$$

Assume that both φ and $\overline{\varphi}$ are of bounded type. Since $T_{\overline{z}} H_\psi = H_\psi T_z$ for all $\psi \in L^\infty$, it follows from Beurling's Theorem that $\ker H_{\overline{\varphi_-}} = \theta_0 H^2$ and $\ker H_{\overline{\varphi_+}} = \theta_+ H^2$ for some inner functions θ_0, θ_+. We thus have $b := \overline{\varphi_-}\theta_0 \in H^2$, and hence we can write

$$\varphi_- = \theta_0 \overline{b}, \text{ and similarly } \varphi_+ = \theta_+ \overline{a} \text{ for some } a \in H^2. \tag{4}$$

In the factorization (4), we will always assume that θ_0 and b are coprime and θ_+ and a are coprime. In (4), $\theta_0 \overline{b}$ and $\theta_+ \overline{a}$ are called *coprime factorizations* of φ_- and φ_+, respectively. By Kronecker's Lemma [Ni, p. 183], if $f \in H^\infty$ then $\overline{f}$ is a rational function if and only if $\operatorname{rank} H_{\overline{f}} < \infty$, which implies that

$$\overline{f} \text{ is rational} \iff f = \theta \overline{b} \text{ with a finite Blaschke product } \theta. \tag{5}$$

For a Hilbert space $\mathcal{X}$, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and let $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy space. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ $(= L^\infty \otimes M_n)$ then $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ denotes the block Toeplitz operator with symbol Φ defined by

$$T_\Phi F := P_n(\Phi F) \quad \text{for } F \in H^2_{\mathbb{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$. A block Hankel operator with symbol $\Phi \in L^\infty_{M_n}$ is the operator $H_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ defined by

$$H_\Phi F := J_n P_n^\perp (\Phi F) \quad \text{for } F \in H^2_{\mathbb{C}^n},$$

where $P_n^\perp$ is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $(H^2_{\mathbb{C}^n})^\perp$ and J_n denotes the unitary operator on $L^2_{\mathbb{C}^n}$ given by $J_n(F)(z) := \overline{z} I_n F(\overline{z})$ for $F \in L^2_{\mathbb{C}^n}$ (where I_n is the $n \times n$ identity matrix). If we set $H^2_{\mathbb{C}^n} = H^2 \oplus \cdots \oplus H^2$ then we see that

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \dots & T_{\varphi_{1n}} \\ & \vdots & \\ T_{\varphi_{n1}} & \dots & T_{\varphi_{nn}} \end{bmatrix} \quad \text{and} \quad H_\Phi = \begin{bmatrix} H_{\varphi_{11}} & \dots & H_{\varphi_{1n}} \\ & \vdots & \\ H_{\varphi_{n1}} & \dots & H_{\varphi_{nn}} \end{bmatrix},$$

where

$$\Phi = \begin{bmatrix} \varphi_{11} & \dots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \dots & \varphi_{nn} \end{bmatrix} \in L^\infty_{M_n}.$$

For $\Phi \in L^\infty_{M_n}$, we write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}). \tag{6}$$

For $\Phi \in L^\infty_{M_n}$, we also write

$$\Phi_+ := P_n \Phi \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \left(P_n^\perp \Phi\right)^* \in H^2_{M_n}.$$

Thus we can write $\Phi = \Phi_-^* + \Phi_+$. However, it will often be convenient to allow the constant term in Φ_-. When this is the case, $\Phi_-(0)^*$ will not be zero; however, we will still ensure that $\Phi(0) = \Phi_+(0) + \Phi_-(0)^*$.

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ $(= H^\infty \otimes M_{n\times m})$ is called *inner* if Θ is isometric a.e. on $\mathbb{T}$. The following basic relations can be easily derived:

$$\begin{aligned} &T^*_\Phi = T_{\Phi^*}, \quad H^*_\Phi = H_{\widetilde{\Phi}} \quad (\Phi \in L^\infty_{M_n}); \\ &T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad (\Phi, \Psi \in L^\infty_{M_n}); && (7)\\ &H_\Phi T_\Psi = H_{\Phi\Psi}, \quad H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad (\Phi \in L^\infty_{M_n}, \Psi \in H^\infty_{M_n}). && (8) \end{aligned}$$

For a matrix-valued function $\Phi = [\phi_{ij}] \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry ϕ_{ij} is of bounded type and that Φ is *rational* if each entry ϕ_{ij} is a rational function.

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ $(m \le n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We remark that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$. If $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$.

The *shift* operator S on $H^2_{\mathbb{C}^n}$ is defined by

$$S := \sum_{j=1}^{n} \bigoplus T_z.$$

The following fundamental result known as the Beurling-Lax-Halmos Theorem is useful in the sequel.

The Beurling-Lax-Halmos Theorem. A nonzero subspace M of $H^2_{\mathbb{C}^n}$ is invariant for the shift operator S on $H^2_{\mathbb{C}^n}$ if and only if $M = \Theta H^2_{\mathbb{C}^m}$, where Θ is an inner matrix function in $H^\infty_{M_{n\times m}}$ $(m \le n)$. Furthermore, Θ is unique up to a unitary constant right factor; that is, if $M = \Delta H^2_{\mathbb{C}^r}$ (for Δ an inner function in $H^\infty_{M_{n\times r}}$), then $m = r$ and $\Theta = \Delta W$, where W is a unitary matrix mapping $\mathbb{C}^m$ onto $\mathbb{C}^m$.

As is customarily done, we say that two matrix-valued functions A and B are *equal* if they are equal up to a unitary constant right factor. Observe by (8) that for $\Phi \in L^\infty_{M_n}$, $H_\Phi S = H_\Phi T_{zI_n} = H_{\Phi\cdot zI_n} = H_{zI_n\cdot\Phi} = T^*_{\overline{z}I_n} H_\Phi$, which implies that the kernel of a block Hankel operator H_Φ is an invariant subspace of the shift operator on $H^2_{\mathbb{C}^n}$. Thus, if $\ker H_\Phi \ne \{0\}$, then by the Beurling-Lax-Halmos Theorem,

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function Θ. We note that Θ need not be a square matrix. For example, let θ_i $(i = 0, 1, 2)$ be a scalar inner function such that θ_1 and θ_2 are coprime and let $q \in L^\infty$ be such that $\ker H_q = \{0\}$. Define

$$\Theta := \frac{1}{\sqrt{2}} \begin{bmatrix} \theta_0\theta_1 \\ \theta_0\theta_2 \end{bmatrix} \quad \text{and} \quad \Phi := \begin{bmatrix} \overline{\theta_0\theta_1} & \overline{\theta_0\theta_2} \\ q\theta_2 & -q\theta_1 \end{bmatrix}.$$

Then a straightforward calculation shows that $\ker H_\Phi = \Theta H^2$ (cf. [GHR, Example 2.9]).

The following result was shown in [GHR, Theorem 2.2].

Theorem 1.1. ([GHR]) For $\Phi \in L^\infty_{M_n}$, the following statements are equivalent:

(i) Φ is of bounded type;
(ii) $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some square inner matrix function Θ;
(iii) $\Phi = A\Theta^*$, where $A \in H^\infty_{M_n}$ and A and Θ are right coprime.

For an inner matrix function $\Theta \in H^2_{M_n}$, we write

$$\mathcal{H}_\Theta := H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^n}.$$

In view of Theorem 1.1, if $\Phi \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type then Φ_+ and Φ_- can be written in the form

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*, \tag{9}$$

where Θ_1 and Θ_2 are inner, $A, B \in H^2_{M_n}$, Θ_1 and A are right coprime, and Θ_2 and B are right coprime. In (9), $\Theta_1 A^*$ and $\Theta_2 B^*$ will be called *right coprime factorizations* of Φ_+ and Φ_-, respectively.

In general, it is not easy to check the condition "B and Θ are right coprime". But if $\Theta \equiv \theta I_n$ for a finite Blaschke product θ, then we have a tractable criterion (cf. [CHL2, Lemma 3.3]):

$$\Theta \text{ and } B \text{ are right coprime } \iff B(\alpha) \text{ is invertible for each zero } \alpha \text{ of } \theta. \tag{10}$$

§1.3. Hyponormality of Toeplitz operators

An elegant and useful theorem of C. Cowen [Co4] characterizes the hyponormality of a Toeplitz operator T_φ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. This result makes it possible to answer an algebraic question coming from operator theory – namely, is T_φ hyponormal ? - by studying the function φ itself. Normal Toeplitz operators were characterized by a property of their symbol in the early 1960's by A. Brown and P.R. Halmos [BH], and so it is somewhat surprising that 25 years passed before the exact nature of the relationship between the symbol $\varphi \in L^\infty$ and the positivity of the self-commutator $[T_\varphi^*, T_\varphi]$ was understood (via Cowen's Theorem). As Cowen notes in his survey paper [Co3], the intensive study of subnormal Toeplitz operators in the 1970's and early 80's is one explanation for the relatively late appearance of the sequel to the Brown-Halmos work. The characterization of hyponormality via Cowen's Theorem requires one to solve a certain functional equation in the unit ball of H^∞.

Cowen's Theorem. ([Co4], [NT]) For each $\varphi \in L^\infty$, let

$$\mathcal{E}(\varphi) \equiv \{k \in H^\infty : \ ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

Cowen's Theorem has been used in [CHL1], [CL1], [CL2], [FL], [Gu1], [Gu2], [GS], [HKL1], [HKL2], [HL1], [HL2], [HL3], [Le], [NT] and [Zhu], which have been devoted to the study of hyponormality for Toeplitz operators on H^2. Particular attention has been paid to Toeplitz operators with polynomial symbols, rational symbols, and bounded type symbols [HL2], [HL3], [CHL1]. However, the case of arbitrary symbol φ, though solved in principle by Cowen's theorem, is in practice very complicated. Indeed, it may not even be possible to find tractable necessary and sufficient condition for the hyponormality of T_φ in terms of the Fourier coefficients of the symbol φ unless certain assumptions are made about φ. To date, tractable criteria for the cases of trigonometric polynomial symbols (resp. rational symbols) were derived from a Carathéodory-Schur interpolation problem ([Zhu]) (resp. a tangential Hermite–Fejér interpolation problem ([Gu1]) or the classical Hermite–Fejér interpolation problem ([HL3])). Very recently, a tractable and explicit criterion on the hyponormality of Toeplitz operators having bounded type symbols was established via the triangularization theorem for compressions of the shift operator ([CHL1]).

When one studies the hyponormality (also, normality and subnormality) of the Toeplitz operator T_φ one may, without loss of generality, assume that $\varphi(0) = 0$; this is because hyponormality is invariant under translation by scalars.

In 2006, Gu, Hendricks and Rutherford [GHR] characterized the hyponormality of block Toeplitz operators in terms of their symbols. Their characterization for hyponormality of block

Toeplitz operators T_Φ resembles Cowen's Theorem except for an additional condition which is trivially satisfied in the scalar case – the normality of the symbol, i.e., $\Phi^*\Phi = \Phi\Phi^*$.

Theorem 1.2. (**Hyponormality of Block Toeplitz Operators**) (Gu-Hendricks-Rutherford [GHR]) For each $\Phi \in L^\infty_{M_n}$, let

$$\mathcal{E}(\Phi) := \Big\{K \in H^\infty_{M_n} : \ ||K||_\infty \le 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n}\Big\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

In [GHR], the normality of block Toeplitz operator T_Φ was also characterized in terms of the symbol Φ, under a "determinant" assumption on the symbol Φ.

Theorem 1.3. (**Normality of Block Toeplitz Operators**) (Gu-Hendricks-Rutherford [GHR]) Let $\Phi \equiv \Phi_+ + \Phi_-^*$ be normal. If $\det \Phi_+$ is not identically zero then

$$T_\Phi \text{ is normal} \Longleftrightarrow \Phi_+ - \Phi_+(0) = \big(\Phi_- - \Phi_-(0)\big)\, U \ \text{ for some constant unitary matrix } U. \tag{11}$$

Until now, tractable criteria for the hyponormality of block Toeplitz operators T_Φ with matrix-valued trigonometric polynomials, rational functions or bounded type functions Φ have been established via interpolation problems or the so-called triangularization theorem for compressions of the shift operator ([GHR], [HL4], [HL5], [CHL1]).

§1.4. Halmos' Problem 5

In view of the preceding argument, it is natural and significant to elucidate the subnormality of Toeplitz operators. In 1970, P.R. Halmos addressed a problem on subnormality of Toeplitz operators acting on H^2, the so-called Halmos' Problem 5 in his lectures "Ten problems in Hilbert space" [Hal1]:

Halmos' Problem 5. Is every subnormal Toeplitz operator either normal or analytic ?

A Toeplitz operator T_φ is called *analytic* if $\varphi \in H^\infty$. Any analytic Toeplitz operator is easily seen to be subnormal: indeed, M_φ is a normal extension of T_φ, where M_φ is the normal operator of multiplication by φ on L^2. Thus the question is natural because the two classes, the normal and analytic Toeplitz operators, are well understood and are subnormal. In the 1970's, interesting partial (affirmative) answers appeared. Thus, when in 1979 Halmos wrote a report on progress on his ten problems (cf. [Hal2]), he stated that "some very good mathematics had gone into that answer" on Problem 5. He then conjectured that the future of Problem 5 was hopeful in the affirmative direction. However, in 1984, Halmos' Problem 5 was answered in the negative by C. Cowen and J. Long [CoL]: they found an analytic function ψ for which $T_{\psi+\alpha\overline{\psi}}$ $(0 < \alpha < 1)$ is subnormal – in fact, this Toeplitz operator is unitarily equivalent to a subnormal weighted shift W_β with weight sequence $\beta \equiv \{\beta_n\}$, where $\beta_n = (1 - \alpha^{2n+2})^{\frac{1}{2}}$ for $n = 0, 1, 2, \ldots$. A similar result was independently obtained by S. Sun ([Sun1], [Sun2], [Sun3]). Unfortunately, these constructions do not provide an intrinsic connection between subnormality and the theory of Toeplitz operators.

Until now researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. In fact it may not even be possible to find tractable necessary and sufficient condition for the subnormality of T_φ in terms of their symbols unless certain assumptions are made about φ. On the other hand, surprisingly, as C. Cowen notes in [Co2], some analytic Toeplitz operators are unitarily equivalent to non-analytic Toeplitz operators; i.e., the analyticity of Toeplitz operators is not invariant under unitary equivalence. In this sense, we might ask whether Cowen and Long's non-analytic subnormal Toeplitz operator is unitarily equivalent to an analytic Toeplitz operator. It was shown in [CHL2] that Cowen and Long's non-analytic subnormal Toeplitz operator

T_φ is not unitarily equivalent to any analytic Toeplitz operator. Consequently, even if we interpret "*is*" in Halmos' Problem 5 as "*is up to unitary equivalence*," the answer to Halmos' Problem 5 is still negative. Thus we would like to reformulate Halmos' Problem 5 as follows:

Halmos' Problem 5 reformulated. Which Toeplitz operators are subnormal?

Directly connected with Halmos' Problem 5 is the following question:

$$\text{Which subnormal Toeplitz operators are normal or analytic?} \tag{12}$$

In 1976, M.B. Abrahamse proved that the answer to Halmos' question is affirmative for Toeplitz operators with bounded type symbols ([Ab]):

Abrahamse's Theorem ([Ab, Theorem]). Let $\varphi \in L^\infty$ be such that φ or $\overline{\varphi}$ is of bounded type. If T_φ is hyponormal and $\ker\,[T_\varphi^*, T_\varphi]$ is invariant under T_φ then T_φ is normal or analytic.

Consequently, if $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type, then every subnormal Toeplitz operator must be either normal or analytic. Partial answers to question (12) have been obtained by many authors (cf. [AIW], [Co2], [CoL], [CHL1], [CHL2], [CL1], [CL2], [CL3], [ItW], [NT]). More generally, we are interested in the following question:

$$\text{Which subnormal block Toeplitz operators are normal or analytic?} \tag{13}$$

Question (13) is more difficult to answer, in comparison with the scalar-valued case. Indeed, Abrahamse's Theorem does not hold for block Toeplitz operators (even with matrix-valued trigonometric polynomial symbol): For instance, if

$$\Phi := \begin{bmatrix} z+\overline{z} & 0 \\ 0 & z \end{bmatrix},$$

then

$$T_\Phi = \begin{bmatrix} U_+ + U_+^* & 0 \\ 0 & U_+ \end{bmatrix} \quad (U_+ := \text{the unilateral shift on } H^2)$$

is neither normal nor analytic although T_Φ is evidently subnormal.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *quasinormal* if T commutes with T^*T and is said to be *pure* if it has no nonzero reducing subspace on which it is normal. It is well known that quasinormal $\Rightarrow$ subnormal. On the other hand, in [ItW], it was shown that every quasinormal Toeplitz operator is either normal or analytic, i.e., the answer to the Halmos' Problem 5 is affirmative for quasinormal Toeplitz operators. However, this is not true for the cases of matrix-valued symbols: indeed, if

$$\Phi \equiv \begin{bmatrix} \overline{z} & \overline{z}+2z \\ \overline{z}+2z & \overline{z} \end{bmatrix}. \tag{14}$$

then T_Φ is quasinormal, but it is neither normal nor analytic. Since

$$T_\Phi = \begin{bmatrix} U_+^* & U_+^*+2U_+ \\ U_+^*+2U_+ & U_+^* \end{bmatrix},$$

it follows that if $W := \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$, then W is unitary and

$$W^*T_\Phi W = 2\begin{bmatrix} U_+^* + U_+ & 0 \\ 0 & -U_+ \end{bmatrix},$$

which says that T_Φ is unitarily equivalent to a direct sum of the normal operator $2(U_+^* + U_+)$ and the analytic Toeplitz operator $-2U_+$. This phenomenon is not an accident. Indeed, very recently, in [CHKL], it was shown that every pure quasinormal operator with finite rank self-commutator is unitarily equivalent to a Toeplitz operator with a matrix-valued analytic rational symbol and (as a corollary) that every pure quasinormal Toeplitz operator with a matrix-valued rational symbol is unitarily equivalent to an analytic Toeplitz operator.

Also, in [CHKL], the following theorem was obtained:

Theorem 1.4. (Abrahamse's Theorem for Matrix-Valued Rational Symbols) [CHKL] Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function. Thus in view of (9), we may write

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization)}.$$

Assume that Θ has an inner divisor of the form θI_n, where θ is a nonconstant inner function. If

(i) T_Φ is hyponormal;
(ii) $\ker [T_\Phi^*, T_\Phi]$ is invariant for T_Φ,

then T_Φ is normal. Hence in particular, if T_Φ is subnormal then T_Φ is normal.

Theorem 1.4 may fail if we drop the assumption "Θ has a nonconstant diagonal-constant inner divisor." To see this, consider the matrix-valued function in (14):

$$\Phi \equiv \begin{bmatrix} \overline{z} & \overline{z}+2z \\ \overline{z}+2z & \overline{z} \end{bmatrix}.$$

We thus have

$$\Phi_- = \begin{bmatrix} z & z \\ z & z \end{bmatrix} = \left(\tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ -1 & z \end{bmatrix} \right) \left(\tfrac{1}{\sqrt{2}} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \right)^*,$$

where

$$\Theta \equiv \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & z \\ -1 & z \end{bmatrix} \text{ and } B \equiv \tfrac{1}{\sqrt{2}} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \text{ are right coprime.}$$

As we saw in the preceding, T_Φ is quasinormal, and hence subnormal. But clearly, T_Φ is neither normal nor analytic. Here we note that Θ does not have any nonconstant diagonal inner divisor of the form θI_n with a nonconstant inner function θ.

§1.5. A special subnormal Toeplitz completion

Given a partially specified operator matrix with some known entries, the problem of finding suitable operators to complete the given partial operator matrix so that the resulting matrix satisfies certain given properties is called a *completion problem*. Dilation problems are special cases of completion problems: in other words, the dilation of T is a completion of the partial operator matrix $\left[\begin{smallmatrix} T & ? \\ ? & ? \end{smallmatrix}\right]$. A *partial block Toeplitz matrix* is simply an $n \times n$ matrix some of whose entries are specified Toeplitz operators and whose remaining entries are unspecified. A *subnormal completion* of a partial operator matrix is a particular specification of the unspecified entries resulting in a subnormal operator. In particular, to avoid the triviality, we are interested in the cases whose diagonal entries are specified. For example, if ω is a finite Blaschke product, then T_ω is evidently subnormal, so that $\left[\begin{smallmatrix} T_\omega & 0 \\ 0 & T_\omega \end{smallmatrix}\right]$ is itself subnormal. On the other hand,

$$\begin{bmatrix} T_\omega & 1 - T_\omega T_{\overline{\omega}} \\ 0 & T_{\overline{\omega}} \end{bmatrix} \quad (\omega \text{ is a finite Blaschke product}) \tag{15}$$

is a subnormal (even unitary) completion of the 2×2 partial operator matrix

$$\begin{bmatrix} T_\omega & ? \\ ? & T_{\overline{\omega}} \end{bmatrix}.$$

A *subnormal Toeplitz completion* of a partial block Toeplitz matrix is a subnormal completion whose unspecified entries are Toeplitz operators. Then the following question comes up at once: Does there exist a subnormal Toeplitz completion of $\left[\begin{smallmatrix} T_\omega & ? \\ ? & T_{\overline{\omega}} \end{smallmatrix}\right]$? Evidently, (15) is not such a completion. To answer this question, let

$$\Phi \equiv \begin{bmatrix} \omega & \varphi \\ \psi & \overline{\omega} \end{bmatrix} \quad (\varphi, \psi \in L^\infty).$$

If T_Φ is hyponormal then by Theorem 1.2, Φ should be normal. Thus a straightforward calculation shows that

$$|\varphi| = |\psi| \quad \text{and} \quad \overline{\omega}(\varphi + \overline{\psi}) = \omega(\varphi + \overline{\psi}),$$

which implies that $\varphi = -\overline{\psi}$. Thus a direct calculation shows that

$$[T_\Phi^*, T_\Phi] = \begin{bmatrix} * & * \\ * & T_\omega T_{\overline{\omega}} - 1 \end{bmatrix},$$

which is not positive semi-definite because $T_\omega T_{\overline{\omega}} - 1$ is not. Therefore, there are no hyponormal Toeplitz completions of $\begin{bmatrix} T_\omega & ? \\ ? & T_{\overline{\omega}} \end{bmatrix}$. The following question seems to be more difficult: Does there exist a subnormal Toeplitz completion of

$$\begin{bmatrix} T_{\overline{\omega}} & ? \\ ? & T_{\overline{\omega}} \end{bmatrix} \quad (\omega \text{ is a finite Blaschke product}) \ ?$$

Special cases of this question were successfully considered in [CHL1] and [CHL3]. In the next section, we consider a subnormal Toeplitz completion problem.

2. Subnormal Toeplitz completions

In this section we consider the following:

Problem A. Complete the unspecified *rational* Toeplitz operators of the partial block Toeplitz matrix

$$G := \begin{bmatrix} T_{\overline{\omega_1}} & ? \\ ? & T_{\overline{\omega_2}} \end{bmatrix} \quad (\omega_1 \text{ and } \omega_2 \text{ are finite Blaschke products}) \tag{16}$$

to make G subnormal.

To answer Problem A, we need several auxiliary lemmas. We write

$$b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z} \quad (\alpha \in \mathbb{D}).$$

We begin with:

Lemma 2.1. Suppose $\varphi, \psi \in L^\infty$. Then

$$[T_{\varphi\circ b_\alpha}^*,\ T_{\psi\circ b_\alpha}] \cong [T_\varphi^*, T_\psi] \quad (\cong \text{ denotes unitary equivalence}).$$

In particular, $T_{\varphi\circ b_\alpha}$ is hyponormal if and only if T_φ is hyponormal.

Proof. By a well-known fact due to C. Cowen [Co1, Theorem 1], there exists a unitary operator V such that

$$T_{\varphi\circ b_\alpha} = V^* T_\varphi V \ \text{ and } \ T_{\psi\circ b_\alpha} = V^* T_\psi V.$$

We thus have $[T_{\varphi\circ b_\alpha}^*,\ T_{\psi\circ b_\alpha}] = V^*[T_\varphi^*, T_\psi]V$, which gives the result. □

Lemma 2.2. Let $\varphi, \psi \in L^\infty$ be rational functions and let ω_1 and ω_2 be finite Blaschke products. If

$$\Phi := \begin{bmatrix} \overline{\omega_1} & \varphi \\ \psi & \overline{\omega_2} \end{bmatrix}$$

is such that T_Φ is hyponormal then $\omega_1 = \omega_2$.

Proof. We first observe $(b_\alpha \circ b_{-\alpha})(z) = z$. Thus, in view of Lemma 2.1 we may assume that $\omega_1(0) = 0$. Then this lemma follows from a slight variation of the proof of [CHKL, Lemma 5.1], in which $\omega_1 = z^p$ and $\omega_2 = z^q$. □

In view of Lemma 2.2, for the problem (16), it suffices to consider the case

$$\Phi := \begin{bmatrix} \overline{\omega} & \varphi \\ \psi & \overline{\omega} \end{bmatrix} \quad (\varphi, \psi \in L^\infty \text{ are rational; } \omega \text{ is a finite Blaschke product}))$$

Lemma 2.3. Suppose $\Phi := \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is a matrix-valued rational function. Then we may write (cf. [CHL3, Lemma 3.1])

$$\Phi_+ = A^*\Delta_0\Delta \quad \text{and} \quad \Phi_- = B^*\Delta,$$

where $\Delta_0\Delta \equiv \theta I_n$ with an inner function θ, B and Δ are left coprime and $A, B \in H^2_{M_n}$. If $\ker[T_\Phi^*, T_\Phi]$ is invariant under T_Φ and $K \in \mathcal{E}(\Phi)$, then

$$\text{cl ran}\, H_{A\Delta^*} \subseteq \ker(I - T_{\widetilde{K}} T_{\widetilde{K}}^*).$$

(For the definition of $\widetilde{K}$, see (6).)

Proof. This follows from formula (16) in [CHL2], together with a careful analysis that the proof of (16) in [CHL2] does not employ the diagonal-*constant*-ness of Δ. □

Lemma 2.4. Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function such that

$$\Phi_- := \begin{bmatrix} \omega & \psi_- \\ \varphi_- & \omega \end{bmatrix},$$

where ω is a finite Blaschke product of the form

$$\omega = \prod_{i=1}^p b_i^{q_i} \quad \left(b_i(z) := \frac{z - \alpha_i}{1 - \overline{\alpha_i} z} \ (\alpha_i \neq \alpha_j \text{ if } i \neq j) \text{ and } q_i \geq 1 \right).$$

If

$$\Phi_- = \Theta B^* \quad \text{(right coprime factorization)},$$

then Θ has an inner divisor of the form $b_i I_2$ for some $i = 1, 2, \cdots, p$, except in the following two cases:

(i) $m_i + n_i = 2q_i$ for all $i = 1, 2, \cdots, p$;
(ii) $m_{i_0} + n_{i_0} > 2q_{i_0}$ and $m_{i_0} n_{i_0} = 0$ for some i_0,

in the representation

$$\varphi_- \equiv \theta_0 \overline{a} = \Bigl(\prod_{i=1}^p b_i^{m_i}\Bigr)\theta_0' \overline{a} \quad \text{and} \quad \psi_- \equiv \theta_1 \overline{b} = \Bigl(\prod_{i=1}^p b_i^{n_i}\Bigr)\theta_1' \overline{b} \quad \text{(coprime factorizations)}$$

($m_i, n_i = 0, 1, \cdots$ and $(\theta_0'\theta_1')(\alpha_i) \neq 0$ for all $i = 1, 2, \cdots, p$).

Proof. By Theorem 1.1, $\ker H_{\Phi_-^*} = \Theta H^2_{\mathbb{C}^2}$. We observe that for $f, g \in H^2$,

$$\Phi_-^* \begin{bmatrix} f \\ g \end{bmatrix} \in H^2_{\mathbb{C}^2} \iff \begin{bmatrix} \overline{\omega} & \overline{\theta_0} a \\ \overline{\theta_1} b & \overline{\omega} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \in H^2_{\mathbb{C}^2},$$

which implies that if $\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\Phi_-^*}$, then

$$\Bigl(\prod_{i=1}^p \overline{b_i}^{q_i}\Bigr) f + \Bigl(\prod_{i=1}^p \overline{b_i}^{m_i}\Bigr)\overline{\theta_0'} a g \in H^2 \quad \text{and} \quad \Bigl(\prod_{i=1}^p \overline{b_i}^{n_i}\Bigr)\overline{\theta_1'} b f + \Bigl(\prod_{i=1}^p \overline{b_i}^{q_i}\Bigr) g \in H^2. \tag{17}$$

We split the proof into two cases.

Case 1 ($0 \leq m_{i_0} + n_{i_0} < 2q_{i_0}$ for some $i_0 = 1, 2, \cdots, d$): In this case, $n_{i_0} < q_{i_0}$ or $m_{i_0} < q_{i_0}$. Suppose that $m_{i_0} < q_{i_0}$. Then by the first statement of (17) we have

$$\Bigl(\prod_{i \neq i_0} \overline{b_i}^{q_i - m_i}\Bigr) \overline{b_{i_0}}^{q_{i_0} - m_{i_0}} \theta_0' f \in H^2,$$

which implies that $f = b_{i_0}^{q_{i_0}-m_{i_0}} f_1$ for some $f_1 \in H^2$. In turn, by the second statement of (17) we have

$$\Bigl(\prod_{i\neq i_0} \overline{b_i}^{n_i}\Bigr)\overline{b_{i_0}}^{m_{i_0}+n_{i_0}-q_{i_0}}\overline{\theta_1'}bf_1 + \Bigl(\prod_{i=1}^{p} \overline{b_i}^{q_i}\Bigr)g \in H^2.$$

Thus if $m_{i_0}+n_{i_0}-q_{i_0} \le 0$, then $g = b_{i_0}^{q_{i_0}} g_1$ for some $g_1 \in H^2$ and if instead $m_{i_0}+n_{i_0}-q_{i_0} > 0$, then

$$\overline{b_{i_0}}^{2q_{i_0}-m_{i_0}-n_{i_0}}\Bigl(\prod_{i\neq i_0} \overline{b_i}^{q_i-n_i}\Bigr)\theta_1' g \in H^2\,,$$

which implies that $g = b_{i_0}^{2q_{i_0}-m_{i_0}-n_{i_0}} g_2$ for some $g_2 \in H^2$. Therefore $b_{i_0}I_2$ is an inner divisor of Θ.

If instead $n_{i_0} < q_{i_0}$ then the same argument as the above gives that $b_{i_0}I_2$ is an inner divisor of Θ.

Case 2 ($m_{i_0}+n_{i_0} > 2q_{i_0}$ and $m_{i_0}n_{i_0} \neq 0$ for some i_0):

(a) Suppose $m_{i_0} \ge q_{i_0}+1$. If $\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\Phi_-^*}$, then by the first statement of (17) we have

$$\Bigl(\prod_{i\neq i_0} \overline{b_i}^{m_i-q_i}\Bigr)\overline{b_{i_0}}^{m_{i_0}-q_{i_0}}\overline{\theta_0'}ag \in H^2,$$

which implies that $g = b_{i_0}^{m_{i_0}-q_{i_0}} g_1$ for some $g_1 \in H^2$. In turn, by the second statement of (17) we have

$$\Bigl(\prod_{i\neq i_0} \overline{b_i}^{n_i}\Bigr)\overline{b_{i_0}}^{n_{i_0}}\overline{\theta_1'}bf + \Bigl(\prod_{i\neq i_0} \overline{b_i}^{q_i}\Bigr)\overline{b_{i_0}}^{2q_{i_0}-m_{i_0}} g_1 \in H^2.$$

Thus if $2q_{i_0} \le m_{i_0}$, then $f = b_{i_0}^{n_{i_0}} f_1$ for some $f_1 \in H^2$ and if instead $2q_{i_0} > m_{i_0}$, then

$$\Bigl(\prod_{i\neq i_0} \overline{b_i}^{n_i-q_i}\Bigr)\overline{b_{i_0}}^{m_{i_0}+n_{i_0}-2q_{i_0}}\overline{\theta_1'}bf \in H^2\,,$$

which implies that $f = b_{i_0}^{m_{i_0}+n_{i_0}-2q_{i_0}} f_2$ for some $f_2 \in H^2$. Therefore $b_{i_0}I_2$ is an inner divisor of Θ.

(b) Suppose $m_{i_0} < q_{i_0}+1$. Then $n_{i_0} \ge q_{i_0}+1$ and the same argument as the Case 2(a) gives that $b_{i_0}I_2$ is an inner divisor of Θ.

From Case 1 and Case 2, we can conclude that Θ has an inner divisor of the form b_iI_2 for some $i = 1, 2, \cdots, p$ except the cases $m_i + n_i = 2q_i$ for all $i = 1, 2, \cdots, p$ and $m_{i_0}+n_{i_0} > 2q_{i_0}$ with $m_{i_0}n_{i_0} = 0$ for some i_0. This completes the proof. $\square$

Lemma 2.5. Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be a matrix-valued rational function such that

$$\Phi_- := \begin{bmatrix} \omega & \varphi_- \\ \psi_- & \omega \end{bmatrix},$$

where ω is a finite Blaschke product of the form

$$\omega = \prod_{i=1}^{p} b_i^{q_i} \quad \Bigl(b_i(z) := \frac{z-\alpha_i}{1-\overline{\alpha_i}z},\ q_i \ge 1\Bigr),$$

$$\varphi_- \equiv \theta_0\overline{a} = \Bigl(\prod_{i=1}^{p} b_i^{m_i}\Bigr)\theta_0'\overline{a} \quad \text{and} \quad \psi_- \equiv \theta_1\overline{b} = \Bigl(\prod_{i=1}^{p} b_i^{n_i}\Bigr)\theta_1'\overline{b} \quad \text{(coprime factorizations)}$$

($m_i, n_i = 0, 1, \cdots$ and $(\theta_0'\theta_1')(\alpha_i) \neq 0$ for all $i = 1, 2, \cdots, p$). If $\alpha_{i_0} = 0$, $m_{i_0} > 2q_{i_0}$ and $n_{i_0} = 0$ for some i_0, then

$$\ker H_{\Phi_-^*} \subseteq \frac{1}{\sqrt{|\alpha|^2+1}}\begin{bmatrix} z^{m_{i_0}-q_{i_0}}\theta_0' & -\overline{\alpha}z^{m_{i_0}-q_{i_0}+1}\theta_0' \\ \alpha\theta_1' & z\theta_1' \end{bmatrix} H^2_{\mathbb{C}^2} \quad \Bigl(\alpha := -\frac{a'(0)}{\theta_1''(0)}\Bigr),$$

where

$$a' := \Big(\prod_{i\neq i_0} b_i^{M_i-m_i}\Big)a \quad \text{and} \quad \theta_1'' := \Big(\prod_{i\neq i_0} b_i^{M_i-q_i}\Big)\theta_1'$$

$(M_i := \max(m_i, q_i)$ for $i \neq i_0)$.

Proof. Observe that for $f, g \in H^2$,

$$\Phi_-^* \begin{bmatrix} f \\ g \end{bmatrix} \in H^2_{\mathbb{C}^2} \iff \begin{bmatrix} \overline{\omega} & \overline{\theta_1 b} \\ \overline{\theta_0 a} & \overline{\omega} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \in H^2_{\mathbb{C}^2}\,,$$

which implies that if $\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\Phi_-^*}$, then

$$\Big(\prod_{i=1}^p \overline{b_i}^{q_i}\Big) f + \Big(\prod_{i=1}^p \overline{b_i}^{n_i}\Big)\overline{\theta_1'} b g \in H^2 \quad \text{and} \quad \Big(\prod_{i=1}^p \overline{b_i}^{m_i}\Big)\overline{\theta_0'} a f + \Big(\prod_{i=1}^p \overline{b_i}^{q_i}\Big) g \in H^2. \tag{18}$$

It follows from the first statement of (18) that $g = \theta_1' g_1$ for some $g_1 \in H^2$. In turn

$$\Big(\prod_{i=1}^p \overline{b_i}^{q_i}\Big) f + \Big(\prod_{i=1}^p \overline{b_i}^{n_i}\Big) b g_1 \in H^2$$

Since $n_{i_0} = 0$, we have $f = z^{q_{i_0}} f_1$ for some $f_1 \in H^2$. Thus, by the second statement of (18) we have

$$\Big(\prod_{i\neq i_0} \overline{b_i}^{m_i}\Big)\overline{z}^{m_{i_0}-q_{i_0}}\overline{\theta_0'} a f_1 + \Big(\prod_{i=1}^p \overline{b_i}^{q_i}\Big) g \in H^2\,, \tag{19}$$

so that

$$\Big(\prod_{i\neq i_0} \overline{b_i}^{m_i-q_i}\Big)\overline{z}^{m_{i_0}-2q_{i_0}}\overline{\theta_0'} a f_1 \in H^2.$$

Since $m_{i_0} > 2q_{i_0}$, it follows that $f_1 = \theta_0' z^{m_{i_0}-2q_{i_0}} f_2$ for some $f_2 \in H^2$. Thus, by (19) we have

$$\Big(\prod_{i\neq i_0} \overline{b_i}^{m_i}\Big)\overline{z}^{q_{i_0}} a f_2 + \Big(\prod_{i=1}^p \overline{b_i}^{q_i}\Big)\theta_1' g_1 \in H^2. \tag{20}$$

Then it follows from (20) that

$$\Big(\prod_{i\neq i_0} b_i^{M_i-m_i}\Big) a f_2 + \Big(\prod_{i\neq i_0} b_i^{M_i-q_i}\Big)\theta_1' g_1 \in z^{q_{i_0}} H^2. \tag{21}$$

Write

$$a' := \Big(\prod_{i\neq i_0} b_i^{M_i-m_i}\Big)a \quad \text{and} \quad \theta_1'' := \Big(\prod_{i\neq i_0} b_i^{M_i-q_i}\Big)\theta_1'.$$

Then we have $a'(0) \neq 0$ and $\theta_1''(0) \neq 0$, and by (21) we have

$$g_1(0) = \alpha f_2(0) \quad \Big(\alpha := -\frac{a'(0)}{\theta_1''(0)}\Big).$$

Therefore, we have

$$\begin{bmatrix} f \\ g \end{bmatrix} \in \ker H_{\Phi_-^*} \Longrightarrow f = z^{m_{i_0}-q_{i_0}}\theta_0' f_2,\ g = \theta_1' g_1,\ \text{and}\ g_1(0) = \alpha f_2(0). \tag{22}$$

Put

$$\Omega := \frac{1}{\sqrt{|\alpha|^2+1}} \begin{bmatrix} z^{m_{i_0}-q_{i_0}}\theta_0' & -\overline{\alpha} z^{m_{i_0}-q_{i_0}+1}\theta_0' \\ \alpha\theta_1' & z\theta_1' \end{bmatrix}.$$

Then Ω is inner, and for $h_1, h_2 \in H^2$,

$$\Omega \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{\sqrt{|\alpha|^2+1}} \begin{bmatrix} z^{m_{i_0}-q_{i_0}}\theta_0' h_1 - \overline{\alpha} z^{m_{i_0}-q_{i_0}+1}\theta_0' h_2 \\ \alpha\theta_1' h_1 + z\theta_1' h_2 \end{bmatrix}$$
$$= \frac{1}{\sqrt{|\alpha|^2+1}} \begin{bmatrix} z^{m_{i_0}-q_{i_0}}\theta_0' \big(h_1 - \overline{\alpha} z h_2\big) \\ \theta_1' \big(\alpha h_1 + z h_2\big) \end{bmatrix}.$$

Since $(\alpha h_1 + z h_2)(0) = \alpha h_1(0) = \alpha (h_1 - \overline{\alpha} z h_2)(0)$, it follows from (22) that

$$\ker H_{\Phi_-^*} \subseteq \Omega H^2_{\mathbb{C}^2},$$

which gives the result. □

To answer Problem A, we recall ([CHL2, Lemma 3.2]) that if $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type, we may write, as in (9),

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta B^* \quad \text{(right coprime factorizations)}.$$

If T_Φ is hyponormal, then

$$\Theta_1 = \Theta\Theta_0 \quad \text{for some inner matrix function } \Theta_0; \tag{23}$$

in other words, Θ is a left inner divisor of Θ_1.

We are ready for:

Theorem 2.6. Let $\varphi, \psi \in L^\infty$ be rational functions and consider

$$G := \begin{bmatrix} T_{\overline{\omega_1}} & T_\varphi \\ T_\psi & T_{\overline{\omega_2}} \end{bmatrix} \quad (\omega_i \text{ is a finite Blaschke product for } i = 1, 2)\,. \tag{24}$$

Then the following statements are equivalent:

1. G is normal;
2. G is subnormal;
3. G is 2-hyponormal;
4. G is hyponormal and $\ker\,[G^*, G]$ is invariant for G;
5. $\omega_1 = \omega_2 =: \omega$ and the following condition holds:

$$\varphi = e^{i\delta_1}\omega + \zeta \quad \text{and} \quad \psi = e^{i\delta_2}\varphi \quad (\zeta \in \mathbb{C};\ \delta_1, \delta_2 \in [0, 2\pi))\,, \tag{25}$$

except in the following case:

$$m_i + n_i = 2q_i \ \text{ for some } i = 1, 2, \cdots, p\,, \tag{26}$$

in the representation

$$\omega := \prod_{i=1}^p b_i^{q_i} \quad \Big(b_i(z) := \frac{z - \alpha_i}{1 - \overline{\alpha_i} z},\ q_i \ge 1\Big)\,,$$

$$\varphi_- \equiv \theta_0 \overline{a} = \Big(\prod_{i=1}^p b_i^{m_i}\Big)\theta_0' \overline{a} \quad \text{and} \quad \psi_- \equiv \theta_1 \overline{b} = \Big(\prod_{i=1}^p b_i^{n_i}\Big)\theta_1' \overline{b} \quad \text{(coprime factorizations)}$$

($m_i, n_i = 0, 1, \cdots$ and $(\theta_0'\theta_1')(\alpha_i) \neq 0$ for all $i = 1, 2, \cdots, p$).

Proof. Clearly, (1) ⇒ (2) and (2) ⇒ (3). Also (3) ⇒ (4) is evident because $\ker\,[T^*, T]$ is invariant under T for every 2-hyponormal operator $T \in \mathcal{B}(\mathcal{H})$ (cf. [CL2]). Moreover, (5) ⇒ (1) follows from a straightforward calculation.

(4) ⇒ (5): By Lemma 2.2, $\omega_1 = \omega_2 =: \omega$. Thus we may write

$$\Phi \equiv \begin{bmatrix} \overline{\omega} & \varphi \\ \psi & \overline{\omega} \end{bmatrix} \equiv \Phi_-^* + \Phi_+ = \begin{bmatrix} \omega & \psi_- \\ \varphi_- & \omega \end{bmatrix}^* + \begin{bmatrix} 0 & \varphi_+ \\ \psi_+ & 0 \end{bmatrix}$$

and assume that T_Φ is hyponormal and $\ker [T_\Phi, T_\Phi^*]$ is invariant for T_Φ. Since, by Theorem 1.2, Φ is normal, we have

$$|\varphi| = |\psi|, \tag{27}$$

and also there exists a function $K \equiv \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right] \in H^\infty_{M_2}$ such that $\Phi_-^* - K\Phi_+^* \in H^2_{M_2}$, i.e.,

$$\begin{bmatrix} \overline{\omega} & \overline{\varphi_-} \\ \overline{\psi_-} & \overline{\omega} \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} 0 & \overline{\psi_+} \\ \overline{\varphi_+} & 0 \end{bmatrix} \in H^2_{M_2},$$

which implies that φ_+ and ψ_+ are not identically zero and hence $\det \Phi_+$ is not identically zero.

We now split the proof into three cases.

Case 1 ($m_{i_0} = n_{i_0} = 0$ for some i_0): In this case, by Lemma 2.4 and Theorem 1.4, we can conclude that T_Φ is normal. Since $\det \Phi_+$ is not identically zero, it follows from Theorem 1.3 that $\Phi_+ - \Phi_- U \in M_n(\mathbb{C})$ for some constant unitary matrix $U \equiv \left[\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right]$. We observe

$$\begin{aligned} \Phi_+ - \Phi_- U \in M_n(\mathbb{C}) &\Longleftrightarrow \begin{bmatrix} 0 & \varphi_+ \\ \psi_+ & 0 \end{bmatrix} - \begin{bmatrix} \omega & \theta_1 \overline{b} \\ \theta_0 \overline{a} & \omega \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \in M_n(\mathbb{C}) \\ &\Longrightarrow \begin{cases} c_1\omega + c_3\theta_1\overline{b} = \xi_1 \\ c_4\omega + c_2\theta_0\overline{a} = \xi_2 \\ \varphi_+ = c_2\omega + c_4\theta_1\overline{b} + \xi_3 \\ \psi_+ = c_3\omega + c_1\theta_0\overline{a} + \xi_4 \end{cases} \qquad (\xi_i \in \mathbb{C} \text{ for } i = 1, \cdots, 4), \end{aligned} \tag{28}$$

which gives

$$\overline{c_1} H_{\overline{\omega}} = -\overline{c_3} H_{\overline{\theta_1 b}} \quad \text{and} \quad \overline{c_4} H_{\overline{\omega}} = -\overline{c_2} H_{\overline{\theta_0 a}}. \tag{29}$$

Thus if $c_1 \neq 0$ then $c_3 \neq 0$ and hence $\omega = \theta_1$, which is a contradiction because $\omega(\alpha_{i_0}) = 0$, but $\theta_1(\alpha_{i_0}) \neq 0$. Thus $c_1 = 0$ and similarly, $c_4 = 0$. Since U is unitary, it follows that $|c_2| = |c_3| = 1$, and hence $\theta_1 \overline{b}$ and $\theta_0 \overline{a}$ are constants. Thus, again by (28), we have

$$\varphi = \varphi_+ = e^{i\delta_1}\omega + \beta_1 \quad \text{and} \quad \psi = \psi_+ = e^{i\delta_2}\omega + \beta_2 \quad (\delta_1, \delta_2 \in [0, 2\pi);\ \beta_1, \beta_2 \in \mathbb{C}).$$

Since $|\varphi| = |\psi|$, it follows that

$$\varphi = e^{i\delta_1}\omega + \zeta \quad \text{and} \quad \psi = e^{i\delta_2}\varphi \quad (\delta_1, \delta_2 \in [0, 2\pi);\ \zeta \in \mathbb{C})).$$

Case 2 $\big($(i) $0 < m_{i_0} + n_{i_0} < 2q_{i_0}$; or (ii) $m_{i_0} + n_{i_0} > 2q_{i_0}$ ($m_{i_0} n_{i_0} \neq 0$) for some $i_0\big)$: In this case, by Lemma 2.4 and Theorem 1.4, we can conclude that T_Φ is normal. By case assumption, we have $m_{i_0} \neq q_{i_0}$ or $n_{i_0} \neq q_{i_0}$. Suppose that $m_{i_0} \neq q_{i_0}$. Then $\omega \neq \theta_0$, and hence by (29) we have $c_2 = c_4 = 0$. Therefore U is not unitary, a contradiction.

If instead $n_{i_0} \neq q_{i_0}$ then the same argument as above gives that U is not unitary, a contradiction. Thus this case cannot occur.

Case 3 ($m_i + n_i > 2q_i$ ($m_i n_i = 0$) for all $i = 1, \cdots, p$): Fix i_0 ($1 \leq i_o \leq p$); we may, without loss of generality, assume that $n_{i_0} = 0$ (and hence, $m_{i_0} > 2q_{i_0}$). By Lemma 2.1, we may also assume that $b_{i_0} = z$. It follows from Theorem 1.2 that there exists a matrix function $K \equiv \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right] \in \mathcal{E}(\Phi)$, so that

$$\begin{bmatrix} \overline{\omega} & \overline{\varphi_-} \\ \overline{\psi_-} & \overline{\omega} \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} 0 & \overline{\psi_+} \\ \overline{\varphi_+} & 0 \end{bmatrix} \in H^2_{M_2},$$

which implies that

$$\begin{cases} \overline{\omega} - k_2\overline{\varphi_+} \in H^2, & \overline{\theta}_1 b - k_4\overline{\varphi_+} \in H^2 \\ \overline{\omega} - k_3\overline{\psi_+} \in H^2, & \overline{\theta}_0 a - k_1\overline{\psi_+} \in H^2. \end{cases} \tag{30}$$

Since $||K||_\infty \leq 1$ and hence $||k_i||_\infty \leq 1$ for each $i = 1, \cdots, 4$, the following Toeplitz operators are all hyponormal (by Cowen's Theorem):

$$T_{\overline{\omega} + \varphi_+}, \quad T_{\overline{\theta}_1 b + \varphi_+}, \quad T_{\overline{\omega} + \psi_+}, \quad T_{\overline{\theta}_0 a + \psi_+}. \tag{31}$$

Put $M_i := \max(m_i, q_i)$ and $N_i := \max(n_i, q_i)$. Then by (31) and a scalar-valued version of (23), we can see that

$$\varphi_+ = z^{q_{i_0}} \prod_{i \neq i_0} b_i^{N_i} \theta_1' \theta_3 \overline{d} \quad \text{and} \quad \psi_+ = z^{m_{i_0}} \prod_{i \neq i_0} b_i^{M_i} \theta_0' \theta_2 \overline{c} \qquad \text{(coprime factorizations)},$$

where θ_2 and θ_3 are finite Blaschke products. Thus, in particular, $c(0) \neq 0$ and $d(0) \neq 0$. Thus, by (30), we can see that

$$k_3(0) = 0 \quad \text{and} \quad k_4(0) = 0: \tag{32}$$

indeed, in (30),

$$\begin{aligned} \overline{\omega} - k_3 \overline{\psi_+} \in H^2 &\Longrightarrow \overline{z}^{q_{i_0}} \prod_{i \neq i_0} \overline{b_i}^{q_i} - k_3 \overline{z}^{m_{i_0}} \prod_{i \neq i_0} \overline{b_i}^{M_i} \overline{\theta_0'} \overline{\theta_2} c \in H^2 \\ &\Longrightarrow z^{m_{i_0} - q_{i_0}} \prod_{i \neq i_0} b_i^{M_i - q_i} \theta_0' \theta_2 - k_3 c \in z^{m_{i_0}} H^2 \\ &\Longrightarrow k_3(0) = 0 \quad (\text{since } m_{i_0} > 2q_{i_0}) \end{aligned}$$

and

$$\begin{aligned} \overline{\theta}_1 b - k_4 \overline{\varphi_+} \in H^2 &\Longrightarrow \Big(\prod_{i=1}^{d} \overline{b_i}^{n_i} \Big) \overline{\theta_1'} b - k_4 \overline{z}^{q_{i_0}} \prod_{i \neq i_0} \overline{b_i}^{N_i} \overline{\theta_1'} \overline{\theta_3} d \in H^2 \\ &\Longrightarrow z^{q_{i_0}} \Big(\prod_{i \neq i_0} b_i^{N_i - n_i} \Big) \theta_3 b - k_4 d \in z^{q_{i_0}} H^2 \\ &\Longrightarrow k_4(0) = 0\,, \end{aligned}$$

which proves (32). Write

$$\theta_2 = z^{l_2} \theta_2' \quad \text{and} \quad \theta_3 = z^{l_3} \theta_3' \quad (\theta_2'(0) \neq 0,\ \theta_3'(0) \neq 0).$$

Then we can write

$$\Phi_+ = \begin{bmatrix} 0 & z^{q_{i_0} + l_3} \prod_{i \neq i_0} b_i^{N_i} \theta_1' \theta_3' \overline{d} \\ z^{m_{i_0} + l_2} \prod_{i \neq i_0} b_i^{M_i} \theta_0' \theta_2' \overline{c} & 0 \end{bmatrix}.$$

On the other hand, write

$$a' := \Big(\prod_{i \neq i_0} b_i^{M_i - m_i} \Big) a, \quad \theta_1'' := \Big(\prod_{i \neq i_0} b_i^{M_i - q_i} \Big) \theta_1'$$

and

$$\alpha := -\frac{a'(0)}{\theta_1''(0)} \quad \text{and} \quad \nu := \frac{1}{\sqrt{|\alpha|^2 + 1}}.$$

Note that

$$\widetilde{\Phi_-} = \begin{bmatrix} \widetilde{\omega} & \widetilde{\theta}_0 \overline{\widetilde{a}} \\ \widetilde{\theta}_1 \overline{\widetilde{b}} & \widetilde{\omega} \end{bmatrix}.$$

Since $\widetilde{\Phi_-^*}$ is of bounded type, it follows from Theorem 1.1 that there exists a square inner matrix function Δ such that $\ker H_{\widetilde{\Phi_-^*}} = \widetilde{\Delta} H^2_{\mathbb{C}^2}$ and

$$\widetilde{\Phi_-^*} = \widetilde{B} \widetilde{\Delta}^* \quad \text{(right coprime factorization)}.$$

Thus, by Lemma 2.5 we have

$$\ker H_{\widetilde{\Phi_-^*}} = \widetilde{\Delta} H^2_{\mathbb{C}_2} \subseteq \widetilde{\Omega} H^2_{\mathbb{C}^2} \quad \text{and} \quad \Phi_- = B^* \Delta \quad \text{(left coprime factorization)}\,, \tag{33}$$

where

$$\Omega = \nu \begin{bmatrix} z^{m_{i_0} - q_{i_0}} \theta_0' & \alpha \theta_1' \\ -\overline{\alpha} z^{m_{i_0} - q_{i_0} + 1} \theta_0' & z \theta_1' \end{bmatrix}.$$

Since $\widetilde{\Delta}H^2_{\mathbb{C}_2} \subseteq \widetilde{\Omega}H^2_{\mathbb{C}^2}$, it follows that $\widetilde{\Omega}$ is a left inner divisor of $\widetilde{\Delta}$. Thus, we can write

$$\widetilde{\Delta} = \widetilde{\Omega}\widetilde{\Omega}_1 \quad \text{for some } \Omega_1, \quad \text{so that} \quad \Delta = \Omega_1\Omega.$$

We suppose that $q_{i_0} + l_3 \le m_{i_0} + l_2$ and write $r := (m_{i_0} + l_2) - (q_{i_0} + l_3) \ge 0$. Then there exist finite Blaschke products θ_4 and θ_5 with $\theta_i(0) \ne 0$ $(i = 4, 5)$ such that

$$\Phi_+ = \prod_{i\ne i_0} b_i^{\max(M_i,N_i)} (z^{m_{i_0}+l_2}\theta_1'\theta_3'\theta_0'\theta_2')I_2 \begin{bmatrix} 0 & \theta_5\theta_1'\theta_3'c \\ z^r\theta_4\theta_0'\theta_2'd & 0 \end{bmatrix}^* \equiv (\theta I_2)A^*,$$

where $\theta := \prod_{i\ne i_0} b_i^{\max(M_i,N_i)} (z^{m_{i_0}+l_2}\theta_1'\theta_3'\theta_0'\theta_2')$. Since $H_{A\Delta^*} = H_{A\Omega^*\Omega_1^*}$, it follows that

$$\mathrm{ran} H_{A\Delta^*} \supseteq \mathrm{ran} H_{A\Omega^*}. \tag{34}$$

Observe that

$$\begin{aligned} A\Omega^* &= \nu \begin{bmatrix} 0 & \theta_5\theta_1'\theta_3'c \\ z^r\theta_4\theta_0'\theta_2'd & 0 \end{bmatrix} \begin{bmatrix} z^{m_{i_0}-q_{i_0}}\theta_0' & \alpha\theta_1' \\ -\overline{\alpha}z^{m_{i_0}-q_{i_0}+1}\theta_0' & z\theta_1' \end{bmatrix}^* \\ &= \nu \begin{bmatrix} \overline{\alpha}\theta_5\theta_3'c & \overline{z}\theta_5\theta_3'c \\ z^{r-m_{i_0}+q_{i_0}}\theta_4\theta_2'd & -\alpha z^{r-m_{i_0}+q_{i_0}-1}\theta_4\theta_2'd \end{bmatrix}. \end{aligned}$$

If $r \le m_{i_0} - q_{i_0}$, then we have

$$H_{A\Omega^*} \begin{bmatrix} 0 \\ z^{m_{i_0}-q_{i_0}-r} \end{bmatrix} = \nu \begin{bmatrix} H_{\overline{z}}(z^{m_{i_0}-q_{i_0}-r}\theta_5\theta_3'c) \\ -\alpha H_{\overline{z}}(\theta_4\theta_2'd) \end{bmatrix}.$$

Since $(\theta_4\theta_2'd)(0) \ne 0$, it follows from Lemma 2.3, (33) and (34) that

$$\begin{bmatrix} \beta \\ 1 \end{bmatrix} \in \mathrm{cl\ ran}\, H_{A\Omega^*} \subseteq \mathrm{cl\ ran}\, H_{A\Delta^*} \subseteq \ker(I - T_{\widetilde{K}}T^*_{\widetilde{K}}) \quad \text{for some } \beta \in \mathbb{C}. \tag{35}$$

It thus follows from (32) and (35) that

$$\begin{aligned} \begin{bmatrix} \beta \\ 1 \end{bmatrix} = T_{\widetilde{K}}T^*_{\widetilde{K}} \begin{bmatrix} \beta \\ 1 \end{bmatrix} &= \begin{bmatrix} T_{\widetilde{k}_1} & T_{\widetilde{k}_3} \\ T_{\widetilde{k}_2} & T_{\widetilde{k}_4} \end{bmatrix} \begin{bmatrix} T_{\overline{\widetilde{k}}_1} & T_{\overline{\widetilde{k}}_2} \\ T_{\overline{\widetilde{k}}_3} & T_{\overline{\widetilde{k}}_4} \end{bmatrix} \begin{bmatrix} \beta \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} T_{\widetilde{k}_1} & T_{\widetilde{k}_3} \\ T_{\widetilde{k}_2} & T_{\widetilde{k}_4} \end{bmatrix} \begin{bmatrix} (\beta k_1(0) + k_2(0)) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{k}_1(\beta k_1(0) + k_2(0)) \\ \widetilde{k}_2(\beta k_1(0) + k_2(0)) \end{bmatrix}, \end{aligned}$$

which implies that k_1 is a constant and k_2 is a nonzero constant. Again by (30),

$$\begin{aligned} \overline{\omega} - k_2\overline{\varphi_+} \in H^2 &\Longrightarrow \omega\overline{z}^{q_{i_0}} \prod_{i\ne i_0} \overline{b_i}^{N_i}\overline{\theta_1'\theta_3}d \in H^2 \\ &\Longrightarrow q_i \ge n_i \ (i \ne i_0) \text{ and } \overline{\theta_1'\theta_3}d \in H^2 \\ &\Longrightarrow n_i = 0 \ (i \ne i_0) \text{ and } \theta_1'\theta_3 = 1, \end{aligned} \tag{36}$$

where the last implication follows from the observation that if $n_i \ne 0$ then by the case assumption, $m_i = 0$ and hence, $2q_i < n_i \le q_i$, a contradiction. We thus have $n_i = 0$ for all $i = 1, \cdots, p$. Since $\theta_1' = 1$, it follows that

$$\theta_1 = 1 \text{ and hence, } \psi_- = 0.$$

In turn, $m_i > 2q_i$ for all $i = 1, \cdots, p$, so that θ_0 is nonconstant, and hence $\varphi_- = \theta_0\overline{a} \neq 0$. Since by (30), $\overline{\theta_0}a - k_1\overline{\psi_+} \in H^2$, it follows that $k_1 \neq 0$. We thus have

$$\begin{aligned}\overline{\theta_0}a - k_1\overline{\psi_+} \in H^2 &\Longrightarrow \overline{\theta_0}a - k_1\overline{z}^{m_{i_0}} \prod_{i\neq i_0} \overline{b_i}^{M_i}\overline{\theta_0'\theta_2 c} \in H^2\\ &\Longrightarrow \prod_{i=1}^{p} \overline{b_i}^{m_i} a - k_1\overline{z}^{m_{i_0}} \prod_{i\neq i_0} \overline{b_i}^{M_i}\overline{\theta_2 c} \in H^2 \qquad (37)\\ &\Longrightarrow \overline{\theta_2 c} \in H^2 \quad (\text{because } m_i > 2q_i \text{ and hence, } M_i = m_i)\\ &\Longrightarrow \theta_2 = 1\,.\end{aligned}$$

Therefore, we have

$$\varphi_+ = z^q \prod_{i\neq i_0} b_i^{N_i}\theta_1'\theta_3\overline{d} = \omega\overline{d} \quad (q := q_{i_0}) \quad \text{and} \quad \psi_+ = z^{m_{i_0}} \prod_{i\neq i_0} b_i^{M_i}\theta_0'\theta_2\overline{c} = \theta_0\overline{c}.$$

Since by (27), $|\varphi| = |\psi|$, we have

$$|\omega\overline{d} + \overline{\theta_0}a| = |\varphi_+ + \overline{\varphi_-}| = |\psi_+| = |\theta_0\overline{c}| \quad (\text{where } a \in \mathcal{H}_{\theta_0},\ d \in \mathcal{H}_{z\omega},\ c \in \mathcal{H}_{z\theta_0})\,,$$

which implies

$$\omega\theta_0(\omega\overline{d} + \overline{\theta_0}a)(\overline{\omega}d + \theta_0\overline{a}) = \omega\theta_0 c\overline{c}\,,$$

so that

$$ad = z\Big((\theta_0\overline{c})(\overline{z}\omega)c - (\theta_0\overline{d})(\overline{z}\omega)d - (\theta_0\overline{a})(\theta_0\overline{d})(\overline{z}\omega^2) - (\theta_0\overline{a})(\overline{z}\omega)a\Big)\,. \qquad (38)$$

Since $a \in \mathcal{H}_{\theta_0}$, $c \in \mathcal{H}_{z\theta_0}$, $d \in \mathcal{H}_{z\omega}$ and $m_i \geq 2q_i$ for all $i = 1, \cdots, p$, it follows that $\theta_0\overline{a} \in H^2$, $\theta_0\overline{c} \in H^2$ and $\theta_0\overline{d} = (\prod_{i=1}^p b_i^{m_i})\,\theta_0'\overline{d} = \big(\prod_{i=1}^p b_i^{m_i-q_i}\theta_0'\big)\,(\omega\overline{d}) \in H^2$. Thus, (38) implies that $ad = zh$ for some $h \in H^2$, and hence $(ad)(0) = 0$, a contradiction. Therefore this case cannot occur.

If instead $r > m_{i_0} - q_{i_0}$, then the same argument as before leads to a contradiction. Moreover, by the same argument as in the case $q_{i_0} + l_3 \leq m_{i_0} + l_2$, the case $q_{i_0} + l_3 > m_{i_0} + l_2$ cannot occur either.

Therefore, Case 3 cannot occur. This proves the implication (4) $\Rightarrow$ (5).

This completes the proof. □

Remark 2.7. From the proof of Theorem 2.6 we can see that if G is given by (24) then G is subnormal if and only if G is normal, except in the case (26). However we need not expect that the exceptional case (26) implies normality of G. For example, if

$$\Phi := \begin{bmatrix} \overline{\omega} & \overline{\omega} + 2\omega \\ \overline{\omega} + 2\omega & \overline{\omega} \end{bmatrix} \quad (\omega \text{ is a finite Blaschke product})$$

then T_Φ satisfies the case (26) (where $m_i = n_i = q_i$ and $a = b = \theta_0' = \theta_1' = 1$). A straightforward calculation shows that T_Φ is not normal. Since

$$T_\Phi = \begin{bmatrix} T_{\overline{\omega}} & T_{\overline{\omega}} + 2T_\omega \\ T_{\overline{\omega}} + 2T_\omega & T_{\overline{\omega}} \end{bmatrix}\,,$$

it follows that if $W := \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$, then W is unitary and

$$W^*T_\Phi W = 2\begin{bmatrix} T_{\overline{\omega}} + T_\omega & 0 \\ 0 & -T_\omega \end{bmatrix}\,,$$

which says that T_Φ is unitarily equivalent to a direct sum of the normal operator $2(T_{\overline{\omega}} + T_\omega)$ and the analytic Toeplitz operator $-2T_\omega$. From this viewpoint, we might conjecture that every subnormal rational Toeplitz operator is unitarily equivalent to a direct sum of a normal operator and an analytic Toeplitz operator. However we have been unable to settle this conjecture.

Acknowledgment. The authors are deeply indebted to the referee for many helpful comments that helped improved the presentation and mathematical content of the paper.

References

[Ab] M.B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43** (1976), 597–604.

[AIW] I. Amemiya, T. Ito, and T.K. Wong, *On quasinormal Toeplitz operators*, Proc. Amer. Math. Soc. **50**(1975), 254–258.

[At] A. Athavale *On joint hyponormality of operators*, Proc. Amer. Math. Soc. **103**(1988), 417–423.

[Br] J. Bram, *Subnormal operators*, Duke Math. J. **22**(1955), 75–94.

[BH] A. Brown and P.R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213**(1963/1964), 89–102.

[Con] J.B. Conway, The Theory of Subnormal Operators, *Math. Surveys and Monographs*, **36**(1991), Amer. Math. Soc. Providence, Rhodie Island.

[CoS] J. B. Conway and W. Szymanski, *Linear combination of hyponormal operators*, Rocky Mountain J. Math. **18**(1988), 695–705.

[Co1] C.C. Cowen, *On equivalence of Toeplitz operators*, J. Operator Theory **7**(1982), 167–172.

[Co2] C.C. Cowen, *More subnormal Toeplitz operators*, J. Reine Angew. Math. **367**(1986), 215–219.

[Co3] C.C. Cowen, *Hyponormal and subnormal Toeplitz operators*, Surveys of Some Recent Results in Operator Theory, I (J.B. Conway and B.B. Morrel, eds.), Pitman Research Notes in Mathematics, Volume 171, Longman, 1988, pp. (155–167).

[Co4] C.C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103**(1988), 809–812.

[CoL] C.C. Cowen and J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351**(1984), 216–220.

[Cu1] R.E. Curto, *Fredholm and invertible n-tuples of operators. The deformation problem*, Trans. Amer. Math. Soc. **266**(1981), 129–159

[Cu2] R.E. Curto, *Quadratically hyponormal weighted shifts*, Integral Equations Operator Theory, **13**(1990), 49–66.

[CuF1] R.E. Curto and L.A. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. **17**(1991), 603–635.

[CuF2] R.E. Curto and L.A. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, Integral Equations Operator Theory, **17**(1993), 202–246.

[CuF3] R.E. Curto and L.A. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem II*, Integral Equations Operator Theory, **18**(1994), 369–426.

[CHKL] R.E. Curto, I.S. Hwang, D. Kang and W.Y. Lee, *Subnormal and quasinormal Toeplitz operator with matrix-valued rational symbols*, (preprint 2013).

[CHL1] R.E. Curto, I.S. Hwang and W.Y. Lee, *Hyponormality and subnormality of block Toeplitz operators*, Adv. Math. **230**(2012), 2094-2151.

[CHL2] R.E. Curto, I.S. Hwang and W.Y. Lee, *Which subnormal Toeplitz operators are either normal or analytic ?*, J. Funct. Anal. **263(8)**(2012), 2333-2354.

[CHL3] R.E. Curto, I.S. Hwang and W.Y. Lee, *Abrahamse's Theorem for matrix-valued symbols and subnormal Toeplitz completions*, (preprint 2012) (arXiv:1301.6901,2013).

[CLL] R.E. Curto, S.H. Lee and W.Y. Lee, *Subnormality and 2-hyponormality for Toeplitz operators*, Integral Equations Operator Theory, **44**(2002), 138–148.

[CL1] R.E. Curto and W.Y. Lee, *Joint hyponormality of Toeplitz pairs*, Memoirs Amer. Math. Soc. **712**, Amer. Math. Soc., Providence, 2001.

[CL2] R.E. Curto and W.Y. Lee, *Towards a model theory for 2–hyponormal operators*, Integral Equations Operator Theory **44**(2002), 290–315.

[CL3] R.E. Curto and W.Y. Lee, *Subnormality and k-hyponormality of Toeplitz operators: A brief survey and open questions*, Operator Theory and Banach Algebras (Rabat, 1999), 73–81, Theta, Bucharest, 2003.

[CMX] R.E. Curto, P.S. Muhly and J. Xia, *Hyponormal pairs of commuting operators*, Contributions to Operator Theory and Its Applications(Mesa, AZ, 1987) (I. Gohberg, J.W. Helton and L. Rodman, eds.), Operator Theory: Advances and Applications, vol. 35, Birkhäuser, Basel–Boston, 1988, 1–22.

[CP1] R.E. Curto and M. Putinar, *Existence of non-subnormal polynomially hyponormal operators*, Bull. Amer. Math. Soc. (N.S.), **25**(1991), 373–378.

[CP2] R.E. Curto and M. Putinar, *Nearly subnormal operators and moment problems*, J. Funct. Anal. **115**(1993), 480–497.

[DPY] R.G. Douglas, V.I. Paulsen, and K. Yan, *Operator theory and algebraic geometry* Bull. Amer. Math. Soc. (N.S.) **20**(1989), 67–71.

[FL] D.R. Farenick and W.Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348**(1996), 4153–4174.

[Gu1] C. Gu, *A generalization of Cowen's characterization of hyponormal Toeplitz operators*, J. Funct. Anal. **124**(1994), 135–148.

[Gu2] C. Gu, *On a class of jointly hyponormal Toeplitz operators*, Trans. Amer. Math. Soc. **354**(2002), 3275–3298.

[GHR] C. Gu, J. Hendricks and D. Rutherford, *Hyponormality of block Toeplitz operators*, Pacific J. Math. **223**(2006), 95–111.

[GS] C. Gu and J.E. Shapiro, *Kernels of Hankel operators and hyponormality of Toeplitz operators*, Math. Ann. **319**(2001), 553–572.

[Hal1] P.R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76**(1970), 887–933.

[Hal2] P.R. Halmos, *Ten years in Hilbert space*, Integral Equations Operator Theory **2**(1979), 529–564.

[HKL1] I.S. Hwang, I.H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols*, Math. Ann. **313(2)**(1999), 247–261.

[HKL2] I.S. Hwang, I.H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols: An extremal case*, Math. Nach. **231**(2001), 25–38.

[HL1] I.S. Hwang and W.Y. Lee, *Hyponormality of trigonometric Toeplitz operators*, Trans. Amer. Math. Soc. **354**(2002), 2461–2474.

[HL2] I.S. Hwang and W.Y. Lee, *Hyponormality of Toeplitz operators with rational symbols*, Math. Ann. **335** (2006), 405–414.

[HL3] I.S. Hwang and W.Y. Lee, *Hyponormal Toeplitz operators with rational symbols*, J. Operator Theory **56**(2006), 47–58.

[HL4] I.S. Hwang and W.Y. Lee, *Block Toeplitz Operators with rational symbols*, J. Phys. A: Math. Theor. **41(18)**(2008), 185207.

[HL5] I.S. Hwang and W.Y. Lee, *Block Toeplitz Operators with rational symbols (II)*, J. Phys. A: Math. Theor. **41(38)**(2008), 385206.

[Le] W.Y. Lee, *Cowen sets for Toeplitz operators with finite rank selfcommutators*, J. Operator Theory **54(2)**(2005), 301–307.

[ItW] T. Ito and T.K. Wong, *Subnormality and quasinormality of Toeplitz operators*, Proc. Amer. Math. Soc. **34**(1972), 157–164.

[McCP] S. McCullough and V. Paulsen, *A note on joint hyponormality*, Proc. Amer. Math. Soc. **107**(1989), 187–195.

[NT] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338**(1993), 753–769.

[Ni] N.K. Nikolskii, Treatise on the shift operator, *Springer*, New York, 1986.

[Sun1] S. Sun, *On hyponormal weighted shift*, Chinese Ann. Math. Ser. B **5** (1984), no. 1, 101–108. (A Chinese summary appears in Chinese Ann. Math. Ser. A **5** (1984), no. 1, 124.)

[Sun2] S. Sun, *On Toeplitz operators in the θ-class*, Sci. Sinica Ser. A **28** (1985), no. 3, 235–241.

[Sun3] S. Sun, *On hyponormal weighted shift, II*, Chinese Ann. Math. Ser. B **6** (1985), no. 3, 359-361. (A Chinese summary appears in Chinese Ann. Math. Ser. A **6** (1985), no. 4, 516.)

[Zhu] K. Zhu, *Hyponormal Toeplitz operators with polynomial symbols*, Integral Equations Operator Theory **21**(1996), 376–381.

Raúl E. Curto
Department of Mathematics, University of Iowa, Iowa City, IA 52242, U.S.A.
e-mail: `raul-curto@uiowa.edu`

In Sung Hwang
Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
e-mail: `ihwang@skku.edu`

Woo Young Lee
Department of Mathematics, Seoul National University, Seoul 151-742, Korea
e-mail: `wylee@snu.ac.kr`