Elementary properties of ∞-isometries on a Hilbert space

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Abstract

Inspired by recent works on m-isometries for a positive integer m, in this paper we introduce the classes of ∞-isometries and ∞-unitaries on a Hilbert space. We show that an ∞-isometry on a finite dimensional complex Hilbert space H with dimension N is in fact an $(2N-1)$-isometry. We describe the spectra of such operators, study the quasinilpotent perturbations of ∞-isometries and characterize when tensor products of ∞-isometries are also ∞-isometries. As a surpring byproduct, we obtain a generalization of Nagy-Foias-Langer decomposition of a contraction into an unitary and a completely nonunitary contraction.

1 Introduction

Since a systematic study on m-isometries by Agler and Stankus [3], [4] and [5], the theory of m-isometries has been highly developed. The theory for m-isometries on Hilbert spaces has rich connections to Toeplitz operators, classical function theory, and other areas of

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mathematics. In particular, a class of 2-isometries arises from non-stationary stochastic process of Brownian motion (see [4]). The work of Richter [27] and [28] on analytic 2-isometries has a connection with the invariant subspaces of the shift operator on the Dirichlet space, see also related papers [24], [25] and [30] in this direction. On the other hand, the definition of m-isometries depends on the degree m of the polynomial $(yx-1)^m$ in two variables. Thus we may ask what happens if $m \to \infty$. This stimulates a new notion of ∞-isometries. The aim of this paper is to explore elementary properties of ∞-isometries. In particular, we give a complete characterization of ∞-isometries on a finite dimensional Hilbert space.

Let H be a complex Hilbert space and let $B(H)$ denote the set of all bounded linear operators acting on H. If $T \in B(H)$, we write $\sigma(T)$ and $\sigma_{ap}(T)$ for the spectrum and the approximate spectrum of T, respectively. If H_0 is a subspace of H, P_{H_0} is the projection from H onto H_0. An operator $T \in B(H)$ is called an m-isometry for a positive integer m (as in Agler and Stankus [3]) if

$$\begin{aligned}\beta_m(T) := (yx-1)^m|_{y=T^*;\ x=T} &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}y^k x^k|_{y=T^*;\ x=T}\\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k = 0.\end{aligned}$$

Note that

$$\beta_{m+1}(T) = T^*\beta_m(T)T - \beta_m(T). \tag{1}$$

Thus if T is an m-isometry, then T is an n-isometry for all $n \geq m$. We say T is a strict m-isometry if T is an m-isometry but not an $(m-1)$-isometry. We now introduce the class of ∞-isometries.

Definition 1.1 *Let $T \in B(H)$. The operator T is called an ∞-isometry if*

$$\limsup_{m\to\infty} \|\beta_m(T)\|^{1/m} = 0.$$

Also T is called a finite-isometry if T is an m-isometry for some $m \geq 1$.

If H_0 is an invariant subspace of T, then

$$\beta_m(T|H_0) = P_{H_0}\beta_m(T)|H_0 \tag{2}$$

where $T|H_0$ is the restriction of T to H_0. This property is one of the main motivations of hereditary functional calculus as in [1], [2], [3] and [20]. So, if T on H is an ∞-isometry, then $T|H_0$ is also an ∞-isometry.

The first motivation of studying ∞-isometries comes from recent interests in m-isometries on Hilbert spaces, Banach spaces and metric spaces [6], [8], [14], [18], [21], [26], and [31]. The second motivation is that ∞-isometries seem to be natural limits of m-isometries as $m \to \infty$, see Proposition 4.6 below. However, the main motivation is that ∞-isometries enjoy many properties of m-isometries, as we will demonstrate in this paper. This new class of operators also poses some interesting questions and challenges. For example, one of the main tools in [3], [6] and [28] for studying an m-isometry T is the nonnegative covariance operator $\beta_{m-1}(T)$ which is not available in an ∞-isometry. We hope that this study will deepen our understanding of m-isometries.

First we give some examples. Recall a unilateral weighted shift T on l_2 is defined by $Te_j = w_j e_{j+1}, j \geq 0$, where $\{e_j, j \geq 0\}$ is the standard basis of l_2.

Example 1.2 *Let Q be the weighted shift on l_2 with weights $w_j = \frac{1}{j+1}$. Then Q is quasinilpotent and $T = I + Q$ is an ∞-isometry by Theorem 4.4 below. But T is not a finite-isometry. A direct computation of showing $\beta_m(T) \neq 0$ for any $m \geq 1$ seems to be difficult. Note that $\sigma(T) = \{1\}$. Recall that if S is an m-isometry and $\sigma(S)$ consists of a finite number of points, then S is the direct sum of operators of the form $\lambda I + Q_0$, where Q_0 is a nilpotent operator, see Proposition 11 in [17]. Hence if $T = I + Q$ is a finite-isometry, then Q is a nilpotent operator, which is a contradiction.*

Example 1.3 *Let T_n be an $n \times n$ Jordan block*

$$T_n = \begin{bmatrix} \lambda_n & \frac{1}{n} & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{n} \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} = \lambda_n I_n + \frac{1}{n} J_n,$$

where $|\lambda_n| = 1$ and

$$J_n = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then, by a direct calculation (for example, using the first formula in Lemma 4.1 below with $T = \lambda_n I_n$ and $Q = \frac{1}{n} J_n$),

$$\begin{aligned} &\beta_{2n-1}(T_n) = 0 \\ \text{and } &\beta_{2n-2}(T_n) = \binom{2n-2}{n-1} \frac{1}{n^2} \lambda_n^{n-1} T^{*(n-1)} J_n^{*(n-1)} J_n^{(n-1)} \neq 0. \end{aligned}$$

That is, T_n is a strict $(2n-1)$-isometry. Let

$$T = T_1 \oplus T_2 \oplus T_3 \oplus \cdots.$$

Then T is an ∞-isometry but not a finite-isometry. Furthermore, $\sigma(T) = \{\lambda_n : n \geq 1\}^-$. To see this, let

$$S_n = T_1 \oplus \cdots \oplus T_n \oplus \lambda_{n+1} I \oplus \lambda_{n+2} I \oplus \cdots.$$

Then S_n is a strict $(2n-1)$-isometry and $S_nS_l = S_lS_n$ for all $n,l \geq 1$. It is also clear that $S_n \to T$ in operator norm. Therefore, by Proposition 4.6 below, T is an ∞-isometry. Clearly, T is not a finite-isometry since

$$\beta_m(T) = \beta_m(T_1) \oplus \beta_m(T_2) \oplus \beta_m(T_3) \oplus \cdots \neq 0$$

for any $m \geq 1$.

2 The ∞-isometric matrices

In this section, we discuss basic spectral properties of an ∞-isometry. As applications of those properties, we show that if H is a finite dimensional Hilbert space, then $T \in B(H)$ is an ∞-isometry if and only if T is an m-isometry for some $m \geq 1$.

The spectrum of an m-isometry was described in [3] on a Hilbert space and in [6] on a Banach space. Similarly, we have the following generalization. Let D be the open unit disk and ∂D be the unit circle.

Proposition 2.1 *If T is an ∞-isometry, then $\sigma_{ap}(T) \subseteq \partial D$. Therefore, either $\sigma(T) = D^-$ or $\sigma(T) \subseteq \partial D$. In particular, T is left invertible.*

Proof. Let $\lambda \in \sigma_{ap}(T)$ and let $h_i \in H$ be a sequence of unit vectors such that $\|(T-\lambda I)\, h_i\| \to 0$ as $i \to \infty$. It is clear that $\left\|T^k h_i\right\|^2 \to |\lambda|^{2k}$ as $i \to \infty$. Hence

$$\begin{aligned}\langle \beta_m(T)h_i, h_i\rangle &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \left\|T^k h_i\right\|^2 \\ &\to \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} |\lambda|^{2k} = (|\lambda|^2 - 1)^m.\end{aligned}$$

Thus

$$\|\beta_m(T)\| \geq |\langle \beta_m(T)h_i, h_i\rangle| \quad \text{and hence,} \quad \|\beta_m(T)\|^{1/m} \geq \left| |\lambda|^2 - 1 \right|.$$

Therefore $\limsup_{m\to\infty} \|\beta_m(T)\|^{1/m} = 0$ implies that $|\lambda| = 1$. ■

By Example 1.3, any compact set $K \subseteq \partial D$ could be the spectrum of an ∞-isometry.

Definition 2.2 *We say T is an ∞-unitary if both T and T^* are ∞-isometries. Similarly, for $m \geq 1$, T is an m-unitary if both T and T^* are m-isometries. T is a finite-unitary if T is an m-unitary for some $m \geq 1$.*

Corollary 2.3 *If T is an ∞-unitary, then $\sigma(T) \subseteq \partial D$.*

Proof. The proof is the same as the proof of Corollary 1.22 in [3]. We prove by contradiction. If $\sigma(T) \subsetneq \partial D$, then by Proposition 2.1, $\sigma(T) = D^-$ and $0 \notin \sigma_{ap}(T)$. Thus T is onto and

$0 \notin \sigma_{ap}(T^*)$. By applying Proposition 2.1 to T^*, we have $\sigma(T^*) \subseteq \partial D$. This is a contraction to $\sigma(T) = D^-$. ■

Note that examples in Example 1.2 and Example 1.3 are ∞-unitaries.

We next show that the following result about eigenvalues for an m-isometry does not extend to ∞-isometries, see part (a) of Theorem 1 in [15]. The operator in this example is just the adjoint of the operator in Example 1.2.

Theorem 2.4 *[15] Let T be an m-isometry. If λ is an eigenvalue of T, then $\overline{\lambda}$ is an eigenvalue of T^*. Similarly, if $\lambda \in \sigma_{ap}(T)$, then $\overline{\lambda} \in \sigma_{ap}(T^*)$.*

Example 2.5 *Let $T = \lambda I + Q^*$, where $|\lambda| = 1$ and Q is the weighted shift on l_2 with weights $w_j = \frac{1}{j+1}$. Then T is an ∞-isometry. Furthermore, $(T - \lambda I)\, e_0 = Q^* e_0 = 0$, but $\left(T^* - \overline{\lambda} I\right) e_0 = e_1 \neq 0$. It is easy to see that λ is an eigenvalue of T, but $\overline{\lambda}$ is not an eigenvalue of T^*.*

However a similar result for eigenvectors or approximate eigenvectors of an m-isometry in Theorem 1 of [15] does extend to an ∞-isometry. See also Lemma 19 in [2] for a related result.

Proposition 2.6 *Let T be an ∞-isometry.*

(a) Eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

(b) If λ and μ are two distinct approximate eigenvalues of T, and $\{x_n\}$ and $\{y_n\}$ are sequence of unit vectors such that $(T - \lambda I)x_n \to 0$ and $(T - \mu I)y_n \to 0$, then $\langle x_n, y_n \rangle \to 0$.

Proof. We first prove (a). Let λ and μ be two distinct eigenvalues of T. By Proposition 2.1, $|\lambda| = |\mu| = 1$, so $\lambda\overline{\mu} - 1 \neq 0$. Let x and y be two unit vectors such that $(T - \lambda I)x = 0$ and $(T - \mu I)y = 0$. Then

$$\begin{aligned}
\langle \beta_m(T)x, y \rangle &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \left\langle T^k x, T^k y \right\rangle \\
&= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \lambda^k \overline{\mu}^k \langle x, y \rangle = (\lambda\overline{\mu} - 1)^m \langle x, y \rangle .
\end{aligned}$$

It follows that

$$\left| (\lambda\overline{\mu} - 1)^m \langle x, y \rangle \right|^{1/m} = \left| \langle \beta_m(T)x, y \rangle \right|^{1/m} \leq \left\| \beta_m(T) \right\|^{1/m} .$$

By taking limsup as $m \to \infty$ of the inequality above, we see that $\langle x, y \rangle = 0$.

The proof of (b) is similar. Let λ and μ be two distinct approximate eigenvalues of T. By Proposition 2.1, $|\lambda| = |\mu| = 1$, so $\lambda\overline{\mu} - 1 \neq 0$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of unit vectors such that $(T - \lambda I)x_n \to 0$ and $(T - \mu I)y_n \to 0$. To prove $\langle x_n, y_n \rangle \to 0$, let

$\langle x_{n_j}, y_{n_j}\rangle$ be any convergent subsequence of $\langle x_n, y_n\rangle$ such that $\langle x_{n_j}, y_{n_j}\rangle \to a$, we shall show that $a = 0$. Note that for each fix $m \geq 1$,

$$\begin{aligned} |(\lambda\overline{\mu}-1)^m a| &= \lim_{n_j\to\infty} \left|(\lambda\overline{\mu}-1)^m \langle x_{n_j}, y_{n_j}\rangle\right| \\ &= \left|\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\lambda^k\overline{\mu}^k \lim_{n_j\to\infty}\langle x_{n_j}, y_{n_j}\rangle\right| \\ &= \left|\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} \lim_{n_j\to\infty}\langle T^k x_{n_j}, T^k y_{n_j}\rangle\right| \\ &= \lim_{n_j\to\infty}\left|\langle \beta_m(T)x_{n_j}, y_{n_j}\rangle\right| \leq \|\beta_m(T)\|. \end{aligned}$$

Therefore

$$\begin{aligned} |\lambda\overline{\mu}-1| \lim_{m\to\infty} |a|^{1/m} &= \limsup_{m\to\infty} |(\lambda\overline{\mu}-1)^m a|^{1/m} \\ &\leq \limsup_{m\to\infty} \|\beta_m(T)\|^{1/m} = 0. \end{aligned}$$

This implies $a = 0$ and hence $\langle x_n, y_n\rangle \to 0$. ∎

We next show that the above proposition can be significantly strengthened. We first need a lemma.

Lemma 2.7 *For any two complex numbers λ and μ, the following holds:*

$$\beta_m(T) = \sum_{m_1+m_2+m_3=m}\binom{m}{m_1, m_2, m_3}(T^*-\overline{\mu}I)^{m_1}T^{m_1}\overline{\mu}^{m_2}(T-\lambda I)^{m_2}(\lambda\overline{\mu}-1)^{m_3}.$$

Proof. Using the multinomial formula,

$$\begin{aligned} \beta_m(T) &= (yx-1)^m|_{y=T^*;\ x=T} \\ &= \Big((y-\overline{\mu})\,x + \overline{\mu}\,(x-\lambda) + (\lambda\overline{\mu}-1)\Big)^m|_{y=T^*;\ x=T} \\ &= \sum_{m_1+m_2+m_3=m}\binom{m}{m_1, m_2, m_3}(y-\overline{\mu})^{m_1}x^{m_1}\overline{\mu}^{m_2}(x-\lambda)^{m_2}(\lambda\overline{\mu}-1)^{m_3}|_{y=T^*;\ x=T} \\ &= \sum_{m_1+m_2+m_3=m}\binom{m}{m_1, m_2, m_3}(T^*-\overline{\mu}I)^{m_1}T^{m_1}\overline{\mu}^{m_2}(T-\lambda I)^{m_2}(\lambda\overline{\mu}-1)^{m_3}. \end{aligned}$$

The proof is complete. ∎

Lemma 2.8 *Let T be an ∞-isometry. If λ and μ are two distinct eigenvalues of T, then* $\ker(T-\lambda I)^k \perp \ker(T-\mu I)^l$ *for all $k, l \geq 1$.*

Proof. We prove the lemma by induction. For the case $k = l = 1$. Let v_1, v_2 be unit vectors such that $(T - \lambda I)v_1 = (T - \mu I)v_2 = 0$. Then by Lemma 2.7,

$$\begin{aligned}
&\langle \beta_m(T)v_1, v_2\rangle \\
&= \left\langle \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} (\lambda\overline{\mu} - 1)^{m_3} T^{m_1} \overline{\mu}^{m_2} (T - \lambda I)^{m_2} v_1, (T - \mu I)^{m_1} v_2 \right\rangle \\
&= (\lambda\overline{\mu} - 1)^m \langle v_1, v_2\rangle ,
\end{aligned}$$

since $(T - \lambda I)^{m_2} v_1 = (T - \mu I)^{m_1} v_2 = 0$ for all $m_1, m_2 \neq 0$. By assumption $\lambda\overline{\mu} \neq 1$, thus

$$\begin{aligned}
\limsup_{m\to\infty} \left|(\lambda\overline{\mu} - 1)^m \langle v_1, v_2\rangle\right|^{1/m} &= \limsup_{m\to\infty} \left|\langle \beta_m(T)v_1, v_2\rangle\right|^{1/m} \\
&\leq \limsup_{m\to\infty} \|\beta_m(T)\|^{1/m} = 0,
\end{aligned} \tag{3}$$

which gives $\langle v_1, v_2\rangle = 0$.

We now fix $l = 1$ and use induction on k. Assume $\ker(T - \lambda I)^k \perp \ker(T - \mu I)$. Let $v_1 \in \ker(T - \lambda I)^{k+1}$, $v_2 \in \ker(T - \mu I)$. We will show that $\langle v_1, v_2\rangle = 0$, and hence $\ker(T - \lambda I)^k \perp \ker(T - \mu I)$ for all $k \geq 1$. Note that

$$\begin{aligned}
&\langle \beta_m(T)v_1, v_2\rangle \\
&= \left\langle \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} (\lambda\overline{\mu} - 1)^{m_3} T^{m_1} \overline{\mu}^{m_2} (T - \lambda I)^{m_2} v_1, (T - \mu I)^{m_1} v_2 \right\rangle \\
&= \left\langle \sum_{m_2+m_3=m} \binom{m}{0, m_2, m_3} (\lambda\overline{\mu} - 1)^{m_3} \overline{\mu}^{m_2} (T - \lambda I)^{m_2} v_1, v_2 \right\rangle \\
&= (\lambda\overline{\mu} - 1)^m \langle v_1, v_2\rangle ,
\end{aligned}$$

since if $m_1 \geq 1$, $(T - \mu I)^{m_1} v_2 = 0$. Moreover, if $m_2 \geq k + 1$, $(T - \lambda I)^{m_2} v_1 = 0$; and if $1 \leq m_2 \leq k$, $(T - \lambda I)^k (T - \lambda I)^{m_2} v_1 = (T - \lambda I)^{k+m_2} v_1 = 0$, so $(T - \lambda I)^{m_2} v_1 \in \ker(T - \lambda I)^k$, thus $(T - \lambda I)^{m_2} v_1 \perp v_2$ and $\langle (T - \lambda I)^{m_2} v_1, v_2\rangle = 0$ by the inductive hypothesis. Thus the only term left is when $m_1 = m_2 = 0; m_3 = m$. An argument similar to (3) shows that $\langle v_1, v_2\rangle = 0$. This proves that

$$\ker(T - \lambda I)^k \perp \ker(T - \mu I) \text{ for all } k \geq 1. \tag{4}$$

By symmetry, we also have

$$\ker(T - \lambda I) \perp \ker(T - \mu I)^l \text{ for all } l \geq 1. \tag{5}$$

We will show that $\ker(T - \lambda I)^k \perp \ker(S - \mu I)^l$ for any $k, l \geq 1$ by using induction on l. Assume for fixed l,

$$\ker(T - \lambda I)^k \perp \ker(T - \mu I)^l \text{ for all } k \geq 1. \tag{6}$$

We now show that $\ker(T-\lambda I)^k \perp \ker(T-\mu I)^{l+1}$ for all $k \geq 1$ as well. We do this by using induction on k. For $k=1$, this is just (5). Now we assume

$$\ker(T-\lambda I)^n \perp \ker(S-\mu I)^{l+1}. \tag{7}$$

We will show

$$\ker(T-\lambda I)^{n+1} \perp \ker(T-\mu I)^{l+1}.$$

Let $v_1 \in \ker(T-\lambda I)^{n+1}$ and $v_2 \in \ker(T-\mu I)^{l+1}$. A similar computation yields:

$$\begin{aligned} 0 &= \langle \beta_m(T)v_1, v_2\rangle \\ &= \left\langle \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} (\lambda\overline{\mu}-1)^{m_3} T^{m_1} \overline{\mu}^{m_2} (T-\lambda I)^{m_2} v_1, (T-\mu I)^{m_1} v_2 \right\rangle. \end{aligned} \tag{8}$$

Now, if $m_1 \geq 1$ and $m_2 \geq 0$, $(T-\mu I)^{m_1} v_2 \in \ker(T-\mu I)^l$ and $(T-\lambda I)^{m_2} v_1 \in \ker(T-\lambda I)^n$. Thus by (6), these terms in (8) are zero. If $m_1 = 0$ and $m_2 > 0$, $(T-\mu I)^{m_1} v_2 \in \ker(T-\mu I)^{l+1}$ and $(T-\lambda I)^{m_2} v_1 \in \ker(T-\lambda I)^n$. Then by the inductive hypothesis (7), these terms (8) are also zero. The only term left in (8) is when $m_1 = m_2 = 0; m_3 = m$. Finally, an argument similar to (3) shows that $\langle v_1, v_2\rangle = 0$. ■

Remark 2.9 *Similarly, if T is an ∞-isometry, and if λ and μ are two distinct approximate eigenvalues of T, and $\{x_n\}$ and $\{y_n\}$ are sequence of unit vectors such that $(T-\lambda I)^k x_n \to 0$ and $(T-\mu I)^l y_n \to 0$ for some fixed $k, l \geq 1$, then $\langle x_n, y_n\rangle \to 0$.*

Theorem 2.10 *Assume H is a finite dimensional complex Hilbert space with $\dim(H) = N$. Then $T \in B(H)$ is an ∞-isometry if and only if T is an $(2N-1)$-isometry.*

Proof. Assume T is an ∞-isometry on a finite dimensional H. By Proposition 2.1, it is $\sigma(T) \subseteq \partial D$. Let $p(\lambda) = \prod_{i=1}^k (\lambda - \lambda_i)^{p_i}$ be the characteristic polynomial of T, where λ_i are distinct eigenvalues of T. By Lemma 2.8, T is unitarily equivalent to direct sum

$$T \cong \bigoplus_{i=1}^{k} (\lambda_i I + Q_i)$$

where Q_i is a nilpotent matrix of order p_i. A direct calculation (or by Lemma 4.1) shows that $\beta_m(T) = 0$ for $m = \max\{2p_i - 1, i = 1, 2, \cdots, k\}$. ■

An m-isometry on a finite dimensional Hilbert space H is the direct sum of operators of the form $\lambda I + Q$ as in [2] where $|\lambda| = 1$ and Q is a nilpotent matrix. See [2] also for more general hereditary roots than just m-isometries on a finite dimensional H.

3 The decomposition of an ∞-isometry

In this section, for an ∞-isometry T, we identify invariant subspaces of T such that the restriction of T to those invariant subspaces are finite-isometries. In particular, we decompose an ∞-unitary T into direct sum of m-unitaries for $m \in \{1, 2, 3, \cdots\} \cup \{\infty\}$. The first result identifies some invariant subspace H_m for an invertible T such that $T|H_m$ is an m-isometry. Let

$$H_m := \bigcap_{n \geq m} \ker(\beta_n(T)).$$

It follows from the definition that

$$H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots .$$

Proposition 3.1 *Let $T \in B(H)$. Assume $\ker(T^*) = \{0\}$. Then H_m is an invariant subspace for T and $T|H_m$ is an m-isometry.*

Proof. We first prove H_m is invariant for T. If $x \in H_m$, then for any $n \geq m$,

$$\beta_n(T)x = 0 \text{ and } \beta_{n+1}(T)x = 0.$$

Then by (1),

$$[\beta_{n+1}(T) + \beta_n(T)]\, x = T^*\beta_n(T)Tx = 0.$$

Since T^* is injective, $\beta_n(T)Tx = 0$ for any $n \geq m$. That is, $Tx \in H_m$. Now by (2),

$$\beta_m(T|H_m) = P_{H_m}\beta_m(T)|H_m = P_{H_m}0 = 0.$$

Therefore, $T|H_m$ is an m-isometry. ■

If T is an m-isometry, then $\beta_n(T) = 0$ and $\ker(\beta_n(T)) = H$ for $n \geq m$, so we have the following result.

Corollary 3.2 *If $T \in B(H)$ is an invertible m-isometry, then for $n = 1, 2, \ldots, m-1$,*

$$H_n = \bigcap_{m-1 \geq j \geq n} \ker(\beta_j(T))$$

is invariant for T and $T|H_n$ is an n-isometry.

Proposition 1.6 in [3] states that if $T \in B(H)$ is an m-isometry, then H_{m-1} is the maximal invariant subspace M such that $T|M$ is an $(m-1)$-isometry. It is natural to ask if H_n in Proposition 3.1 is in fact the maximal invariant subspace M such that $T|M$ is an n-isometry for $n = 1, 2, \ldots, m-2$. Next, we will identify some other invariant subspaces.

For $m \geq 1$, another subspace K_m is defined by

$$K_m(T) \equiv K_m := \bigcap_{i \geq 0} \ker(\beta_m(T)T^i). \tag{9}$$

It follows from (1) that

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots .$$

Similarly we have the following result.

Proposition 3.3 *Let $T \in B(H)$. Then K_m is invariant for T and $T|K_m$ is an m-isometry.*

Proof. If $x \in K_m$, then $\beta_m(T)T^i x = 0$ for all $i \geq 0$. Thus $\beta_m(T)T^i Tx = \beta_m(T)T^{i+1}x = 0$ for all $i \geq 0$. So $Tx \in K_m$ and K_m is invariant for T. Furthermore $\beta_m(T|K_m) = P_{K_m}\beta_m(T)|K_m = 0$ because $K_m \subseteq \ker(\beta_m(T))$. ■

In fact, there is a subspace K_∞ defined by

$$K_\infty(T) \equiv K_\infty := \left\{ h \in H : \limsup_{m\to\infty} \left\|\beta_m(T)T^i h\right\|^{1/m} = 0 \text{ for all } i \geq 0 \right\}.$$

Similarly, we define

$$\begin{aligned} &K_\infty(T^*) := \left\{ h \in H : \limsup_{m\to\infty} \left\|\beta_m(T^*)T^{*i} h\right\|^{1/m} = 0 \text{ for all } i \geq 0 \right\} \\ &\text{and } U_\infty := K_\infty(T)\bigcap K_\infty(T^*). \end{aligned} \tag{10}$$

Proposition 3.4 *Let $T \in B(H)$. Then K_∞ is invariant for T and $T|K_\infty$ is an ∞-isometry.*

Proof. By definition, if $x, y \in K_\infty$, then for $i \geq 0$ and any two complex numbers a and b,

$$\begin{aligned} &\limsup_{m\to\infty} \left\|\beta_m(T)T^i(ax+by)\right\|^{1/m} \\ &\leq \limsup_{m\to\infty} (|a| \left\|\beta_m(T)T^i x\right\| + |b| \left\|\beta_m(T)T^i x\right\|)^{1/m} \\ &\leq \limsup_{m\to\infty} (|a| \left\|\beta_m(T)T^i x\right\|)^{1/m} + \limsup_{m\to\infty} (|b| \left\|\beta_m(T)T^i x\right\|)^{1/m} = 0. \end{aligned}$$

Hence K_∞ is a subspace. The proof that K_∞ is invariant for T is also straightforward. To prove $T|K_\infty$ is an ∞-isometry, we need to show

$$\limsup_{m\to\infty} \|\beta_m(T|K_\infty)\|^{1/m} = 0. \tag{11}$$

Note that $\beta_m(T|K_\infty) = P_{K_\infty}\beta_m(T)|K_\infty$. Hence for $h \in K_\infty$,

$$\begin{aligned} \limsup_{m\to\infty} \|\beta_m(T|K_\infty)h\|^{1/m} &= \limsup_{m\to\infty} \|P_{K_\infty}\beta_m(T)h\|^{1/m} \\ &\leq \limsup_{m\to\infty} \|\beta_m(T)h\|^{1/m} = 0. \end{aligned} \tag{12}$$

We will use the uniform boundedness principle to prove (11). But a straightforward application of the uniform boundedness principle to the sequence of operators $\{\beta_m(T|K_\infty)\}$ only gives

$$\|\beta_m(T|K_\infty)\| \leq C \text{ for } m \geq 1 \quad \text{and} \quad \limsup_{m\to\infty} \|\beta_m(T|K_\infty)\|^{1/m} \leq 1,$$

where C is some constant. However, note that for any $\delta > 0$ and $h \in K_\infty$, by (12),

$$\limsup_{m\to\infty} \|\delta^m \beta_m(T|K_\infty)h\|^{1/m} = \delta \limsup_{m\to\infty} \|\beta_m(T|K_\infty)h\|^{1/m} = 0.$$

By applying the uniform boundedness principle to the sequence of operators $\{\delta^m \beta_m(T|K_\infty)\}$, we have

$$\|\delta^m \beta_m(T|K_\infty)\| \le C \quad (\text{for } m \ge 1) \quad \text{and} \quad \|\beta_m(T|K_\infty)\|^{1/m} \le \frac{C^{1/m}}{\delta},$$

where C is some constant. Therefore

$$\limsup_{m\to\infty} \|\beta_m(T|K_\infty)\|^{1/m} \le \limsup_{m\to\infty} \frac{C^{1/m}}{\delta} = \frac{1}{\delta}.$$

Since $\delta > 0$ is arbitrary, this proves that $\limsup_{m\to\infty} \|\beta_m(T|K_\infty)\|^{1/m} = 0$. ■

Decomposition theorems for operators are important in operator theory, see the book [22]. The Nagy-Foias-Langer decomposition theorem for a contraction says every contraction is a direct sum of an unitary and a completely nonunitary contraction, see Theorem 5.1 in [22] for details. The following result generalizes Nagy-Foias-Langer decomposition theorem in two ways. First, it is valid for an arbitrary operator. Second, it is valid for m-unitaries for all $m \ge 1$ and for ∞-unitaries. This also gives a different (probably more direct) proof of Nagy-Foias-Langer decomposition theorem since the proof of Nagy-Foias-Langer decomposition theorem use the assumption of T being a contraction in an essential way.

Theorem 3.5 *Let $T \in B(H)$.*

(a) Then U_m defined by the formula

$$U_m := K_m(T)\bigcap K_m(T^*) = \bigcap_{i\ge 0} \left[\ker(\beta_m(T)T^i)\bigcap \ker(\beta_m(T^*)T^{i*})\right] \tag{13}$$

is the unique maximal reducing subspace on which T is an m-unitary. Furthermore, $T = T_1 \oplus T_2$ with respect to the decomposition $H = U_m \oplus U_m^\perp$, where T_1 is an m-unitary and T_2 is an operator which has no direct m-unitary summand.

(b) The space U_∞ defined by (10) is the unique maximal reducing subspace on which T is an ∞-unitary. Furthermore, $T = T_1 \oplus T_2$ with respect to the decomposition $H = U_\infty \oplus U_\infty^\perp$, where T_1 is an ∞-unitary and T_2 is an operator which has no direct ∞-unitary summand.

Proof. We first prove (a). By Proposition 3.3, $K_m(T)$ defined in (9) is invariant for T and $K_m(T^*)$ is invariant for T^*, thus U_m is reducing for T. It also follows that $T|U_m$ is an m-unitary. Now if $M \subseteq H$ is reducing for T and $T|M$ is an m-unitary, then $\beta_m(T)|M = P_M\beta_m(T)|M = \beta_m(T|M) = 0$. So $M \subseteq \ker(\beta_m(T))$. Since M is invariant for T, $T^iM \subseteq M \subseteq \ker(\beta_m(T))$ for each $i \ge 0$. This implies that $M \subseteq \ker(\beta_m(T)T^i)$. Hence $M \subseteq K_m(T)$. Similarly, since $M \subseteq H$ is reducing for T^* and $T^*|M$ is an m-unitary, we have $M \subseteq K_m(T^*)$. Thus $M \subseteq U_m$. This proves the maximality of U_m. The decomposition $T = T_1 \oplus T_2$ now follows from the maximality of U_m.

For (b), we only prove the maximality of U_∞ since the rest of the proof is similar to that of (a). Assume $M \subseteq H$ is reducing for T and $T|M$ is an ∞-unitary. Then $\beta_m(T)|M = P_M\beta_m(T)|M = \beta_m(T|M)$. So, for any $h \in M$ and $i \geq 0$

$$\begin{aligned}\limsup_{m\to\infty} \left\|\beta_m(T)T^i h\right\|^{1/m} &= \limsup_{m\to\infty} \left\|\beta_m(T|M)T^i h\right\|^{1/m} \\ &\leq \limsup_{m\to\infty} \left\|\beta_m(T|M)\right\|^{1/m} \left\|T^i h\right\|^{1/m} = 0.\end{aligned}$$

By the definition of $K_\infty(T), h \in K_\infty(T)$. So, $M \subseteq K_\infty(T)$. Similarly, since $M \subseteq H$ is reducing for T^* and $T^*|M$ is an ∞-unitary, we have $M \subseteq K_\infty(T^*)$. In conclusion $M \subseteq U_\infty$. The proof is complete. ■

The following definition is natural.

Definition 3.6 *If $T \in B(H)$ is an m-unitary for $m \geq 2$, T is called a pure m-unitary if T has no nonzero direct summand which is an $(m-1)$-unitary. Similarly, if $T \in B(H)$ is an ∞-unitary, T is called a pure ∞-unitary if T has no nonzero direct summand which is a finite-unitary.*

For each $m \geq 1$, let U_m denote the unique maximal reducing subspace as in (13) such that $T|U_m$ is an m-unitary. We have the following decomposition theorem for an ∞-unitary.

Theorem 3.7 *Let $T \in B(H)$ be an ∞-unitary. Let $V_1 = U_1, V_n = U_n \ominus U_{n-1}$ for $n \geq 2$ and*

$$V_\infty = H \ominus \bigvee \{U_i, i \geq 1\} = H \ominus \bigvee \{V_i, i \geq 1\}.$$

Then V_i is reducing for T for each $i = 1, 2, \ldots, \infty$, and with respect to the decomposition $H = V_\infty \oplus V_1 \oplus V_2 \oplus V_3 \oplus \cdots$, T has the following form

$$T = V_\infty \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_n \oplus \cdots,$$

where $T|V_\infty$ is a pure ∞-unitary, $T|V_1$ is an unitary, and $T|V_n$ is a pure n-unitary for $n \geq 2$.

For a 2-isometry T, Proposition 1.25 in [3] identifies the unique maximal reducing subspace R_1 such that $T|R_1$ is an isometry. Here we are able to identify R_1 for an arbitrary operator. This also leads to a decomposition theorem for T similar to Nagy-Foias-Langer decomposition for a contraction where the unitary part is replaced by the isometric part.

Theorem 3.8 *Let $T \in B(H)$. Then R_1 defined by the formula*

$$R_1 = R_1(T) := \bigcap_{i,n\geq 0} \ker(\beta_1(T)T^i T^{*n}) \tag{14}$$

is the unique maximal reducing subspace on which T is an isometry. Furthermore, $T = T_1 \oplus T_2$ with respect to the decomposition $H = R_1 \oplus R_1^\perp$, where T_1 is an isometry and T_2 is an operator which has no direct isometry summand.

Proof. We first prove that R_1 is reducing for T. Let $h \in R_1$. That is,

$$\left(\beta_1(T)T^iT^{*n}\right)h = 0 \text{ for all } i, n \geq 0.$$

We need to show both T^*h and Th are in R_1. Clearly

$$\left(\beta_1(T)T^iT^{*n}\right)T^*h = \left(\beta_1(T)T^iT^{*n+1}\right)h = 0 \text{ for all } i, n \geq 0.$$

Hence $T^*h \in R_1$. Note also if $n = 0$, then

$$\left(\beta_1(T)T^iT^{*n}\right)Th = \left(\beta_1(T)T^{i+1}\right)h = 0 \text{ for all } i \geq 0.$$

Furthermore, if $n \geq 1$, then for all $i \geq 0$,

$$\begin{aligned}\left(\beta_1(T)T^iT^{*n}\right)Th &= \left(\beta_1(T)T^iT^{*n-1}T^*T\right)h \\ &= \beta_1(T)T^iT^{*n-1}\left[\beta_1(T) + I\right]h \\ &= \beta_1(T)T^iT^{*n-1}\beta_1(T)h + \beta_1(T)T^iT^{*n-1}h \\ &= 0 + 0 = 0.\end{aligned}$$

Hence, $Th \in R_1$. In conclusion, R_1 is reducing for T. Note that by (9), $R_1 \subseteq K_1$. By Proposition 3.3, $T|K_1$ is an isometry. Thus $T_1 = T|R_1$ is an isometry.

Next, we prove the maximality of R_1. Suppose now M is reducing for T and $T|M$ is an isometry. Then, if $h \in M$,

$$\begin{aligned}\left[\beta_1(T)T^i\right]h &= \left[P_M\beta_1(T|M)\right](T^ih) \\ &= (0)(T^ih) = 0 \text{ for all } i \geq 0.\end{aligned}$$

So, $M \subseteq \ker(\beta_1(T)T^i)$ and $M \subseteq K_1$ as in (9). Since M is reducing for T, $T^{*n}h \in M \subseteq K_1$ for all $n \geq 0$. Equivalently,

$$\left(\beta_1(T)T^iT^{*n}\right)h = \left(\beta_1(T)T^i\right)T^{*n}h = 0 \text{ for all } i, n \geq 0.$$

Therefore, $h \in R_1$ and $M \subseteq R_1$. The proof is complete. ■

The maximal reducing subspace in Proposition 1.25 in [3] for a 2-isometry is defined somewhat differently. Of course, with some work, one can prove that R_1 defined by (14) reduces to the one in [3] for a 2-isometry. It seems rather miraculous the proof worked for R_1 since we are unable to prove the same conclusion for the analogous R_m with $m > 1$, where

$$R_m := \bigcap_{i,n\geq 0} \ker(\beta_m(T)T^iT^{*n}).$$

4 Quasinilpotent perturbations of ∞-isometries

We will use the following Lemma 1 and Lemma 7 from [19] and Lemma 8 in [17]. Recall that T_1 and T_2 in $B(H)$ are double commuting if

$$T_1T_2 = T_2T_1 \text{ and } T_1T_2^* = T_2^*T_1.$$

Lemma 4.1 *Assume T_1 and $T_2 \in B(H)$ are double commuting, and T and $Q \in B(H)$ are commuting. Then*

$$\beta_m(T+Q) = \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} (T^* + Q^*)^{m_1} Q^{*m_2} \beta_{m_3}(T) T^{m_2} Q^{m_1},$$

$$\beta_m(T+Q) = \sum_{m_1+m_2+m_3+m_4=m} \binom{m}{m_1, m_2, m_3, m_4} \cdot T^{*m_1} Q^{*(m_2+m_4)} \beta_{m_3}(T) T^{m_2} Q^{(m_1+m_4)}, \tag{15}$$

$$\beta_m(T_1T_2) = \sum_{k=0}^{m} \binom{m}{k} T_1^{*k} \beta_{m-k}(T_1) T_1^k \beta_k(T_2). \tag{16}$$

Theorem 4.4 below has its inspiration from sub-jordan operators in [1] and [20]. It is directly suggested by the following result for m-isometries.

Theorem 4.2 *Assume T and $Q \in B(H)$ are commuting, and also assume that T is an m-isometry and Q is a nilpotent operator of order n. Then $T+Q$ is an $(m+2n-2)$-isometry.*

The above result was proved for $m = 1$ in [10]. The general and sometimes slightly improved versions were discussed independently in [7], [19] and [23]. We first prove a lemma.

Lemma 4.3 *Assume T and $Q \in B(H)$ are commuting. Then*

$$\|\beta_m(T+Q)\| \le C^m \left(\max_{m \ge n \ge l} \|\beta_n(T)\| + \max_{m \ge n \ge l} \|Q^n\| \right),$$

where $l = \left[\frac{m}{3}\right]$ is the integer part of $\frac{m}{3}$ and

$$C = 2\left([\|T\| + \|Q\|]^2 + 2\|T\| + 1 \right).$$

Proof. By Lemma 4.1,

$$\beta_m(T+Q) = \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} (T^* + Q^*)^{m_1} Q^{*m_2} \beta_{m_3}(T) T^{m_2} Q^{m_1}.$$

Let $l = \left[\frac{m}{3}\right]$, the integer part of $\frac{m}{3}$. For $i = 1, 2, 3$, we define

$$\Omega_i = \sum_{m_1+m_2+m_3=m \text{ and } m_i \ge l} \binom{m}{m_1, m_2, m_3} \left\| (T^* + Q^*)^{m_1} Q^{*m_2} \beta_{m_3}(T) T^{m_2} Q^{m_1} \right\|.$$

Since $m_1 + m_2 + m_3 = m$ implies that $m_i \geq l$ for one of the $i = 1, 2, 3$, we have

$$\begin{aligned}
\|\beta_m(T+Q)\| \leq & \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} \\
& \cdot \|(T^* + Q^*)^{m_1} Q^{*m_2} \beta_{m_3}(T) T^{m_2} Q^{m_1}\| \\
& \leq \Omega_1 + \Omega_2 + \Omega_3.
\end{aligned} \tag{17}$$

We now estimate Ω_i. We will prove the estimate for Ω_3 since the proofs of the estimates for Ω_1 and Ω_2 are similar. Note that

$$\begin{aligned}
\Omega_3 = & \sum_{m_1+m_2+m_3=m \text{ and } m_3 \geq l} \binom{m}{m_1, m_2, m_3} \|(T^* + Q^*)^{m_1} Q^{*m_2} \beta_{m_3}(T) T^{m_2} Q^{m_1}\| \\
& \leq \sum_{m_1+m_2+m_3=m \text{ and } m_3 \geq l} \binom{m}{m_1, m_2, m_3} \\
& \cdot (\|T^*\| + \|Q^*\|)^{m_1} \|Q^*\|^{m_2} \|\beta_{m_3}(T)\| \|T\|^{m_2} \|Q\|^{m_1} \\
& \leq \max_{m \geq n \geq l} \|\beta_n(T)\| \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} \\
& \cdot (\|T^*\| + \|Q^*\|)^{m_1} \|Q^*\|^{m_2} \|T\|^{m_2} \|Q\|^{m_1} \\
& = \max_{m \geq n \geq l} \|\beta_n(T)\| \left([\|T^*\| + \|Q^*\|] \|Q\| + \|Q^*\| \|T\| + 1 \right)^m \\
& \leq (C/2)^m \max_{m \geq n \geq l} \|\beta_n(T)\|,
\end{aligned}$$

where the last equality follows from multinomial formula. Similarly, by noting that

$$\|\beta_k(T)\| \leq (\|T\| + 1)^k \text{ for } k \geq 1,$$

we have

$$\begin{aligned}
\Omega_1 &\leq \max_{m \geq n \geq l} \|Q^n\| \left([\|T^*\| + \|Q^*\|] + \|Q^*\| \|T\| + [\|T\| + 1] \right)^m \\
&\leq \max_{m \geq n \geq l} \|Q^n\| (C/2)^m, \\
\Omega_2 &\leq \max_{m \geq n \geq l} \|Q^{*n}\| \varepsilon^l \left([\|T^*\| + \|Q^*\|] \|Q\| + \|T\| + [\|T\| + 1] \right)^m \\
&\leq \max_{m \geq n \geq l} \|Q^n\| (C/2)^m.
\end{aligned}$$

Therefore by (17),

$$\begin{aligned}
\|\beta_m(T+Q)\| &\leq (C/2)^m \max_{m \geq n \geq l} \|\beta_n(T)\| + 2(C/2)^m \max_{m \geq n \geq l} \|Q^n\| \\
&\leq C^m \left(\max_{m \geq n \geq l} \|\beta_n(T)\| + \max_{m \geq n \geq l} \|Q^n\| \right).
\end{aligned}$$

The proof is complete. ∎

Theorem 4.4 *Assume T and $Q \in B(H)$ are commuting, and also assume that T is an ∞-isometry and Q is a quasinilpotent operator. Then $T+Q$ is an ∞-isometry.*

Proof. Given $1 > \varepsilon > 0$, let N be such that

$$\|\beta_n(T)\| \le \varepsilon^n, \|Q^n\| \le \varepsilon^n \text{ for } n \ge N.$$

Then by Lemma 4.3, for $m \ge 3N$, we have $l = \left[\frac{m}{3}\right] \ge N$ and

$$\|\beta_m(T+Q)\| \le C^m \left(\max_{m \ge n \ge l} \|\beta_n(T)\| + \max_{m \ge n \ge l} \|Q^n\| \right) \le 2C^m \varepsilon^{\left[\frac{m}{3}\right]}$$

for some constant C. Hence $\limsup_{m\to\infty} \|\beta_m(T+Q)\|^{1/m} = 0$ and $T+Q$ is an ∞-isometry. ∎

In view of Theorem 4.2 and Theorem 4.4, we make the following conjecture.

Conjecture 4.5 *Assume T and $Q \in B(H)$ are commuting, and also assume that T is an m-isometry for some $m \ge 1$ and Q is a quasinilpotent operator but not a nilpotent operator. Then $T+Q$ is an ∞-isometry but not a finite-isometry.*

We now prove a limit result alluded in the introduction.

Proposition 4.6 *If $T_n \in B(H)$ is a sequence of commuting ∞-isometries and $T_n \to T$ in operator norm, then T is an ∞-isometry.*

Proof. By the assumption $T_n T_l = T_l T_n$ for all $n, l \ge 1$, we have $TT_n = T_n T$ for all $n \ge 1$. The proof is similar to the previous proof. Given $1 > \varepsilon > 0$, let N be such that

$$\|T - T_N\| \le \varepsilon \text{ and } \|\beta_n(T_N)\| \le \varepsilon^n \text{ for } n \ge N.$$

Then by Lemma 4.3, for $m \ge 3N$, $l = \left[\frac{m}{3}\right] \ge N$ and

$$\begin{aligned}
\|\beta_m(T)\| &= \|\beta_m(T_N + T - T_N)\| \\
&\le C^m \left(\max_{m \ge k \ge [m/3]} \|\beta_k(T_N)\| + \max_{m \ge k \ge [m/3]} \left\|(T - T_N)^k\right\| \right) \\
&\le C^m \left(\max_{m \ge k \ge [m/3]} \|\beta_k(T_N)\| + \max_{m \ge k \ge [m/3]} \|T - T_N\|^k \right) \\
&\le 2C^m \varepsilon^{\left[\frac{m}{3}\right]}
\end{aligned}$$

for some constant C. Hence $\limsup_{m\to\infty} \|\beta_m(T)\|^{1/m} = 0$ and T is an ∞-isometry. ∎

Problem 4.7 *What happens when T_n are not commuting in Proposition 4.6?*

There is a partial converse to Theorem 4.4. Let K be another complex Hilbert space. Let $H \otimes K$ denote the Hilbert space tensor product of H and K. Let I_H and I_K denote the identity operators on H and K respectively. If $T \in B(H)$ is an ∞-isometry and $S \in B(K)$ is a quasinilpotent operators, then $T \otimes I_K$ on $H \otimes K$ is an ∞-isometry and $T_2 = I_H \otimes S$ is a quasinilpotent operators. Thus, by Theorem 4.4, $T \otimes I_K + I_H \otimes S$ is an ∞-isometry on $H \otimes K$. We will prove the converse of this result under a technical assumption. We first prove a lemma.

Lemma 4.8 *Let $T \in B(H)$ and $Q \in B(K)$. If $(T \otimes I_H + I_K \otimes Q)$ is an ∞-isometry on $H \otimes K$ and $0 \in \sigma_{ap}(Q)$, then T is an ∞-isometry.*

Proof. By assumption, $0 \in \sigma_{ap}(Q)$. That is, there exist unit vectors $k_i \in K$ such that $Qk_i \to 0$ as $i \to \infty$. Thus, for any $l > 0$, $Q^l k_i \to 0$ as $i \to \infty$. Note formula (15) in Lemma 4.1 becomes

$$\begin{aligned}
&\beta_m(S \otimes I + I \otimes Q) \\
&= \sum_{m_1+m_2+m_3+m_4=m} \binom{m}{m_1, m_2, m_3, m_4} T^{*m_1} \beta_{m_3}(T) T^{m_2} \otimes Q^{*(m_2+m_4)} Q^{(m_1+m_4)}.
\end{aligned}$$

So, for any unit vector $h \in H$, letting $i \to \infty$, we have

$$\begin{aligned}
&\langle \beta_m(S \otimes I + I \otimes Q)(h \otimes k_i), (h \otimes k_i) \rangle \\
&= \sum_{m_1+m_2+m_3+m_4=m} \binom{m}{m_1, m_2, m_3, m_4} \\
&\cdot \langle T^{*m_1} \beta_{m_3}(T) T^{m_2} h, h \rangle \left\langle Q^{(m_1+m_4)} k_i, Q^{(m_2+m_4)} k_i \right\rangle \\
&\to \langle \beta_m(T) h, h \rangle ,
\end{aligned}$$

since all the terms in the above summation tend to zero except when $m_1 = m_2 = m_4 = 0$ and $m_3 = m$. Note that $(h \otimes k_i)$ is a unit vector in $H \otimes K$. Hence

$$\begin{aligned}
&\|\beta_m(S \otimes I + I \otimes Q)\| \geq |\langle \beta_m(S \otimes I + I \otimes Q)(h \otimes k_i), (h \otimes k_i) \rangle| \text{ and} \\
&\|\beta_m(S \otimes I + I \otimes Q)\| \geq |\langle \beta_m(T) h, h \rangle| .
\end{aligned}$$

Since the unit vector $h \in H$ is arbitrary, $\|\beta_m(S \otimes I + I \otimes Q)\|$ is bigger than or equal to the numerical radius of $\beta_m(T)$. But $\beta_m(T)$ is a self-adjoint operator and thus its numerical radius and the norm are the same. Therefore

$$\|\beta_m(S \otimes I + I \otimes Q)\| \geq \|\beta_m(T)\| .$$

This implies that if $\limsup_{m\to\infty} \|\beta_m(S \otimes I + I \otimes Q)\|^{1/m} = 0$, then $\limsup_{m\to\infty} \|\beta_m(T)\|^{1/m} = 0$. So T is an ∞-isometry on H. ■

Note that for any constant λ,

$$(T + \lambda I_H) \otimes I_K + I_H \otimes (Q - \lambda I_K) = S \otimes I + I \otimes Q.$$

By a translation we can assume $0 \in \sigma_{ap}(Q)$. So a little reflection gives the following interesting corollary.

Corollary 4.9 *Let $T \in B(H)$ and $Q \in B(K)$. Then*

$$\begin{aligned} \|\beta_m(S \otimes I + I \otimes Q)\| &\geq \max\left\{\|\beta_m(T + \lambda I_H)\| : \lambda \in \sigma_{ap}(Q)\right\}, \\ \|\beta_m(S \otimes I + I \otimes Q)\| &\geq \max\left\{\|\beta_m(Q + \lambda I_K)\| : \lambda \in \sigma_{ap}(T)\right\}. \end{aligned}$$

The following technical condition is similar to the one needed for an analogous result for m-isometries (Theorem 12 in [17]). Let Δ be the set of four points on the unit circle ∂D, specifically,

$$\Delta = \pm e^{\pm i\alpha} e^{i\theta} \text{ for some } \alpha, \theta \in [0, 2\pi). \tag{18}$$

Theorem 4.10 *Let $T \in B(H)$ and $Q \in B(K)$. Assume $\sigma\left(T \otimes I_H + I_K \otimes Q\right)$ $\neq \Delta$. Then $(T \otimes I_H + I_K \otimes Q)$ is an ∞-isometry on $H \otimes K$ if and only if one of the following holds.*

(a) There exists a constant λ such that $T + \lambda I$ is an ∞-isometry and $Q - \lambda I$ is a quasinilpotent operator.

(b) There exists a constant λ such that $Q + \lambda I$ is an ∞-isometry and $T - \lambda I$ is a quasinilpotent operator.

Proof. Assume $(T \otimes I_H + I_K \otimes Q)$ is an ∞-isometry. We will prove either (a) or (b) holds. The proof is similar to and slightly simpler than the proof of Theorem 12 in [17]. We first prove that either $\sigma(T)$ or $\sigma(Q)$ is a singleton or both $\sigma(T)$ and $\sigma(Q)$ contain exactly two points.

If $\sigma(Q)$ is not a singleton, let λ_1, λ_2 be two different numbers in $\sigma_{ap}(Q)$. We use I to denote either I_H or I_K. Since

$$(T + \lambda_i I_H) \otimes I_K + I_H \otimes (Q - \lambda_i I_K) = S \otimes I + I \otimes Q, i = 1, 2,$$

it follows that $0 \in \sigma_{ap}\left(Q - \lambda_i I\right)$. By Lemma 4.8, $T + \lambda_i I$ is an ∞-isometry for $i = 1, 2$. Note that

$$\sigma(T + \lambda_2 I) = \sigma(T + \lambda_1 I) + \lambda_2 - \lambda_1.$$

But by Proposition 2.1, either $\sigma(T + \lambda_1 I) = D^-$ or $\sigma(T + \lambda_1 I) \subseteq \partial D$. If $\sigma(T + \lambda_1 I) = D^-$, then $T + \lambda_2 I$ can not be an ∞-isometry since a translation of the unit disk D is not the unit disk. Therefore $\sigma(T + \lambda_1 I) \subseteq \partial D$. It is clear that $\sigma(T + \lambda_2 I) \neq D^-$, so $\sigma(T + \lambda_2 I) \subseteq \partial D$ as well. But $\sigma(T + \lambda_2 I)$ is a translation of $\sigma(T + \lambda_1 I)$ by the number $\lambda_2 - \lambda_1$, thus $\sigma(T - \lambda_1 I)$ consists of at most two points. In the case $\sigma(T + \lambda_1 I)$ is a singleton we are done. In the case $\sigma(T)$ consists of two points α_1, α_2, since $\alpha_1, \alpha_2 \in \sigma_{ap}(T)$, a similar argument shows that $\sigma(Q - \alpha_1 I)$ consists of exactly two points as well.

We first deal with the singleton case. Without loss of generality, let $\sigma(Q) = \{\lambda\}$. Then $Q - \lambda I$ is a quasinilpotent operator. By Lemma 4.8, $T + \lambda I$ is an ∞-isometry. That is, (b) holds.

In this case, both $\sigma(T)$ and $\sigma(Q)$ contain exactly two points. Since

$$\sigma(e^{i\theta}\left(T + \lambda I\right) \otimes I + I \otimes e^{i\theta}(Q - \lambda I)) = e^{i\theta}\sigma\left(T \otimes I + I \otimes Q\right),$$

by a rotation and a translation, we may assume that

$$\sigma(Q) = \sigma_{ap}(Q) = \{0, -\lambda\} \text{ for some } \lambda > 0.$$

By a theorem of Rosenblum [29], $\sigma(T \otimes I + I \otimes Q) = \sigma(T) + \sigma(Q)$. But by Proposition 2.1, $\sigma(T \otimes I + I \otimes Q) \subseteq \partial D$. Hence $\sigma(T) \subseteq \partial D \cap (\lambda + \partial D)$ and $\sigma(T)$ contains exactly two points when $0 < \lambda < 2$. In fact, we have

$$\sigma(T) = \sigma_{ap}(T) = \left\{e^{i\alpha}, e^{-i\alpha}\right\}, \text{ where } e^{i\alpha} = \frac{\lambda}{2} + i\sqrt{1 - \frac{\lambda^2}{4}}.$$

Therefore

$$\begin{aligned}\sigma(T \otimes I + I \otimes Q) &= \sigma(T) + \sigma(Q) \\ &= \left\{e^{i\alpha}, e^{-i\alpha}, e^{i\alpha} - \lambda, e^{-i\alpha} - \lambda\right\} \\ &= \left\{\pm e^{\pm i\alpha}\right\} = \Delta,\end{aligned}$$

which can not happen by our assumption. The proof is complete. ■

See Proposition 14 in [17] for examples where it is shown that the corresponding result for m-isometries may not hold without the assumption $\sigma\left(T \otimes I_H + I_K \otimes Q\right) \neq \Delta$.

The following result gives a partial answer to Conjecture 4.5.

Proposition 4.11 *Let $T \in B(H)$ and $Q \in B(K)$. Assume $\sigma\left(T \otimes I_H + I_K \otimes Q\right) \neq \Delta$. If T is an m-isometry for some $m \geq 1$ and Q is a quasinilpotent operator but not a nilpotent operator. Then $(T \otimes I_H + I_K \otimes Q)$ is an ∞-isometry but not a finite-isometry.*

Proof. It follows from Theorem 4.4 that $(T \otimes I_H + I_K \otimes Q)$ is an ∞-isometry. We prove by using contradiction. Assume $(T \otimes I_H + I_K \otimes Q)$ is a strict n-isometry for some $n \geq 1$. By Theorem 12 in [17], either there exists a constant λ such that $T + \lambda I$ is a k-isometry and $Q - \lambda I$ is a nilpotent operator of order l with $k + 2l - 2 = n$ or there exists a constant λ such that $Q + \lambda I$ is an k-isometry and $T - \lambda I$ is a nilpotent operator of order l. In the first case $\lambda = 0$, and Q is a nilpotent operator of order l, which contradicts the assumption on Q. In the second case, $Q + \lambda I$ is a k-isometry implies that $|\lambda| = 1$. So $\sigma(Q + \lambda I) = \{\lambda\}$. Recall that if S is an m-isometry for some $m \geq 1$ and $\sigma(S)$ consists of a finite number of points, then S is the direct sum of operators of the form $\lambda I + Q_0$, where Q_0 is a nilpotent operator, see Proposition 11 in [17]. Hence, $Q + \lambda I$ is a k-isometry, which implies that Q is a nilpotent operator. This again contradicts the assumption on Q. ■

5 Tensor products of ∞-isometries

The following result is inspired by the work on products of m-isometries in [9]. In fact, a result is proved there for products of m-isometries on Banach spaces. See also [19] and [23] for related results for m-isometries on Hilbert spaces.

Theorem 5.1 *Assume T_1 are T_2 in $B(H)$ are double commuting. If T_1 and T_2 are ∞-isometries, then so is T_1T_2.*

Proof. By Lemma 4.1,

$$\beta_m(T_1T_2) = \sum_{k=0}^{m} \binom{m}{k} T_1^{*k}\beta_{m-k}(T_1)T_1^k\beta_k(T_2).$$

This proof is similar to the proof of Lemma 4.3 and Theorem 4.4 by using the above formula. The proof here is actually slightly simpler. For clarity, we include the proof.

Given $1 > \varepsilon > 0$, let N be such that

$$\|\beta_n(T_1)\| \leq \varepsilon^n \text{ and } \|\beta_n(T_2)\| \leq \varepsilon^n \text{ for } n \geq N.$$

We claim that there exists a constant C such that for $m \geq 2N$

$$\|\beta_m(T_1T_2)\| \leq C^m\varepsilon^{m/2}.$$

Let $l = \left[\frac{m}{2}\right]$ denote the integer part of $\frac{m}{2}$. We write

$$\beta_m(T_1T_2) = I + II,$$

where

$$I := \sum_{k=0}^{l} \binom{m}{k} T_1^{*k}\beta_{m-k}(T_1)T_1^k\beta_k(T_2),$$
$$II := \sum_{k=l+1}^{m} \binom{m}{k} T_1^{*k}\beta_{m-k}(T_1)T_1^k\beta_k(T_2).$$

Note that for $k \leq l = \left[\frac{m}{2}\right]$, $m - k \geq \left[\frac{m}{2}\right] = l \geq N$, so

$$\|\beta_{m-k}(T_1)\| \leq \varepsilon^{m-k} \leq \varepsilon^l.$$

Note also by definition

$$\|\beta_k(T_2)\| \leq (\|T_2\| + 1)^k \text{ for } k \geq 1.$$

Therefore

$$\begin{aligned}
\|I\| &\leq \sum_{k=0}^{l} \binom{m}{k} \left\|T_1^{*k}\right\| \|\beta_{m-k}(T_1)\| \left\|T_1^k\right\| \|\beta_k(T_2)\| \\
&\leq \sum_{k=0}^{l} \binom{m}{k} \|T_1\|^k \varepsilon^{m-k} \|T_1\|^k (\|T_2\| + 1)^k \\
&\leq \varepsilon^l \sum_{k=0}^{m} \binom{m}{k} \|T_1\|^k \|T_1\|^k (\|T_2\| + 1)^k \\
&= \varepsilon^l \left[\|T_1\|^2 (\|T_2\| + 1) + 1\right]^m .
\end{aligned}$$

Similarly, by noting that for $k \geq l+1 \geq N$, $\|\beta_k(T_2)\| \leq \varepsilon^k$, we have

$$\|II\| \leq \varepsilon^l \left[\|T_1\|^2 + (\|T_1\|+1)\right]^m.$$

In conclusion, for $n \geq 2N$

$$\|\beta_m(T_1T_2)\| \leq \varepsilon^{\left[\frac{m}{2}\right]} \left(\left[\|T_1\|^2 (\|T_2\|+1)+1\right]^m + \left[\|T_1\|^2 + (\|T_1\|+1)\right]^m \right).$$

Therefore $\limsup_{m\to\infty} \|\beta_m(T_1T_2)\|^{1/m} = 0$ and T_1T_2 is an ∞-isometry. ■

By applying Theorem 5.1, we have the following result: if $T \in B(H)$ and $S \in B(K)$ are ∞-isometries, then $T \otimes S$ on $H \otimes K$ is an ∞-isometry. To see this, let $T_1 = T \otimes I_K$ and $T_2 = I_H \otimes S$, where I_H and I_K are identity operators. Then T_1 and T_2 are double commuting, and T_1 and T_2 are ∞-isometries. Thus by Theorem 5.1, $T_1T_2 = T \otimes S$ is an ∞-isometry.

Corollary 5.2 *If $T \in B(H)$ and $S \in B(K)$ are ∞-isometries, then $T \otimes S$ on $H \otimes K$ is an ∞-isometry.*

A similar result on tensor products of m-isometries is the following:

Theorem 5.3 *[16] If $T \in B(H)$ is an m-isometry and $S \in B(K)$ is an n-isometry, then $T \otimes S$ on $H \otimes K$ is an $(m+n-1)$-isometry.*

We would like to mention that the result in Theorem 5.3 on tensor products of m-isometries was first formulated in term of elementary operators on the ideal of Hilbert-Schmidt operators in [11] and [12]. This result was proved in Theorem 2.10 of [16]. Simple proofs of slightly improved versions of this result were given in [19] and [23]. A converse to Theorem 5.3 was obtained in Theorem 7 of [17].

We now prove an analogous result of Theorem 7 of [17] for tensor products of ∞-isometries.

Theorem 5.4 *Let $T \in B(H)$ and $S \in B(K)$. Then $T \otimes S$ on $H \otimes K$ is an ∞-isometry if and only if both αT and S/α are ∞-isometries on H and K respectively for some constant α.*

Proof. Note that for $T_1 = T \otimes I_K$ and $T_2 = I_H \otimes S$, formula (16) becomes

$$\beta_m(T \otimes S) = \sum_{k=0}^{m} \binom{m}{k} T^{*k}\beta_{m-k}(T)T^k \otimes \beta_k(S).$$

One way of the theorem is Corollary 5.2 by noting that $(\alpha T) \otimes (S/\alpha) = T \otimes S$.

Now assume $T \otimes S$ is an ∞-isometry. By Proposition 2.1, either $\sigma(T \otimes S) \subseteq \partial D$ or $\sigma(T \otimes S) = D^-$. By a theorem from Brown and Pearcy [13], $\sigma(T \otimes S) = \sigma(T) \cdot \sigma(S)$. In particular $1 = r(T \otimes S) = r(T) \cdot r(S)$. If $\alpha = \frac{1}{r(T)}$, then $r(\alpha T) = r(\frac{1}{\alpha}S) = 1$. Since

$\alpha T \otimes \frac{1}{\alpha} S = T \otimes S$, we may assume $r(T) = r(S) = 1$. We will prove that S is an ∞-isometry and the proof for T is similar.

By the assumption $r(T) = 1$, there is a $\lambda \in \partial D$ such that $\lambda \in \sigma_{ap}(T)$. Let h_i be a sequence of unit vectors in H such that $(T - \lambda)h_i \longrightarrow 0$. Then for any $k \geq 0$, $\left\|T^k h_i\right\|^2 \to |\lambda|^{2k} = 1$ as $i \to \infty$. Thus as in the proof of Proposition 2.1, for each $j > 0$, $k \geq 0$,

$$\begin{aligned} \left\langle \beta_j(T) T^k h_i, T^k h_i \right\rangle &= \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} \left\|T^l T^k h_i\right\|^2 \\ &\to \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} = 0 \text{ as } i \to \infty. \end{aligned} \tag{19}$$

Since by (19), $\left\langle \beta_{m-k}(T) T^k h_i, T^k h_i \right\rangle \to 0$ for $k \neq m$, therefore for any unit vector $x \in K$,

$$\begin{aligned} &\langle \beta_m(T \otimes S)(h_i \otimes x), h_i \otimes x \rangle \\ &= \sum_{k=0}^{m} \binom{m}{k} \left\langle T^{*k} \beta_{m-k}(T) T^k h_i \otimes \beta_k(S)x, h_i \otimes x \right\rangle \\ &= \sum_{k=0}^{m} \binom{m}{k} \left\langle \beta_{m-k}(T) T^k h_i, T^k h_i \right\rangle \otimes \langle \beta_k(S)x, x \rangle \\ &\to \langle \beta_m(S)x, x \rangle \ \text{ as } i \to \infty. \end{aligned}$$

Thus

$$\|\beta_m(T \otimes S)\| \geq |\langle \beta_m(T \otimes S)(h_i \otimes x), h_i \otimes x \rangle| \quad \text{and} \quad \|\beta_m(T \otimes S)\| \geq |\langle \beta_m(S)x, x \rangle| \,.$$

Since $\beta_m(S)$ is a self-adjoint operator, we have

$$\|\beta_m(T \otimes S)\| \geq \|\beta_m(S)\| \,.$$

This implies that if $\limsup_{m\to\infty} \|\beta_m(T \otimes S)\|^{1/m} = 0$, then $\limsup_{m\to\infty} \|\beta_m(S)\|^{1/m} = 0$. So, S is an ∞-isometry on H. ■

A little reflection yields the following interesting corollary.

Corollary 5.5 *Let $T \in B(H)$ and $S \in B(K)$. Then for $m \geq 1$*

$$\|\beta_m(T \otimes S)\| \geq \|\beta_m(r(T)S)\| \ \text{ and } \ \|\beta_m(T \otimes S)\| \geq \|\beta_m(r(S)T)\| \,.$$

We remark that several previous results for ∞-isometries also have corresponding results for ∞-unitaries. We state one of them.

Theorem 5.6 *Let $T \in B(H)$ and $S \in B(K)$. Then $T \otimes S$ on $H \otimes K$ is an ∞-unitary if and only if both αT and S/α are ∞-unitary on H and K respectively for some constant α.*

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