Matrix Functions of Bounded Type: An Interplay Between Function Theory and Operator Theory

Raúl E. Curto
In Sung Hwang
Woo Young Lee

Contents

Abstract

In this paper, we study matrix functions of bounded type from the viewpoint of describing an interplay between function theory and operator theory. We first establish a criterion on the coprime-ness of two singular inner functions and obtain several properties of the Douglas-Shapiro-Shields factorizations of matrix functions of bounded type. We propose a new notion of tensored-scalar singularity, and then answer questions on Hankel operators with matrix-valued bounded type symbols. We also examine an interpolation problem related to a certain functional equation on matrix functions of bounded type; this can be seen as an extension of the classical Hermite-Fejér Interpolation Problem for matrix rational functions. We then extend the H^∞-functional calculus to an $\overline{H^\infty} + H^\infty$-functional calculus for the compressions of the shift. Next, we consider the subnormality of Toeplitz operators with matrix-valued bounded type symbols and, in particular, the matrix-valued version of Halmos's Problem 5; we then establish a matrix-valued version of Abrahamse's Theorem. We also solve a subnormal Toeplitz completion problem of 2×2 partial block Toeplitz matrices. Further, we establish a characterization of hyponormal Toeplitz pairs with matrix-valued bounded type symbols, and then derive rank formulae for the self-commutators of hyponormal Toeplitz pairs.

2010 *Mathematics Subject Classification.* Primary 30J05, 30H10, 30H15, 47A13, 47A56, 47B20, 47B35; Secondary 30J10, 30J15, 30H35, 47A20, 47A57

Key words. Functions of bounded type, matrix functions of bounded type, coprime inner functions, Douglas-Shapiro-Shields factorizations, tensored-scalar singularity, compressions of the shifts, H^∞-functional calculus, interpolation problems, Toeplitz operators, Hankel operators, subnormal, hyponormal, jointly hyponormal, Halmos' Problem 5, Abrahamse's Theorem, Toeplitz pairs.

The work of the first named author was partially supported by NSF Grant DMS-1302666. The work of the second named author was supported by NRF(Korea) grant No. 2016R1A2B4012378. The work of the third named author was supported by NRF(Korea) grant No. 2015R1D1A1A01058036.

Affiliations:

Raúl E. Curto: Department of Mathematics, University of Iowa, Iowa City, IA 52242, U.S.A. email: raul-curto@uiowa.edu

In Sung Hwang: Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea, email: ihwang@skku.edu

Woo Young Lee: Department of Mathematics, Seoul National University, Seoul 151-742, Korea, email: wylee@snu.ac.kr

CHAPTER 1

Introduction

A function $\varphi \in L^\infty$ is said to be of bounded type or in the Nevanlinna class if φ can be written as the quotient of two functions in H^∞. This class of functions has been extensively studied in the literature. However, it seems to be quite difficult to determine whether the given function φ is of bounded type if we only look at its Fourier series expansion. The well known criterion for detecting "bounded type" employs Hankel operators - the function φ is of bounded type if and only if the kernel of the Hankel operator H_φ with symbol φ is nonzero (cf. [**Ab**]). The class of functions of bounded type plays an important role in the study of function theory and operator theory. Indeed, for functions of bounded type, there is a nice connection between function theory and operator theory. In 1970, P.R. Halmos posed the following problem, listed as Problem 5 in his lectures "Ten problems in Hilbert space" [**Hal1**], [**Hal2**]: Is every subnormal Toeplitz operator either normal or analytic ?

Many authors have given partial answers to Halmos' Problem 5. In 1984, Halmos' Problem 5 was answered in the negative by C. Cowen and J. Long [**CoL**] - they found a symbol not of bounded type which is non-analytic and induces a non-normal subnormal Toeplitz operator. To date, researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. The most interesting partial answer to Halmos's Problem 5 was given by M.B. Abrahamse [**Ab**] - every subnormal Toeplitz operator T_φ whose symbol φ is such that φ or $\overline{\varphi}$ is of bounded type is either normal or analytic. Besides that, there are several fruitful interplays between function theory and operator theory, for functions of bounded type.

In the present paper we explore matrix functions of bounded type in $L^\infty_{M_n}$ (the space of $n \times n$ matrix-valued bounded measurable functions on the unit circle); that is, matrix functions whose entries are of bounded type. We concentrate on the connections between function theory and operator theory.

For the function-theoretic aspects, we focus on coprime inner functions, the Douglas-Shapiro-Shields factorization, tensored-scalar singularity, an interpolation problem, and a functional calculus. First of all, we consider the following intrinsic question: How does one determine the coprime-ness of two inner functions in H^∞? This question is easy for Blaschke products. Thus we are interested in the following question: When are two singular inner functions coprime? Naturally, a measure-theoretic problem arises at once since singular inner functions correspond to their singular measures. We answer this question in Chapter 3. As we may expect, we will show that two singular inner functions are coprime if and only if the corresponding singular measures are mutually singular (cf. Theorem 3.7). To prove this, we use a notion of infimum of finite positive Borel measures on the unit circle $\mathbb{T}$, and more generally, on a locally compact Hausdorff space. The key point of this argument is to decide when is the infimum of two finite positive Borel measures nonzero (cf. Theorem 3.3).

To properly understand matrix functions of bounded type, we need to factorize those functions into coprime products of matrix inner functions and the adjoints of matrix H^∞-functions. In general, every matrix function of bounded type can be represented by a left or a right coprime factorization - the so-called Douglas-Shapiro-Shields factorization (cf. Remark 4.2). In particular, this factorization is very helpful and somewhat inevitable for the study of Hankel and Toeplitz operators with such symbols. In Chapter 4, we consider several properties of left or right coprime factorizations for matrix functions of bounded type. First of all, we consider the following question: If A is a matrix H^∞-function and Θ is a matrix inner function, does it follow that A and Θ are left coprime if and only if A and Θ are right coprime? In other words, do the notions of "left" coprime-ness and "right" coprime-ness coincide for A and Θ? The answer to this question is negative in general. However, we can show that the answer is affirmative if Θ is diagonal-constant, that is, Θ is a diagonal inner function, constant along the diagonal (cf. Theorem 4.16). We also show that if Φ is a matrix H^∞-function such that Φ^* is of bounded type and its determinant is not identically zero then the degree of the inner part of its inner-outer factorization is less than or equal to the degree of the inner part of its left coprime factorization (cf. Corollary 4.34). In fact, the degree of the inner part of the left coprime factorization is equal to the degree of the inner part of the right coprime factorization (cf. Lemma 4.31).

On the other hand, it is well known that the composition of two inner functions is again an inner function. But we cannot guarantee that the composition of two Blaschke products is again a Blaschke product. Thus, we are interested in the question: If θ and δ are coprime finite Blaschke products and ω is an inner function, are the compositions $\theta\circ\omega$ and $\delta\circ\omega$ coprime? We prove that the answer is affirmative when the common zeros of θ and δ lie in some "range set" of ω at its singularity - almost everywhere in the open unit disk (cf. Theorem 4.25). Moreover, we show that if Φ is a matrix rational function whose determinant is not identically zero, then for a finite Blaschke product ω the inner part of the right coprime factorization of the composition $\Phi \circ \omega$ is exactly the composition of the inner part of Φ and ω (cf. Theorem 4.38); this is still true for matrix functions of bounded type if ω is a Blaschke factor (cf. Theorem 4.40).

We next ask the question: How does one define a singularity for matrix functions of bounded type? Conventionally, the singularity (or the existence of a pole) of matrix L^∞-functions is defined by a singularity (or a pole) of some entry of the matrix function (cf. [**BGR**], [**BR**]). However, we propose another notion of singularity which is more suitable for our study of Hankel and Toeplitz operators. In Chapter 5, we give a new notion, that of "tensored-scalar singularity." This new definition uses the Hankel operator as a characterization of functions of bounded type via the kernel of the Hankel operator. This notion provides an answer to the question: Under which conditions does it follow that if the product of two Hankel operators with matrix-valued bounded type symbols is zero then either of them is zero? It is well known that the answer to this question for scalar-valued cases is affirmative, but is negative for matrix-valued cases unless certain assumptions are made about the symbols. Here we show that if either of the two symbols has a tensored-scalar singularity then the answer is affirmative (cf. Theorem 5.4). We also consider the question: If Φ and Ψ are matrix functions of bounded type, when is $H_\Phi^* H_\Phi = H_\Psi^* H_\Psi$ (where H_Φ, H_Ψ are Hankel operators)? We answer this question as follows: If Φ or Ψ has a tensored-scalar singularity then two products are equal only when the co-analytic parts of Φ and Ψ coincide up to a unitary constant left factor (cf. Theorem 5.5).

As transitional aspects from function theory to operator theory, in Chapter 6 we consider an interpolation problem for matrix functions of bounded type and a functional calculus for the compressions of the shift operator. We consider an interpolation problem involving the following matrix-valued functional equation: When is $\Phi - K\Phi^*$ a matrix H^∞-function (where Φ is a matrix L^∞-function and K is an unknown matrix H^∞-function)? In other words, when does there exist a matrix H^∞-function K such that $\Phi - K\Phi^*$ is a matrix H^∞-function? If Φ is a matrix-valued *rational* function, this interpolation problem reduces to the classical Hermite-Fejér Interpolation Problem. We here examine this interpolation problem for the matrix functions of bounded type (cf. Theorems 6.4 and 6.5). On the other hand, it is well known that the functional calculus for polynomials of the compressions of the shift results in the Hermite-Fejér matrix via the classical Hermite-Fejér Interpolation Problem. We also extend the polynomial calculus to the H^∞-functional calculus (the so-called Sz.-Nagy-Foiaş functional calculus) via the Triangularization Theorem, and then extend it to a $\overline{H^\infty} + H^\infty$-functional calculus for the compressions of the shift.

Chapters 7 - 9 are devoted to operator-theoretic aspects. In Chapter 7 we consider the subnormality of Toeplitz operators with matrix-valued symbols and, in particular, the matrix-valued version of Halmos's Problem 5 ([**Hal1**], [**Hal2**]): Which subnormal Toeplitz operators with matrix-valued symbols are either normal or analytic? In 1976, M.B. Abrahamse [**Ab**] showed that if $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type, if T_φ is hyponormal, and if the kernel of the self-commutator of T_φ is invariant under T_φ then T_φ is either normal or analytic. The aim of this chapter is to establish a matrix-valued version of Abrahamse's Theorem. In fact, a straightforward matrix-valued version of Abrahamse's Theorem may fail. Recently, it was shown in [**CHL1**] that if Φ and Φ^* are matrix functions of bounded type with the constraint that the inner part of the right coprime factorization of the co-analytic part Φ_- is diagonal-constant, then a matrix-valued version of Abrahamse's Theorem holds for T_Φ. Also, it was shown in [**CHKL**] that if Φ is a matrix-valued *rational* function then the above "diagonal-constant" condition can be weakened to the condition of "having a nonconstant diagonal-constant inner divisor." From the results of Chapter 5 (cf. Lemma 5.2), we can see that those conditions of [**CHL1**] and [**CHKL**] are special cases of the condition of "having a tensored-scalar singularity." Indeed, in Chapter 7, we will show that if the symbol has a tensored-scalar singularity then we get a full-fledged matrix-valued version of Abrahamse's Theorem (cf. Theorem 7.3). In particular, if the symbol is scalar-valued then it vacuously has a tensored-scalar singularity, so that the matrix-valued version reduces to the original Abrahamse's Theorem.

Given a partially specified operator matrix with some known entries, the problem of finding suitable operators to complete the given partial operator matrix so that the resulting matrix satisfies certain given properties is called a *completion problem.* A subnormal completion of a partial operator matrix is a particular specification of the unspecified entries resulting in a subnormal operator. In Chapter 8, we solve the following "subnormal Toeplitz completion" problem: find the unspecified Toeplitz entries of the partial block Toeplitz matrix

$$A := \begin{pmatrix} T_{\overline{b}_\alpha} & ? \\ ? & T_{\overline{b}_\beta} \end{pmatrix} \qquad (\alpha, \beta \in \mathbb{D})$$

so that A becomes subnormal, where b_λ is a Blaschke factor whose zero is λ. We can here show that the unspecified entries ? have symbols that are matrix functions of bounded type. Thus this problem reduces to a problem on the subnormality of "bounded type" Toeplitz operators. Recently, in [**CHL2**], we have considered this completion problem for the cases $\alpha = \beta = 0$. The solution given in [**CHL2**, Theorem 5.1] relies upon very intricate and long computations using the symbol involved. However our solution in this chapter provides a shorter and more insightful proof by employing the results of the previous chapter. Our solution also shows that 2-hyponormality, subnormality and normality coincide for this completion, except in a special case (cf. Theorem 8.2).

On the other hand, normal Toeplitz operators were characterized by a property of their symbols in the early 1960's by A. Brown and P.R. Halmos [**BH**]. The exact nature of the relationship between the symbol $\varphi \in L^\infty$ and the hyponormality of the Toeplitz operator T_φ was understood in 1988 via Cowen's Theorem [**Co2**] – this elegant and useful theorem has been used in the works [**CuL1**], [**CuL2**], [**FL**], [**Gu1**], [**Gu2**], [**GS**], [**HKL1**], [**HKL2**], [**HL1**], [**HL2**], [**HL3**], [**Le**], [**NT**], [**Zhu**], and others; these works have been devoted to the study of hyponormality for Toeplitz operators on H^2. Particular attention has been paid to Toeplitz operators with polynomial symbols or rational symbols [**HL1**], [**HL2**], [**HL3**]. However, the case of arbitrary symbol φ, though solved in principle by Cowen's theorem, is in practice very complicated. Indeed, it may not even be possible to find tractable necessary and sufficient condition for the hyponormality of T_φ in terms of the Fourier coefficients of the symbol φ unless certain assumptions are made about it. To date, tractable criteria for the cases of trigonometric polynomial symbols and rational symbols have been derived from a Carathéodory-Schur interpolation problem [**Zhu**] and a tangential Hermite-Fejér interpolation problem [**Gu1**] or the classical Hermite-Fejér interpolation problem [**HL3**]. Recently, C. Gu, J. Hendricks and D. Rutherford [**GHR**] have considered the hyponormality of Toeplitz operators with matrix-valued symbols and characterized it in terms of their symbols. In particular they showed that if T_Φ is a hyponormal Toeplitz operator with matrix-valued symbol Φ, then its symbol Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$. Their characterization resembles Cowen's Theorem except for an additional condition – the normality of the symbol.

In 1988, two important developments took place in the field of operator theory. The first one was the introduction of the notion of "joint hyponormality" for n-tuples of operators and the second one was the characterization of hyponormality of Toeplitz operators in terms of their symbols (via Cowen's Theorem) as we have remarked just above. Since then, it has become natural to consider joint hyponormality for tuples of Toeplitz operators. The notion of joint hyponormality was first formally introduced by A. Athavale [**At**]. He conceived joint hyponormality as a notion at least as strong as requiring that the linear span of the operator coordinates consist of hyponormal operators, the latter notion being called weak joint hyponormality. Joint hyponormality and weak joint hyponormality have been studied by

A. Athavale [**At**], J. Conway and W. Szymanski [**CS**], R. Curto [**Cu**], R. Curto and W.Y. Lee [**CuL1**], R. Curto, P. Muhly, and J. Xia [**CMX**], R. Douglas, V. Paulsen and K. Yan [**DPY**], R. Douglas and K. Yan [**DY**], D. Farenick and R. McEachin [**FM**], C. Gu [**Gu2**], S. McCullough and V. Paulsen [**McCP1**],[**McCP2**], D. Xia [**Xi**], and others. Joint hyponormality originated from questions about commuting normal extensions of commuting operators, and it has also been considered with an aim at understanding the gap between hyponormality and subnormality for single operators. The study of jointly hyponormal Toeplitz tuples started with D. Farenick and R. McEachin [**FM**]. They studied operators that form jointly hyponormal pairs in the presence of the unilateral shift: precisely, they showed that if U is the unilateral shift on the Hardy space H^2, then the joint hyponormality of the pair (U, T) implies that T is necessarily a Toeplitz operator. This result invites us to consider the joint hyponormality for pairs of Toeplitz operators. In Chapter 9, we consider (jointly) hyponormal Toeplitz pairs with matrix-valued bounded type symbols. In their research monograph [**CuL1**], the authors studied hyponormality of pairs of Toeplitz operators (called Toeplitz pairs) when both symbols are trigonometric polynomials. The core of the main result of [**CuL1**] is that the hyponormality of $\mathbf{T} \equiv (T_\varphi, T_\psi)$ (φ, ψ trigonometric polynomials) forces that the co-analytic parts of φ and ψ necessarily coincide up to a constant multiple, i.e.,

$$\varphi - \beta\psi \in H^2 \quad \text{for some } \beta \in \mathbb{C}. \tag{1.1}$$

In [**HL4**], (9.1) was extended for Toeplitz pairs whose symbols are rational functions with some constraint. As a result, the following question arises at once: Does (1.1) still hold for Toeplitz pairs whose symbols are *matrix-valued* trigonometric polynomials or rational functions?

Chapter 9 is concerned with this question. More generally, we give a characterization of hyponormal Toeplitz pairs with bounded type symbols by using the theory established in the previous chapters. Consequently, we will answer the above question (cf. Corollary 9.22). Indeed, (1.1) is still true for matrix-valued trigonometric polynomials under some invertibility and commutativity assumptions on the Fourier coefficients of the symbols (those assumptions always hold vacuously for scalar-valued cases). In fact, this follows from our core idea (Lemma 9.16) that if Φ and Ψ are matrix functions of bounded type whose inner parts of right coprime factorizations of analytic parts commute and whose co-analytic parts have a common tensored-scalar pole then the hyponormality of the Toeplitz pair (T_Φ, T_Ψ) implies that the common tensored-scalar pole has the same order. Consequently, we give a characterization of the (joint) hyponormality of Toeplitz pairs with bounded type symbols (cf. Theorem 9.20). We also consider the self-commutators of the Toeplitz pairs with matrix-valued rational symbols and derive rank formulae for them (cf. Theorem 9.29).

Chapter 10 is devoted to concluding remarks and open questions.

Acknowledgments. The authors are deeply grateful to the referee for many helpful suggestions, which significantly improved both the content and the presentation. In particular, we are indebted to the referee for supplying the present version of the proof of Lemma 4.13, which is more transparent than the original proof.

CHAPTER 2

Preliminaries

The main ingredients of this paper are functions of bounded type, Hankel operators, Toeplitz operators, and hyponormality. First of all, we review the notion of functions of bounded type and a few essential facts about Hankel and Toeplitz operators, and for that we will use [**Ab**], [**BS**], [**Do1**], [**GGK**], [**MAR**], [**Ni2**] and [**Pe**]. Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, and write $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$. For $A, B \in \mathcal{B}(\mathcal{H})$, we let $[A,B] := AB - BA$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $[T^*,T]=0$, hyponormal if $[T^*,T] \geq 0$. For an operator $T \in \mathcal{B}(\mathcal{H})$, we write $\ker T$ and $\operatorname{ran} T$ for the kernel and the range of T, respectively. For a subset $\mathcal{M}$ of a Hilbert space $\mathcal{H}$, $\operatorname{cl}\mathcal{M}$ and $\mathcal{M}^\perp$ denote the closure and the orthogonal complement of $\mathcal{M}$, respectively. Also, let $\mathbb{T} \equiv \partial\mathbb{D}$ be the unit circle (where $\mathbb{D}$ denotes the open unit disk in the complex plane $\mathbb{C}$). Recall that $L^\infty \equiv L^\infty(\mathbb{T})$ is the set of bounded measurable functions on $\mathbb{T}$, that the Hilbert space $L^2 \equiv L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2 \equiv H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n \geq 0\}$. An element $f \in L^2$ is said to be analytic if $f \in H^2$. Let $H^\infty := L^\infty \cap H^2$, i.e., H^∞ is the set of bounded analytic functions on $\mathbb{D}$. Given a function $\varphi \in L^\infty$, the Toeplitz operator T_φ and the Hankel operator H_φ with symbol φ on H^2 are defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi g := JP^\perp(\varphi g) \qquad (g \in H^2), \tag{2.1}$$

where P and $P^\perp$ denote the orthogonal projections from L^2 onto H^2 and $(H^2)^\perp$, respectively, and J denotes the unitary operator from L^2 onto L^2 defined by $J(f)(z) = \overline{z} f(\overline{z})$ for $f \in L^2$.

For $\varphi \in L^\infty$, we write

$$\varphi_+ \equiv P\varphi \in H^2 \quad \text{and} \quad \varphi_- \equiv \overline{P^\perp \varphi} \in zH^2.$$

Thus we may write $\varphi = \overline{\varphi_-} + \varphi_+$.

We recall that a function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class $\mathcal{N}$) if there are functions $\psi_1, \psi_2 \in H^\infty$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

Let BMO denote the set of functions of bounded mean oscillation in L^1. It is well-known that $L^\infty \subseteq BMO \subseteq L^2$ and that if $f \in L^2$, then H_f is bounded on H^2 if and only if $P^\perp f \in BMO$ (where $P^\perp$ is the orthogonal projection of L^2 onto $(H^2)^\perp$) (cf. [**Pe**]). Thus if $\varphi \in L^\infty$, then $\overline{\varphi_-}, \overline{\varphi_+} \in BMO$, so that $H_{\overline{\varphi_-}}$ and $H_{\overline{\varphi_+}}$ are well understood. We recall [**Ab**, Lemma 3] that if $\varphi \in L^\infty$ then

$$\varphi \text{ is of bounded type} \iff \ker H_\varphi \neq \{0\}\,. \tag{2.2}$$

Assume now that both φ and $\overline{\varphi}$ are of bounded type. Then from the Beurling's Theorem, $\ker H_{\overline{\varphi_-}} = \theta_0 H^2$ and $\ker H_{\overline{\varphi_+}} = \theta_+ H^2$ for some inner functions θ_0, θ_+. We thus have $b := \overline{\varphi_-}\theta_0 \in H^2$, and hence we can write

$$\varphi_- = \theta_0 \overline{b} \quad \text{and similarly } \varphi_+ = \theta_+ \overline{a} \text{ for some } a \in H^2. \tag{2.3}$$

By Kronecker's Lemma [**Ni2**, p. 217], if $f \in H^\infty$ then $\overline{f}$ is a rational function if and only if $\operatorname{rank} H_{\overline{f}} < \infty$, which implies that

$$\overline{f} \text{ is rational } \iff f = \theta\overline{b} \ \text{ with a finite Blaschke product } \theta. \tag{2.4}$$

If T_φ is hyponormal then since $T_{\varphi\psi} - T_\varphi T_\psi = H^*_{\overline{\varphi}} H_\psi$ $(\varphi, \psi \in L^\infty)$ and hence,

$$[T^*_\varphi, T_\varphi] = H^*_{\overline{\varphi}} H_{\overline{\varphi}} - H^*_\varphi H_\varphi = H^*_{\overline{\varphi_+}} H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}} H_{\overline{\varphi_-}}, \tag{2.5}$$

it follows that $||H_{\overline{\varphi_+}} f|| \geq ||H_{\overline{\varphi_-}} f||$ for all $f \in H^2$, and hence

$$\theta_+ H^2 = \ker H_{\overline{\varphi_+}} \subseteq \ker H_{\overline{\varphi_-}} = \theta_0 H^2,$$

which implies that θ_0 divides θ_+, i.e., $\theta_+ = \theta_0\theta_1$ for some inner function θ_1. For an inner function θ, we write

$$\mathcal{H}(\theta) := H^2 \ominus \theta H^2.$$

Note that if $f = \theta\overline{a} \in L^2$, then $f \in H^2$ if and only if $a \in \mathcal{H}(z\theta)$; in particular, if $f(0) = 0$ then $a \in \mathcal{H}(\theta)$. Thus, if $\varphi \equiv \overline{\varphi_-} + \varphi_+ \in L^\infty$ is such that φ and $\overline{\varphi}$ are of bounded type such that $\varphi_+(0) = 0$ and T_φ is hyponormal, then we can write

$$\varphi_+ = \theta_0\theta_1\overline{a} \quad \text{and} \quad \varphi_- = \theta_0\overline{b}, \qquad \text{where } a \in \mathcal{H}(\theta_0\theta_1) \text{ and } b \in \mathcal{H}(\theta_0).$$

We turn our attention to the case of matrix functions.

Let $M_{n\times r}$ denote the set of all $n \times r$ complex matrices and write $M_n := M_{n\times n}$. For $\mathcal{X}$ a Hilbert space, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$ and let $L^\infty_{\mathcal{X}} \equiv L^\infty_{\mathcal{X}}(\mathbb{T})$ be the set of $\mathcal{X}$-valued bounded measurable functions on $\mathbb{T}$. We also let $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy space and $H^\infty_{\mathcal{X}} \equiv H^\infty_{\mathcal{X}}(\mathbb{T}) = L^\infty_{\mathcal{X}} \cap H^2_{\mathcal{X}}$. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$.

Definition 2.1. For a matrix-valued function $\Phi \equiv \big(\varphi_{ij}\big) \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry φ_{ij} is of bounded type, and we say that Φ is *rational* if each entry φ_{ij} is a rational function.

Let $\Phi \equiv \big(\varphi_{ij}\big) \in L^\infty_{M_n}$ be such that Φ^* is of bounded type. Then each $\overline{\varphi}_{ij}$ is of bounded type. Thus in view of (2.3), we may write $\varphi_{ij} = \theta_{ij}\overline{b}_{ij}$, where θ_{ij} is inner and θ_{ij} and b_{ij} are coprime, in other words, there does not exist a nonconstant inner divisor of θ_{ij} and b_{ij}. Thus if θ is the least common multiple of $\{\theta_{ij} : i, j = 1, 2, \cdots, n\}$, then we may write

$$\Phi = \big(\varphi_{ij}\big) = \big(\theta_{ij}\overline{b}_{ij}\big) = \big(\theta\overline{a}_{ij}\big) \equiv \theta A^* \quad (\text{where } A \equiv \big(a_{ji}\big) \in H^2_{M_n}). \tag{2.6}$$

In particular, $A(\alpha)$ is nonzero whenever $\theta(\alpha) = 0$ and $|\alpha| < 1$.

For $\Phi \in L^\infty_{M_n}$, we write

$$\Phi_+ := P_n(\Phi) \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \big[P_n^\perp(\Phi)\big]^* \in H^2_{M_n}.$$

Thus we may write $\Phi = \Phi_-^* + \Phi_+$. However, it will often be convenient to allow the constant term in Φ_-. Hence, if there is no confusion we may assume that Φ_- shares the constant term with Φ_+: in this case, $\Phi(0) = \Phi_+(0) + \Phi_-(0)^*$. If

$\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type, then in view of (2.6), we may write

$$\Phi_+ = \theta_1 A^* \quad \text{and} \quad \Phi_- = \theta_2 B^*, \tag{2.7}$$

where θ_1 and θ_2 are inner functions and $A, B \in H^2_{M_n}$. In particular, if $\Phi \in L^\infty_{M_n}$ is rational then the θ_i can be chosen as finite Blaschke products, as we observed in (2.4).

We now introduce the notion of Hankel operators and Toeplitz operators with matrix-valued symbols. If Φ is a matrix-valued function in $L^\infty_{M_n}$, then $T_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ denotes Toeplitz operator with symbol Φ defined by

$$T_\Phi f := P_n(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where P_n is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$. A Hankel operator with symbol $\Phi \in L^\infty_{M_n}$ is an operator $H_\Phi : H^2_{\mathbb{C}^n} \to H^2_{\mathbb{C}^n}$ defined by

$$H_\Phi f := J_n P_n^\perp(\Phi f) \quad \text{for } f \in H^2_{\mathbb{C}^n},$$

where $P_n^\perp$ is the orthogonal projection of $L^2_{\mathbb{C}^n}$ onto $(H^2_{\mathbb{C}^n})^\perp$ and J_n denotes the unitary operator from $L^2_{\mathbb{C}^n}$ onto $L^2_{\mathbb{C}^n}$ given by $J_n(f)(z) := \overline{z} f(\overline{z})$ for $f \in L^2_{\mathbb{C}^n}$. For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}).$$

A matrix-valued function $\Theta \in H^\infty_{M_{n\times m}}$ is called *inner* if $\Theta^*\Theta = I_m$ almost everywhere on $\mathbb{T}$, where I_m denotes the $m \times m$ identity matrix. If there is no confusion we write simply I for I_m. The following basic relations can be easily derived:

$$T_\Phi^* = T_{\Phi^*}, \quad H_\Phi^* = H_{\widetilde{\Phi}} \quad (\Phi \in L^\infty_{M_n}); \tag{2.8}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad (\Phi, \Psi \in L^\infty_{M_n}); \tag{2.9}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}, \quad H_{\Psi\Phi} = T^*_{\widetilde{\Psi}} H_\Phi \quad (\Phi \in L^\infty_{M_n}, \Psi \in H^\infty_{M_n}); \tag{2.10}$$

$$H_\Phi^* H_\Phi - H^*_{\Theta\Phi} H_{\Theta\Phi} = H_\Phi^* H_{\Theta^*} H^*_{\Theta^*} H_\Phi \quad (\Theta \in H^\infty_{M_n} \text{ inner}, \Phi \in L^\infty_{M_n}). \tag{2.11}$$

A matrix-valued trigonometric polynomial $\Phi \in L^\infty_{M_{n\times m}}$ is of the form

$$\Phi(z) = \sum_{j=-m}^{N} A_j z^j \quad (A_j \in M_{n\times m}),$$

where A_N and A_{-m} are called the *outer* coefficients of Φ. For matrix-valued functions $A := \sum_{j=-\infty}^{\infty} A_j z^j \in L^2_{M_{n\times m}}$ and $B := \sum_{j=-\infty}^{\infty} B_j z^j \in L^2_{M_{n\times m}}$, we define the inner product of A and B by

$$\langle A, B \rangle := \int_{\mathbb{T}} \operatorname{tr}(B^* A)\, d\mu = \sum_{j=-\infty}^{\infty} \operatorname{tr}(B_j^* A_j),$$

where $\operatorname{tr}(\cdot)$ denotes the trace of a matrix and define $||A||_2 := \langle A, A \rangle^{\frac{1}{2}}$. We also define, for $A \in L^\infty_{M_{n\times m}}$,

$$||A||_\infty := \text{ess sup}_{t\in\mathbb{T}} ||A(t)|| \quad (||\cdot|| \text{ denotes the spectral norm of a matrix}).$$

Finally, the *shift* operator S on $H^2_{\mathbb{C}^n}$ is defined by $S := T_{zI}$.

The following fundamental result will be useful in the sequel.

The Beurling-Lax-Halmos Theorem. ([**FF**], [**Ni2**]) *A nonzero subspace M of $H^2_{\mathbb{C}^n}$ is invariant for the shift operator S on $H^2_{\mathbb{C}^n}$ if and only if $M = \Theta H^2_{\mathbb{C}^m}$, where*

Θ is an inner matrix function in $H^\infty_{M_{n\times m}}$ $(m \le n)$. Furthermore, Θ is unique up to a unitary constant right factor; that is, if $M = \Delta H^2_{\mathbb{C}^r}$ (for Δ an inner function in $H^\infty_{M_{n\times r}}$), then $m = r$ and $\Theta = \Delta W$, where W is a unitary matrix mapping $\mathbb{C}^m$ onto $\mathbb{C}^m$.

As customarily done, we say that two matrix-valued functions A and B are *equal* if they are equal up to a unitary constant right factor. Observe that, by (2.10), for $\Phi \in L^\infty_{M_n}$,

$$H_\Phi S = H_{\Phi\cdot zI} = H_{zI\cdot\Phi} = S^* H_\Phi,$$

which implies that the kernel of a Hankel operator H_Φ is an invariant subspace of the shift operator on $H^2_{\mathbb{C}^n}$. Thus, if $\ker H_\Phi \ne \{0\}$, then by the Beurling-Lax-Halmos Theorem,

$$\ker H_\Phi = \Theta H^2_{\mathbb{C}^m}$$

for some inner matrix function Θ. We note that Θ need not be a square matrix.

However, we have:

Lemma 2.2. [**GHR**, Theorem 2.2] *For $\Phi \in L^\infty_{M_n}$, the following are equivalent:*

(a) *Φ is of bounded type;*
(b) *$\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some square inner matrix function Θ.*

We recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be subnormal if T has a normal extension, i.e., $T = N|_{\mathcal{H}}$, where N is a normal operator on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\mathcal{H}$ is invariant for N. The Bram-Halmos criterion for subnormality ([**Br**], [**Con**]) states that an operator $T \in \mathcal{B}(\mathcal{H})$ is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \ge 0$ for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$. It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} [T^*,T] & [T^{*2},T] & \dots & [T^{*k},T] \\ [T^*,T^2] & [T^{*2},T^2] & \dots & [T^{*k},T^2] \\ \vdots & \vdots & \ddots & \vdots \\ [T^*,T^k] & [T^{*2},T^k] & \dots & [T^{*k},T^k] \end{pmatrix} \ge 0 \qquad (\text{all } k \ge 1)\,. \tag{2.12}$$

Condition (2.12) provides a measure of the gap between hyponormality and subnormality. In fact the positivity condition (2.12) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (2.12) for all k. For $k \ge 1$, an operator T is said to be *k-hyponormal* if T satisfies the positivity condition (2.12) for a fixed k. Thus the Bram-Halmos criterion can be stated as: T is subnormal if and only if T is k-hyponormal for all $k \ge 1$. The notion of k-hyponormality has been considered by many authors aiming at understanding the bridge between hyponormality and subnormality. In view of (2.12), between hyponormality and subnormality there exists a whole slew of increasingly stricter conditions, each expressible in terms of the joint hyponormality of the tuples $(I, T, T^2, \dots, T^k)$. Given an n-tuple $\mathbf{T} = (T_1, \dots, T_n)$ of operators on $\mathcal{H}$, we let $[\mathbf{T}^*, \mathbf{T}] \in \mathcal{B}(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$ denote the *self-commutator* of $\mathbf{T}$, defined by

$$[\mathbf{T}^*, \mathbf{T}] := \begin{pmatrix} [T_1^*,T_1] & [T_2^*,T_1] & \dots & [T_n^*,T_1] \\ [T_1^*,T_2] & [T_2^*,T_2] & \dots & [T_n^*,T_2] \\ \vdots & \vdots & \ddots & \vdots \\ [T_1^*,T_n] & [T_2^*,T_n] & \dots & [T_n^*,T_n] \end{pmatrix}.$$

By analogy with the case $n = 1$, we shall say ([**At**], [**CMX**]) that $\mathbf{T}$ is *jointly hyponormal* (or simply, *hyponormal*) if $[\mathbf{T}^*, \mathbf{T}]$ is a positive operator on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. $\mathbf{T}$ is said to be *normal* if $\mathbf{T}$ is commuting and every T_i is a normal operator, and *subnormal* if $\mathbf{T}$ is the restriction of a normal n-tuple to a common invariant subspace. Clearly, the normality, subnormality or hyponormality of an n-tuple requires as a necessary condition that every coordinate in the tuple be normal, subnormal or hyponormal, respectively. Normality and subnormality require that the coordinates commute, but hyponormality does not.

In 1988, the hyponormality of the Toeplitz operators T_φ was characterized in terms of their symbols φ via Cowen's Theorem [**Co2**], which follows.

Cowen's Theorem. ([**Co2**], [**NT**]) *For each $\varphi \in L^\infty$, let*

$$\mathcal{E}(\varphi) \equiv \Big\{ k \in H^\infty : \ ||k||_\infty \le 1 \ \textit{and} \ \varphi - k\overline{\varphi} \in H^\infty \Big\}.$$

Then T_φ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

To study hyponormality (resp. normality and subnormality) of the Toeplitz operator T_φ with symbol φ we may, without loss of generality, assume that $\varphi(0) = 0$; this is because hyponormality (resp. normality and subnormality) is invariant under translations by scalars.

In 2006, C. Gu, J. Hendricks and D. Rutherford [**GHR**] have considered the hyponormality of Toeplitz operators with matrix-valued symbols and characterized it in terms of their symbols. Their characterization resembles Cowen's Theorem except for an additional condition – the normality of the symbol.

Lemma 2.3. (Hyponormality of Block Toeplitz Operators) ([**GHR**]) *For each $\Phi \in L^\infty_{M_n}$, let*

$$\mathcal{E}(\Phi) := \Big\{ K \in H^\infty_{M_n} : \ ||K||_\infty \le 1 \quad \textit{and} \quad \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

M. Abrahamse [**Ab**, Lemma 6] showed that if T_φ is hyponormal, if $\varphi \notin H^\infty$, and if φ or $\overline{\varphi}$ is of bounded type then both φ and $\overline{\varphi}$ are of bounded type. However, in contrast to the scalar-valued case, Φ^* may not be of bounded type even though T_Φ is hyponormal, $\Phi \notin H^\infty_{M_n}$ and Φ is of bounded type. But we have a one-way implication:

$$T_\Phi \text{ is hyponormal and } \Phi^* \text{ is of bounded type} \implies \Phi \text{ is of bounded type} \tag{2.13}$$

(see [**GHR**, Corollary 3.5 and Remark 3.6]). Thus, whenever we deal with hyponormal Toeplitz operators T_Φ with symbols Φ satisfying that both Φ and Φ^* are of bounded type (e.g., Φ is a matrix-valued rational function), it suffices to assume that only Φ^* is of bounded type. In spite of this fact, for convenience, we will assume that Φ and Φ^* are of bounded type whenever we deal with bounded type symbols.

Notations

- Let θ be an inner function in H^∞. Then

  $I_\theta := \theta I \equiv \begin{pmatrix} \theta & & \\ & \ddots & \\ & & \theta \end{pmatrix}$ (where I is the identity matrix)

  $\mathcal{Z}(\theta) :=$ the set of all zeros of θ
- $b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z}$ ($\lambda \in \mathbb{D}$), a Blaschke factor
- $H_0^2 \equiv (H^2_{M_n})_0 := zH^2_{M_n}$
- Let $\Theta \in H^\infty_{M_{n\times m}}$ be an inner matrix function. Then

  $\mathcal{H}(\Theta) := H^2_{\mathbb{C}_n} \ominus \Theta H^2_{\mathbb{C}_m}$
  $\mathcal{H}_\Theta := H^2_{M_{n\times m}} \ominus \Theta H^2_{M_m}$
  $\mathcal{K}_\Theta := H^2_{M_{n\times m}} \ominus H^2_{M_n}\Theta$

  If $\Theta = I_\theta$ for an inner function θ, then $\mathcal{H}_\Theta = \mathcal{K}_\Theta$. If there is no confusion then we write, for brevity, $\mathcal{H}_\theta$, $\mathcal{K}_\theta$ for $\mathcal{H}_{I_\theta}$, $\mathcal{K}_{I_\theta}$.
- For $\mathcal{X}$ a closed subspace of $H^2_{M_n}$, $P_{\mathcal{X}}$ denotes the orthogonal projection from $H^2_{M_n}$ onto $\mathcal{X}$.
- If $\Phi \in L^\infty_{M_n}$ and Δ_1 and Δ_2 are inner matrix functions in $H^\infty_{M_n}$, we write

  $$\Phi_{\Delta_1,\Delta_2} := P_{(H^2_{M_n})^\perp}(\Phi_-^*\Delta_1) + P_{H_0^2}(\Delta_2^*\Phi_+);$$
  $$\Phi^{\Delta_1,\Delta_2} := P_{(H^2_{M_n})^\perp}(\Delta_1\Phi_-^*) + P_{H_0^2}(\Phi_+\Delta_2^*),$$

  and abbreviate

  $$\Phi_\Delta \equiv \Phi_{\Delta,\Delta} \quad \text{and} \quad \Phi^\Delta \equiv \Phi^{\Delta,\Delta}.$$

  If $\Delta_i := I_{\delta_i}$ for some inner functions δ_i $(i = 1, 2)$, then $\Phi_{\Delta_1,\Delta_2} = \Phi^{\Delta_1,\Delta_2}$. If there is no confusion then we write, for brevity,

  $$\Phi_{I_{\delta_1},\Delta_2} \equiv \Phi_{\delta_1,\Delta_2}, \quad \Phi_{\Delta_1,I_{\delta_2}} \equiv \Phi_{\Delta_1,\delta_2}, \quad \Phi^{I_\delta} \equiv \Phi^\delta \quad \text{and etc}$$

  (i.e., we write δ for I_δ in this representation).
- For an inner function θ, U_θ denotes the compression of the shift operator $U \equiv T_z$: i.e.,

  $$U_\theta = P_{\mathcal{H}(\theta)}U|_{\mathcal{H}(\theta)}.$$

  More generally, for $A \in L^\infty_{M_n}$ and an inner function $\Theta \in H^\infty_{M_n}$, we write

  $$(T_A)_\Theta = P_{\mathcal{H}(\Theta)}T_A|_{\mathcal{H}(\Theta)},$$

  which is called the compression of T_A to $\mathcal{H}(\Theta)$.

CHAPTER 3

Coprime inner functions

To understand functions of bounded type, we need to factorize those functions into a coprime product of an inner function and the complex conjugate of an H^2-function. Thus we are interested in the following question: When are two inner functions in H^∞ coprime? Naturally, a measure-theoretic problem arises at once, since singular inner functions correspond to their singular measures. In this chapter, we answer this question.

A nonzero sequence $\{\alpha_j\}$ in $\mathbb{D}$ satisfying $\sum_{j=1}^{\infty}(1-|\alpha_j|)<\infty$ is called a Blaschke sequence. If $\{\alpha_j\}$ is a Blaschke sequence and k is an integer, $k \geq 0$, then the function

$$b(z) := z^k \prod_{j=1}^{\infty} \frac{|\alpha_j|}{\alpha_j}\left(\frac{z-\alpha_j}{1-\overline{\alpha}_j z}\right)$$

is called a Blaschke product. The factor z^k in the definition of the Blaschke product is to allow b to have a zero at 0. If $\{\alpha_j\}$ is a finite sequence then b is called a finite Blaschke product.

Recall that an inner function $\theta \in H^\infty$ can be written as

$$\theta(z) = c\, b(z) \exp\left(-\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu(t)\right) \quad (z \in \mathbb{D}),$$

where c is a constant of modulus 1, b is a Blaschke product, and μ is a finite positive Borel measure on $\mathbb{T}$ which is singular with respect to Lebesgue measure. It is evident that if b_1 and b_2 are Blaschke products then

$$b_1 \text{ and } b_2 \text{ are not coprime} \iff \mathcal{Z}(b_1) \cap \mathcal{Z}(b_2) \neq \emptyset\,. \tag{3.1}$$

Thus the difficulty in determining coprime-ness of two inner functions is caused by an inner functions having no zeros in $\mathbb{D}$, which is called *a singular inner function*:

$$\theta(z) \equiv \exp\left(-\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu(t)\right),$$

where μ will be called the *singular measure* of θ. Thus we are interested in the following question:

(3.2) When are two singular inner functions coprime?

Question (3.2) seems to be well known to experts; however, we have not been able to find an answer in the literature. Thus, below we give an answer to Question (3.2).

Let μ be a Borel measure on a locally compact Hausdorff space X. Recall that the support of μ is defined as the set:

$$\text{supp}(\mu) := \Big\{x \in X : \mu(G) > 0 \text{ for every open neighborhood } G \text{ of } x\Big\}.$$

An equivalent definition of support of μ is as the largest closed set $S \subseteq X$ such that for every open subset U of X for which $S \cap U \neq \emptyset$,

$$\mu(S^c) = 0 \quad \text{and} \quad \mu(U \cap S) > 0. \tag{3.3}$$

On the other hand, if there is a Borel set A such that $\mu(E) = \mu(A \cap E)$ for every Borel set E, we say that μ is *concentrated on* A (cf. [**Ru**]). This is equivalent to the condition that $\mu(E) = 0$ whenever $E \cap A = \emptyset$. Also we say that two Borel measures μ and ν are *mutually singular* (and write $\mu \perp \nu$) if there are disjoint Borel sets A and B such that μ is concentrated on A and ν is concentrated on B.

We note that if $\operatorname{supp}(\mu) \cap \operatorname{supp}(\nu) = \emptyset$, then $\mu \perp \nu$. However the converse is not true: for example, if m is the Lebesgue measure on $[0, 1]$ and δ_1 is the Dirac measure at 1, then $m \perp \delta_1$, but $\operatorname{supp}(m) \cap \operatorname{supp}(\delta_1) = \{1\}$.

How does one define the infimum of a family of finite positive Borel measures? Let $\mathcal{S} \equiv \{\mu_1, \mu_2, \cdots\}$ be a countable family of finite positive Borel measures on a locally compact Hausdorff space X. For any Borel set E, define $\mu(E)$ by

$$\mu(E) := \inf \sum_k \mu_k(E_k), \tag{3.4}$$

where the μ_k runs through the family $\mathcal{S}$ and where $\{E_1, E_2, \cdots\}$ runs through all partitions of E into Borel sets (cf. [**Ga**, p.84]). Then we can show that μ is a positive Borel measure satisfying the following:

(i) $\mu \leq \mu_k$ for all $k = 1, 2, \cdots$;
(ii) μ is the maximum positive Borel measure on X satisfying (i) in the sense that: if λ is a positive Borel measure on X satisfying $\lambda \leq \mu_k$ for all $k = 1, 2, \cdots$, then $\lambda \leq \mu$.

The measure μ of (3.4) is called the *infimum* of the family $\mathcal{S}$ of measures, and we write

$$\mu := \inf_k (\mu_k).$$

We note that the infimum of two nonzero measures may be zero. For example, let $X = [0, 1]$ and suppose m is the Lebesgue measure on X and δ_1 is the Dirac measure at 1. Then $\mu(X) \equiv \inf(m, \delta_1)(X) \leq m(\{1\}) + \delta_1([0, 1)) = 0$.

THEOREM 3.1. *Let $\{\mu_k\}$ be a countable family of finite positive Borel measures on a locally compact Hausdorff space X. If μ_k is concentrated on A_k for each $k \in \mathbb{Z}_+$, then $\mu \equiv \inf_k (\mu_k)$ is concentrated on $\bigcap_n A_n$.*

PROOF. Write

$$A \equiv \bigcap_k A_k \quad \text{and} \quad \mu \equiv \inf_k (\mu_k).$$

Let E be a Borel set and suppose $\{E_n\}$ is a partition of E into Borel sets. Let

$$F_n := E_n \cap A \quad \text{and} \quad A_n^{(k)} := E_n \setminus A_k.$$

Then we can see that

$$E = \Big(\cup_n F_n\Big) \bigcup \Big(\cup_{n,k} A_n^{(k)}\Big).$$

Let $\{C_n\}$ be the collection consisting of $\{F_n, A_n^{(k)}\}_{n,k}$. We then have

$$\begin{aligned}\mu(E) &\le \inf \sum_k \mu_k(C_k)\\ &\le \inf \sum_k \mu_k(F_k) \qquad (3.5)\\ &= \inf \sum_k \mu_k(E_k \cap A).\end{aligned}$$

Since $E \cap A = \bigcup_n (E_n \cap A)$, it follows from (3.5) that

$$\mu(E) \le \mu(E \cap A).$$

This completes the proof. □

Corollary 3.2. *Let $\{\mu_k\}$ be a countable family of finite positive Borel measures on a locally compact Hausdorff space X. If $\mu_i \perp \mu_j$ for some i, j, then* $\inf_k (\mu_k) = 0$.

Proof. Immediate from Theorem 3.1. □

Theorem 3.3. *Let μ_1 and μ_2 be finite positive Borel measures on a Borel σ-algebra $\mathfrak{B}$ in a locally compact Hausdorff space X. Then*

$$\mu_1 \perp \mu_2 \Longleftrightarrow \inf(\mu_1, \mu_2) = 0.$$

Proof. ($\Rightarrow$) This follows from Theorem 3.1.

($\Leftarrow$) Suppose μ_1 and μ_2 are finite positive Borel measures. By the Lebesgue decomposition of μ_1 relative to μ_2, there exists a unique pair $\{\mu_a, \mu_s\}$ of finite positive measures on $\mathfrak{B}$ such that

$$\mu_1 = \mu_a + \mu_s, \quad \mu_a \ll \mu_2, \quad \mu_s \perp \mu_2.$$

Let $h \in L^1(\mu_2)$ be the Radon-Nikodym derivative of μ_a with respect to μ_2: that is,

$$\mu_a(E) = \int_E h\, d\mu_2 \quad (E \in \mathfrak{B}). \tag{3.6}$$

Note that h is a nonnegative measurable function. Assume that μ_1 and μ_2 are not mutually singular. Then $h \ne 0$ $[\mu_2]$. Define

$$H := \Big\{x \in X : 0 < h(x) \le 1\Big\}.$$

There are two cases to consider.

Case 1 ($\mu_2(H) \ne 0$): Define

$$\lambda(E) := \int_E h \cdot \chi_H \, d\mu_2 \quad (E \in \mathfrak{B}).$$

Since $h \cdot \chi_H$ is a nonnegative measurable function, it follows that λ is a positive measure. Observe that

$$\lambda(H) = \int_H h d\mu_2 > 0.$$

For each $E \in \mathfrak{B}$, we have, by (3.6),

$$\lambda(E) = \int_E h \cdot \chi_H d\mu_2 \le \int_E h d\mu_2 = \mu_a(E) \le \mu_1(E)$$

and by definition of H,

$$\lambda(E) = \int_E h \cdot \chi_H d\mu_2 \leq \int_E d\mu_2 = \mu_2(E).$$

But since $\lambda(H) > 0$, it follows that $0 \neq \lambda \leq \inf(\mu_1, \mu_2)$, which implies $\inf(\mu_1, \mu_2) \neq 0$.

Cases 2 ($\mu_2(H) = 0$): For $m = 2, 3, 4, \cdots$, define

$$H_m := \Big\{x \in X : 1 < h(x) \leq m\Big\}.$$

Since $h \neq 0$ $[\mu_2]$, there exists $N \in \{2, 3, \cdots\}$ such that $\mu_2(H_N) \neq 0$. Define

$$\lambda(E) := \frac{1}{N} \int_E h \cdot \chi_{H_N} d\mu_2 \quad (E \in \mathfrak{B}).$$

Then λ is a positive measure. Observe that

$$\lambda(H_N) = \frac{1}{N} \int_{H_N} h\, d\mu_2 > \frac{1}{N} \int_{H_N} d\mu_2 = \frac{1}{N} \mu_2(H_N) > 0.$$

For each $E \in \mathfrak{B}$,

$$\lambda(E) = \frac{1}{N} \int_E h \cdot \chi_{H_N} d\mu_2 \leq \frac{1}{N} \int_E h d\mu_2 \leq \int_E h d\mu_2 = \mu_a(E) \leq \mu_1(E),$$

and by definition of H_N,

$$\lambda(E) = \frac{1}{N} \int_E h \cdot \chi_{H_N} d\mu_2 \leq \int_E d\mu_2 = \mu_2(E).$$

Since $\lambda(H_N) > 0$, it follows that $0 \neq \lambda \leq \inf(\mu_1, \mu_2)$, which implies $\inf(\mu_1, \mu_2) \neq 0$. This completes the proof. □

Corollary 3.4. *Let $\{\mu_1, \mu_2, \cdots, \mu_m\}$ be a finite collection of finite positive Borel measures on a locally compact Hausdorff space X. Then the followings are equivalent:*

(i) $\mu \equiv \inf(\mu_1, \mu_2, \cdots, \mu_m) \neq 0$
(ii) *If μ_k is concentrated on A_k for each $k = 1, 2 \cdots, m$, then $\bigcap_{k=1}^m A_k \neq \emptyset$*

Proof. This follows at once from Theorems 3.1 and 3.3. □

Remark 3.5. Corollary 3.4 may fail for a *countable* collection of finite positive Borel measures. For example, let λ be a nonzero finite positive Borel measure on a locally compact Hausdorff space X and let

$$\mu_k := \frac{\lambda}{k} \quad (k \in \mathbb{Z}_+).$$

Then clearly, $\mu \equiv \inf_k(\mu_k) = 0$. Suppose that μ_k is concentrated on A_k for each $k \in \mathbb{Z}_+$. Then λ is concentrated on A_k for each $k \in \mathbb{Z}_+$. Write

$$E := \bigcap_{k=1}^{\infty} A_k \quad \text{and} \quad E_n := \bigcap_{k=1}^{n} A_k.$$

Then

$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots \quad \text{and} \quad E = \bigcap_{n=1}^{\infty} E_n.$$

We now claim that

$$\lambda(E_n) = \lambda(X) \quad \text{for each } n \in \mathbb{Z}_+. \tag{3.7}$$

To prove (3.7), we use mathematical induction. Since λ is concentrated on $A_1 = E_1$, it follows that $\lambda(E_1) = \lambda(E_1 \cap X) = \lambda(X)$. Suppose $\lambda(E_k) = \lambda(X)$ $(k \geq 1)$. Since λ is concentrated on A_{k+1}, we have

$$\lambda(E_{k+1}) = \lambda(E_k \cap A_{k+1}) = \lambda(E_k) = \lambda(X),$$

which proves (3.7). Since λ is a finite measure, $\lambda(X) = \lambda(E_n)$ converges to $\lambda(E)$, so that $\lambda(E) = \lambda(X) \neq 0$, and hence $E \neq \emptyset$. □

Theorem 3.6. *Let $\mu_1,\ \mu_2$ be finite positive regular Borel measures on a locally compact Hausdorff space X such that*

$$S \equiv \operatorname{supp}(\mu_1) \cap \operatorname{supp}(\mu_2) \neq \phi.$$

If there exists $x \in S$ and an open neighborhood N of x such that

$$m \leq \left\{ \frac{\mu_2(N_x)}{\mu_1(N_x)} : x \in N_x,\ \textit{an open subset of } N \right\} \leq M$$

for some $m, M > 0$, then $\mu \equiv \inf(\mu_1, \mu_2) \neq 0$.

Remark. Since $x \in S$, $\mu_i(N_x) \neq 0$ for each $i = 1, 2$.

Proof of Theorem 3.6. Suppose that there exists $x \in S$ and an open neighborhood N of x such that

$$m \leq \left\{ \frac{\mu_2(N_x)}{\mu_1(N_x)} : x \in N_x,\ \text{an open subset of } N \right\} \leq M,$$

for some $0 < m < 1 < M < \infty$. We will show that

$$\mu(N) \neq 0. \tag{3.8}$$

Note that μ_1 and μ_2 are regular Borel measures. Let G be a Borel set and $\epsilon > 0$. Suppose $x \in G$. Then there exists an open set $V \supseteq G$ such that

$$\mu_1(N \cap G) + \epsilon > \mu_1(N \cap V) \geq \frac{1}{M}\mu_2(N \cap V).$$

Thus

$$\begin{aligned}
\mu_1(N \cap G) + \mu_2(N \cap G^c) + \epsilon &\geq \frac{1}{M}\mu_2(N \cap V) + \mu_2(N \cap G^c) \\
&\geq \frac{1}{M}\Big(\mu_2(N \cap G) + \mu_2(N \cap G^c)\Big) \\
&\geq \frac{1}{M}\mu_2(N),
\end{aligned}$$

which gives

$$\mu_1(N \cap G) + \mu_2(N \cap G^c) \geq \frac{1}{M}\mu_2(N).$$

If $x \notin G$, then similarly, we have

$$\mu_1(N \cap G) + \mu_2(N \cap G^c) \geq m\,\mu_1(N).$$

Thus, it follows that

$$\mu(N) = \inf(\mu_1, \mu_2)(N) \geq \min\left(\frac{1}{M}\mu_2(N), m\mu_1(N)\right) > 0,$$

which proves (3.8). □

We now have:

THEOREM 3.7. *Let $\theta_1, \theta_2 \in H^\infty$ be singular inner functions with singular measures μ_1 and μ_2, respectively. Then θ_1 and θ_2 are coprime if and only if $\mu_1 \perp \mu_2$.*

PROOF. ($\Rightarrow$) For $i = 1, 2$, write

$$\theta_i(z) = \exp\left(-\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu_i(t)\right) \quad (z \in \mathbb{D}),$$

where each μ_i is a finite positive Borel measure on $\mathbb{T}$ which is singular with respect to Lebesgue measure. Assume that μ_1 and μ_2 are not mutually singular. Thus, by Theorem 3.3, $\mu \equiv \inf(\mu_1, \mu_2) \neq 0$. Since

$$\mu(E) \leq \mu_1(E) \quad \text{and} \quad \mu(E) \leq \mu_2(E) \quad \text{for each Borel set } E \subseteq \mathbb{T}, \tag{3.9}$$

it follows that μ is absolutely continuous with respect to both μ_1 and μ_2. In particular, μ is a finite positive Borel measure on $\mathbb{T}$. Also, since μ_1 is singular with respect to Lebesgue measure, μ is singular with respect to Lebesgue measure. Thus, it follows from (3.9) that $\mu_i' := \mu_i - \mu$ $(i = 1, 2)$ is a finite positive Borel measure which is singular with respect to Lebesgue measure. Since $\mu_i'(\mathbb{T}) \leq \mu_i(\mathbb{T}) < \infty$, we can see that μ_i' is regular for each $i = 1, 2$. Observe that

$$\exp\left(-\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu_1(t)\right) = \exp\left(-\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu(t)\right) \cdot \exp\left(-\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu_1'(t)\right)$$

Thus,

$$\theta(z) := \exp\left(-\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu(t)\right) \qquad (z \in \mathbb{D}).$$

is a nonconstant inner divisor of θ_1. Similarly we can show that θ is also a nonconstant inner divisor of θ_2. Hence θ_1 and θ_2 are not coprime.

($\Leftarrow$) Assume that θ_1 and θ_2 are not coprime. Thus there exists a nonconstant common inner divisor ω of θ_1 and θ_2: i.e.,

$$\theta_1 = \omega\theta_1' \quad \text{and} \quad \theta_2 = \omega\theta_2'. \tag{3.10}$$

Note that ω is also a singular inner function, so that we may write

$$\omega(z) = \exp\left(-\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu(t)\right),$$

where μ is a nonzero finite positive Borel measure on $\mathbb{T}$ which is singular with respect to Lebesgue measure. If μ_1' and μ_2' are singular measures corresponding to θ_1' and θ_2', respectively, then it follows from (3.10) that

$$d\mu_i = d(\mu + \mu_i') \quad (i = 1, 2).$$

Thus we can see that $\mu(E) \leq \mu_i(E)$ for every Borel set E of $\mathbb{T}$. Since μ is nonzero, it follows from Theorem 3.1 that μ_1 and μ_2 are not mutually singular. $\square$

REMARK 3.8. By a similar argument as in the proof of Theorem 3.7, we can show that if θ_1 and θ_2 are singular inner functions with singular measures μ_1 and μ_2, respectively and if $\mu := \inf(\mu_1, \mu_2)$, then

$$\theta(z) = \exp\left(-\int_{\mathbb{T}} \frac{t+z}{t-z} d\mu(t)\right).$$

is the greatest common inner divisor of θ_1 and θ_2

CHAPTER 4

Douglas-Shapiro-Shields factorizations

To understand matrix functions of bounded type, we need to factor those functions into a coprime product of matrix inner functions and the adjoints of matrix H^∞-functions; this is the so-called Douglas-Shapiro-Shields factorization. This factorization is very helpful and somewhat unavoidable for the study of Hankel and Toeplitz operators with bounded type symbols. In this chapter, we consider several properties of the Douglas-Shapiro-Shields factorization for matrix functions of bounded type.

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ $(m \le n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant, and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We note that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi \neq 0$, then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$: indeed, if $\Phi = \Delta A$ with $\Delta \in H^2_{M_{n\times r}}$ $(r < n)$ then $\operatorname{rank} \Phi(z) \le \operatorname{rank} \Delta(z) \le r < n$, so that $\det \Phi(z) = 0$. If $\Phi \in H^2_{M_n}$ is such that $\det \Phi \neq 0$, then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$.

Let $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ be a family of inner matrix functions. The greatest common left inner divisor Θ_d and the least common left inner multiple Θ_m of the family $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ are the inner functions defined by

$$\Theta_d H^2_{\mathbb{C}^p} = \bigvee_{i\in J} \Theta_i H^2_{\mathbb{C}^n} \quad \text{and} \quad \Theta_m H^2_{\mathbb{C}^q} = \bigcap_{i\in J} \Theta_i H^2_{\mathbb{C}^n}.$$

Similarly, the greatest common right inner divisor Θ'_d and the least common right inner multiple Θ'_m of the family $\{\Theta_i \in H^\infty_{M_n} : i \in J\}$ are the inner functions defined by

$$\widetilde{\Theta}'_d H^2_{\mathbb{C}^r} = \bigvee_{i\in J} \widetilde{\Theta}_i H^2_{\mathbb{C}^n} \quad \text{and} \quad \widetilde{\Theta}'_m H^2_{\mathbb{C}^s} = \bigcap_{i\in J} \widetilde{\Theta}_i H^2_{\mathbb{C}^n}.$$

The Beurling-Lax-Halmos Theorem guarantees that Θ_d and Θ_m exist and are unique up to a unitary constant right factor, and Θ'_d and Θ'_m are unique up to a unitary constant left factor. We write

$$\Theta_d = \text{left-g.c.d.}\,\{\Theta_i : i \in J\}, \quad \Theta_m = \text{left-l.c.m.}\,\{\Theta_i : i \in J\},$$
$$\Theta'_d = \text{right-g.c.d.}\,\{\Theta_i : i \in J\}, \quad \Theta'_m = \text{right-l.c.m.}\,\{\Theta_i : i \in J\}.$$

If $n = 1$, then left-g.c.d. $\{\cdot\}$ = right-g.c.d. $\{\cdot\}$ (simply denoted g.c.d. $\{\cdot\}$) and left-l.c.m. $\{\cdot\}$ = right-l.c.m. $\{\cdot\}$ (simply denoted l.c.m. $\{\cdot\}$). In general, it is not true that left-g.c.d. $\{\cdot\}$ = right-g.c.d. $\{\cdot\}$ and left-l.c.m. $\{\cdot\}$ = right-l.c.m. $\{\cdot\}$.

However, we have:

LEMMA 4.1. *Let $\Theta_i := I_{\theta_i}$ for an inner function θ_i $(i \in J)$.*

(a) left-g.c.d. $\{\Theta_i : i \in J\}$ = right-g.c.d. $\{\Theta_i : i \in J\} = I_{\theta_d}$, *where* $\theta_d =$ g.c.d. $\{\theta_i : i \in J\}$.

(b) left-l.c.m. $\{\Theta_i : i \in J\}$ = right-l.c.m. $\{\Theta_i : i \in J\} = I_{\theta_m}$, *where* $\theta_m =$ l.c.m. $\{\theta_i : i \in J\}$.

PROOF. See [**CHL2**, Lemma 2.1]. □

In view of Lemma 4.1, if $\Theta_i = I_{\theta_i}$ for an inner function θ_i $(i \in J)$, we can define the greatest common inner divisor Θ_d and the least common inner multiple Θ_m of the Θ_i by

$$\Theta_d \equiv \text{g.c.d.}\,\{\Theta_i : i \in J\} := I_{\theta_d}, \quad \text{where } \theta_d = \text{g.c.d.}\,\{\theta_i : i \in J\}$$

and

$$\Theta_m \equiv \text{l.c.m.}\,\{\Theta_i : i \in J\} := I_{\theta_m}, \quad \text{where } \theta_m = \text{l.c.m.}\,\{\theta_i : i \in J\}.$$

Both Θ_d and Θ_m are *diagonal-constant* inner functions, i.e., diagonal inner functions, and constant along the diagonal.

REMARK 4.2. By contrast with scalar-valued functions, in (2.6), I_θ and A need not be (right) coprime. If Ω = left-g.c.d. $\{I_\theta, A\}$ in the representation (2.6), that is,

$$\Phi = \theta A^*\,,$$

then $I_\theta = \Omega\Omega_\ell$ and $A = \Omega A_\ell$ for some inner matrix Ω_ℓ (where $\Omega_\ell \in H^2_{M_n}$ because $\det(I_\theta) \neq 0$) and some $A_l \in H^2_{M_n}$. Therefore if $\Phi^* \in L^\infty_{M_n}$ is of bounded type then we can write

$$\Phi = {A_\ell}^*\Omega_\ell, \quad \text{where } A_\ell \text{ and } \Omega_\ell \text{ are left coprime.} \tag{4.1}$$

In this case, $A_\ell^*\Omega_\ell$ is called the *left coprime factorization* of Φ and write, briefly,

$$\Phi = A_\ell^*\Omega_\ell \ \ \text{(left coprime)}.$$

Similarly, we can write

$$\Phi = \Omega_r A_r^*, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime.} \tag{4.2}$$

In this case, $\Omega_r A_r^*$ is called the *right coprime factorization* of Φ and we write, succinctly,

$$\Phi = \Omega_r A_r^* \ \ \text{(right coprime)}.$$

We often say that (4.2) is the *Douglas-Shapiro-Shields factorization* of Φ and (4.1) is the *left Douglas-Shapiro-Shields factorization* of Φ (cf. [**DSS**], [**Fu**]). We also say that Ω_ℓ and Ω_r are called the *inner parts* of those factorizations.

REMARK 4.3. ([**GHR**, Corollary 2.5]; [**CHL2**, Remark 2.2]) As a consequence of the Beurling-Lax-Halmos Theorem, we can see that

$$\Phi = \Omega_r A_r^* \ \text{(right coprime)} \iff \ker H_{\Phi^*} = \Omega_r H^2_{\mathbb{C}^n}. \tag{4.3}$$

If M is a nonzero closed subspace of $\mathbb{C}^n$ then the matrix function

$$b_\lambda P_M + (I - P_M) \quad (P_M := \text{the orthogonal projection of } \mathbb{C}^n \text{ onto } M)$$

is called a *Blaschke-Potapov factor*; an $n \times n$ matrix function D is called *a finite Blaschke-Potapov product* if D is of the form

$$D = \nu \prod_{m=1}^{M} \Big(b_m P_m + (I - P_m) \Big),$$

where ν is an $n \times n$ unitary constant matrix, b_m is a Blaschke factor, and P_m is an orthogonal projection in $\mathbb{C}^n$ for each $m = 1, \cdots, M$. In particular, a scalar-valued function D reduces to a finite Blaschke product $D = \nu \prod_{m=1}^{M} b_m$. It is known (cf. [**Po**]) that an $n \times n$ matrix function D is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product. If $\Phi \in L^\infty_{M_n}$ is rational then Ω_ℓ and Ω_r in (4.1) and (4.2) can be chosen as finite Blaschke-Potapov products. We can see (cf. [**CHL2**]) that every inner divisor of $I_{b_\lambda} \in H^\infty_{M_n}$ is a Blaschke-Potapov factor.

In what follows, we examine the left and right coprime factorizations of matrix H^2-functions. We begin with:

Lemma 4.4. *Let $\Theta \in H^\infty_{M_{n\times m}}$ be an inner matrix function and $A \in H^2_{M_{n\times m}}$. Then the following hold:*

(a) $A \in \mathcal{K}_\Theta \Longleftrightarrow \Theta A^* \in H^2_0$;
(b) $A \in \mathcal{H}_\Theta \Longleftrightarrow A^* \Theta \in H^2_0$;
(c) $P_{H^2_0}(\Theta A^*) = \Theta \Big(P_{\mathcal{K}_\Theta} A \Big)^*$.

Proof. Immediate from a direct calculation. □

Lemma 4.5. *Let $\Theta \in H^\infty_{M_{n\times m}}$ be an inner matrix function and $B \in H^2_{M_{n\times m}}$. Then the following hold:*

(a) $P_{\mathcal{K}_\Theta} B = \widetilde{P_{\mathcal{H}_{\widetilde{\Theta}}} \widetilde{B}}$ *if $n = m$;*
(b) $P_{\mathcal{K}_\Theta}(\Lambda B) = \Lambda (P_{\mathcal{K}_\Theta} B)$ *for any constant matrix $\Lambda \in M_n$;*
(c) $P_{\mathcal{H}_\Theta}(B \Lambda) = (P_{\mathcal{H}_\Theta} B)\Lambda$ *for any constant matrix $\Lambda \in M_m$.*

In particular, if $n = m$ and $\Theta = I_\theta$ for an inner function θ and Λ commutes with B then

$$\Lambda (P_{\mathcal{K}_\Theta} B) = (P_{\mathcal{K}_\Theta} B)\Lambda.$$

Proof. Write $B := B_1 + B_2$, where $B_1 := P_{\mathcal{K}_\Theta} B$ and $B_2 := B_3 \Theta$ for some $B_3 \in H^2_{M_n}$. Then $\widetilde{B} = \widetilde{B_1} + \widetilde{\Theta}\widetilde{B_3}$. If $n = m$, then $\widetilde{\Theta}$ is an inner matrix function. Since $B_1 \in \mathcal{K}_\Theta$, by Lemma 4.4 (a), we have $\Theta B_1^* \in H^2_0$, so that $\widetilde{B}_1^* \widetilde{\Theta} = \widetilde{\Theta B_1^*} \in H^2_0$. Thus it follows from Lemma 4.4 (b) that $\widetilde{B}_1 \in \mathcal{H}_{\widetilde{\Theta}}$, and hence

$$P_{\mathcal{H}_{\widetilde{\Theta}}} \widetilde{B} = \widetilde{B}_1 = \widetilde{P_{\mathcal{K}_\Theta} B},$$

giving (a). Observe that

$$P_{\mathcal{K}_\Theta}(\Lambda B) = P_{\mathcal{K}_\Theta}(\Lambda (B_1 + B_3 \Theta)) = P_{\mathcal{K}_\Theta}(\Lambda B_1).$$

Since $B_1 \in \mathcal{K}_\Theta$, it follows from Lemma 4.4 (a) that $\Theta B_1^* \in H^2_0$. Thus

$$\Theta (\Lambda B_1)^* = \Theta B_1^* \Lambda^* \in H^2_0,$$

which implies $\Lambda B_1 \in \mathcal{K}_\Theta$, and hence $P_{\mathcal{K}_\Theta}(\Lambda B) = \Lambda B_1 = \Lambda(P_{\mathcal{K}_\Theta} B)$. This proves (b). The statement (c) follows from (a) and (b). The last assertion follows at once from (b) and (c) because $\mathcal{H}_\Theta = \mathcal{K}_\Theta$ if $\Theta = I_\theta$. □

LEMMA 4.6. *Let $\Theta \in H^\infty_{M_n}$ and $\Delta \in H^\infty_{M_{n\times m}}$ be inner matrix functions. Then the following hold:*

(a) $\mathcal{K}_{\Theta\Delta} = \mathcal{K}_\Delta \bigoplus \mathcal{K}_\Theta \Delta$;
(b) $\mathcal{H}_{\Theta\Delta} = \mathcal{H}_\Delta \bigoplus \Delta \mathcal{H}_\Theta$.

PROOF. The inclusion $\mathcal{K}_\Delta \subseteq \mathcal{K}_{\Theta\Delta}$ is obvious; it also follows from Lemma 4.4(a) that $\mathcal{K}_\Theta \Delta \subseteq \mathcal{K}_{\Theta\Delta}$. Thus

$$\mathcal{K}_\Delta \bigoplus \mathcal{K}_\Theta \Delta \subseteq \mathcal{K}_{\Theta\Delta}.$$

For the reverse inclusion, suppose $M \in \mathcal{K}_{\Theta\Delta}$. Write $M_1 := P_{\mathcal{K}_\Delta} M$. Then $M - M_1 = Q\Delta$ for some $Q \in H^2_{M_{n\times m}}$. Since $M_1 \in \mathcal{K}_\Delta \subseteq \mathcal{K}_{\Theta\Delta}$, we have $Q\Delta = M - M_1 \in \mathcal{K}_{\Theta\Delta}$. Thus, by Lemma 4.4(a), $\Theta Q^* = (\Theta\Delta)(Q\Delta)^* \in H^2_0$, and hence, again by Lemma 4.4(a), $Q \in \mathcal{K}_\Theta$, which gives (a). The argument for (b) is similar. □

LEMMA 4.7. *Suppose $A, B, C \in H^\infty_{M_n}$ with $AB = BA$. If A and B are left coprime and A and C are left coprime then A and BC are left coprime.*

PROOF. If A and B are left coprime and A and C are left coprime then it follows (cf. [**FF**, p.242]) that

$$H^2_{\mathbb{C}^n} = AH^2_{\mathbb{C}^n} \bigvee BH^2_{\mathbb{C}^n} = AH^2_{\mathbb{C}^n} \bigvee CH^2_{\mathbb{C}^n}.$$

Then the limit argument together with the commutativity of A and B shows that

$$H^2_{\mathbb{C}^n} = AH^2_{\mathbb{C}^n} \bigvee BCH^2_{\mathbb{C}^n},$$

which gives the result. □

COROLLARY 4.8. *Let $A, B \in H^\infty_{M_n}$ and θ be an inner function. If A and B are left coprime, and A and I_θ are left coprime, then A and θB are left coprime.*

PROOF. Immediate from Lemma 4.7. □

COROLLARY 4.9. *Suppose that $A, B, C \in H^\infty_{M_n}$ with $AB = BA$. If A and B are right coprime and A and C are right coprime then A and CB are right coprime.*

PROOF. This follows from Lemma 4.7 together with the fact that A and B are right coprime if and only if $\widetilde{A}$ and $\widetilde{C}$ are left coprime. □

We recall the inner-outer factorization of vector-valued functions. Let D and E be Hilbert spaces. If F is a function with values in $\mathcal{B}(E, D)$ such that $F(\cdot)e \in H^2_D$ for each $e \in E$, then F is called a strong H^2-function. The strong H^2-function F is called an *inner* function if $F(\cdot)$ is an isometric operator from D into E. Write $\mathcal{P}_E$ for the set of all polynomials with values in E, i.e., $p(\zeta) = \sum_{k=0}^n \widehat{p}(k)\zeta^k$, $\widehat{p}(k) \in E$. Then the function $Fp = \sum_{k=0}^n F\widehat{p}(k)z^k$ belongs to H^2_D. The strong H^2-function F is called *outer* if $\mathrm{cl}\, F \cdot \mathcal{P}_E = H^2_D$. Note that every $F \in H^2_{M_n}$ is a strong H^2-function. We then have an analogue of the scalar Factorization Theorem.

Inner-Outer Factorization. (cf. [**Ni1**]) Every strong H^2-function F with values in $\mathcal{B}(E, D)$ can be expressed in the form

$$F = F_i F_e,$$

where F_e is an outer function with values in $\mathcal{B}(E, D')$ and F_i is an inner function with values in $\mathcal{B}(D', D)$, for some Hilbert space D'.

If $\dim D = \dim E < \infty$, then outer operator functions can be detected by outer scalar functions.

Helson-Lowdenslager Theorem. (cf. [**Ni2**]) If $\dim D < \infty$ and F is a strong H^2-function with values in $\mathcal{B}(D)$, then F is outer if and only if $\det F$ is outer.

We then have:

Lemma 4.10. *Let $\Phi \in H^2_{M_n}$. Then $\det \Phi \neq 0$ if and only if every left inner divisor of Φ is square.*

Proof. We assume to the contrary that every left inner divisor of Φ is square, but $\det \Phi = 0$. Since Φ is a strong H^2-function, we have an inner-outer factorization of Φ of the form

$$\Phi = \Phi_i \, \Phi_e \, ,$$

where $\Phi_i \in H^2_{M_{n\times r}}$ is inner and $\Phi_e \in H^2_{M_{r\times n}}$ is outer. By assumption, $r = n$. We thus have $0 = \det \Phi = \det \Phi_i \det \Phi_e$, which implies that $\det \Phi_e = 0$. By the Helson-Lowdenslager Theorem, 0 should be an outer function, a contradiction. The converse was noticed in p.10. □

We introduce a notion of the "characteristic scalar inner" function of a matrix inner function (cf. [**He**, p. 81], [**SFBK**]).

Definition 4.11. Let $\Delta \in H^\infty_{M_n}$ be inner. We say that Δ *has a scalar inner multiple* $\theta \in H^\infty$ if there exists $\Delta_0 \in H^\infty_{M_n}$ such that

$$\Delta\Delta_0 = \Delta_0\Delta = I_\theta.$$

We define

$$m_\Delta := \text{g.c.d.} \Big\{ \theta : \ \theta \text{ is a scalar inner multiple of } \Delta \Big\} .$$

In view of Lemma 4.1, m_Δ is the minimal θ so that $\Delta\Delta_0 = \Delta_0\Delta = I_\theta$ for some $\Delta_0 \in H^\infty_{M_n}$. We say that m_Δ is the *characteristic scalar inner function* of Δ.

Remark. In Definition 4.11, it is enough to assume $\Delta\Delta_0 = I_\theta$. For, given two matrices A and B such that $AB = \lambda I$, with $\lambda \neq 0$, it is straightforward to verify that $BA = AB$.

It is well known (cf. [**SFBK**, Proposition 5.6.1]) that if $\Delta \in H^\infty_{M_n}$ is inner then Δ has a scalar inner multiple. Thus m_Δ is well-defined. We would like to remark that the notion of m_Δ arises in the Sz.-Nagy–Foiaş theory of contraction operators T of class C_0 (completely nonunitary contractions T for which there exists a nonzero function $u \in H^\infty$ such that $u(T) = 0$): the *minimal function* m_T of the C_0-contraction operator T amounts to our m_{Θ_T}, where Θ_T is the characteristic function of T (cf. [**Be**], [**SFBK**]).

If $\Delta \equiv \big(\delta_{ij}\big) \in H^\infty_{M_n}$ is inner, we may ask, how do we find m_Δ? The following lemma gives an answer. To see this, we first observe that if $\Delta \equiv \big(\delta_{ij}\big) \in H^\infty_{M_n}$ is inner then $\ker H_{\Delta^*} = \Delta H^2_{\mathbb{C}^n}$ and hence, by Lemma 2.2, Δ^* is of bounded type, and we may write $\delta_{ij} = \theta_{ij}\bar{b}_{ij}$ (θ_{ij} inner; θ_{ij} and $b_{ij} \in H^\infty$ coprime).

LEMMA 4.12. *Let $\Delta \in H^\infty_{M_n}$ be inner. Thus we may write $\Delta \equiv \big(\theta_{ij}\bar{b}_{ij}\big)$, where θ_{ij} is an inner function and θ_{ij} and $b_{ij} \in H^\infty$ are coprime for each $i, j = 1, 2, \cdots, n$. Then*

$$m_\Delta = \text{l.c.m.}\Big\{\theta_{ij} : i, j = 1, 2, \cdots, n\Big\}.$$

PROOF. Observe that

$$\Delta = \big(\theta_{ij}\bar{b}_{ij}\big) = \big(\theta\bar{a}_{ij}\big) = \theta A^* \quad (A \in H^2_{M_n}). \tag{4.4}$$

Since Δ is inner, it follows that $A^*A = I$, so that $\Delta A = I_\theta$. This says that m_Δ is an inner divisor of θ. Thus, we may write

$$\zeta m_\Delta = \theta \quad (\zeta \text{ scalar-inner}). \tag{4.5}$$

We now want to show that ζ is constant. Let $I_{m_\Delta} = \Delta\Delta_0$ for some inner function Δ_0. It follows from (4.5) that $\zeta\Delta\Delta_0 = I_\theta = \Delta A$, and hence $\zeta\Delta_0 = A$. Since by (4.5), ζ is an inner divisor of θ, it follows that

$$\big(\theta_{ij}\bar{b}_{ij}\big) = \big(\theta\bar{a}_{ij}\big) = (\theta\bar{\zeta})\Delta_0^*.$$

But since θ_{ij} and b_{ij} are coprime, θ_{ij} is inner divisor of $\theta\bar{\zeta}$ for each $i, j = 1, 2, \cdots, n$. Thus θ is an inner divisor of $\theta\bar{\zeta}$, and therefore ζ is constant. This completes the proof. □

We observe:

LEMMA 4.13. *If $\theta \in H^2$ is a nonconstant inner function, then*

$$\bigcap_{n=1}^\infty \theta^n H^2 = \{0\}.$$

PROOF. If θ is a nonconstant inner function, then the wandering subspace construction from the Halmos proof of the Beurling-Lax-Halmos theorem implies that (cf. [**SFBK**, p. 204])

$$H^2 = \bigoplus_{j=0}^\infty \theta^j \mathcal{H}(\theta)$$

and more generally,

$$\theta^n H^2 = \bigoplus_{j=n}^\infty \theta^j \mathcal{H}(\theta).$$

Thus any $f \in H^2$ has an orthogonal decomposition as $f = \sum_{j=0}^\infty \theta^j f_j$ with $f_j \in \mathcal{H}(\theta)$. If $f \in \theta^n H^2$, then $f_j = 0$ for $0 \le j < n$. Hence, if $f \in \cap_{n=0}^\infty \theta^n H^2$, then $f_j = 0$ for $j = 0, 1, 2, \ldots$, or $f = 0$. □

LEMMA 4.14. (The Local Rank) Let E and D be Hilbert spaces, $\dim E < \infty$, and let Φ be a strong H^2-function with values in $\mathcal{B}(D,E)$. Denote $\text{Rank}\,\Phi := \max_{\zeta\in\mathbb{D}} \text{rank}\Phi(\zeta)$, where $\text{rank}\Phi(\zeta) := \dim\Phi(\zeta)(D)$. Then $\big\{f \in H^2_D : \Phi(\zeta)f(\zeta) \equiv 0\big\} = \vartheta H^2_{D'}$ (ϑ inner) for some Hilbert space D'. In this case, $\dim D = \dim D' + \text{Rank}\,\Phi$.

PROOF. See [**Ni1**, p.44]. □

LEMMA 4.15. *Let $A \in H^\infty_{M_n}$ and $\Theta = I_\theta$ for some nonconstant inner function θ. If Θ and A are left (or right) coprime, then* $\det A \neq 0$.

PROOF. Suppose Θ and A are left coprime. Then by the Beurling-Lax-Halmos Theorem, we have

$$\Theta H^2_{\mathbb{C}^n} \bigvee AH^2_{\mathbb{C}^n} = H^2_{\mathbb{C}^n} \quad (\text{cf. [\textbf{FF}, p.242]}). \tag{4.6}$$

Assume to the contrary that $\det A = 0$. Then $\det\widetilde{A} = \widetilde{\det A} = 0$, and hence $\text{Rank}\,\widetilde{A} = m < n$. Since $\widetilde{A} \in H^\infty_{M_n}$ is a strong H^2-function, it follows from Lemma 4.14 that

$$\ker\widetilde{A} \equiv \Big\{f \in H^2_{\mathbb{C}^n} : \widetilde{A}(\zeta)f(\zeta) \equiv 0\Big\} = \vartheta H^2_{D'},$$

where ϑ is inner and $\dim D' = n - m$. Let $U : H^2_{\mathbb{C}^{n-m}} \to H^2_{D'}$ be unitary and put $\widetilde{\Omega} := \vartheta U \in H^2_{M_{n\times(n-m)}}$. Then $\ker\widetilde{A} = \widetilde{\Omega} H^2_{\mathbb{C}^{n-m}}$, so that $\widetilde{A}\widetilde{\Omega} = 0$, and hence $\Omega A = 0$. It thus follows from (4.6) that

$$\Omega\Theta H^2_{\mathbb{C}^n} = \Omega\Big(\Theta H^2_{\mathbb{C}^n} \bigvee AH^2_{\mathbb{C}^n}\Big) = \Omega H^2_{\mathbb{C}^n}. \tag{4.7}$$

Write $\Omega := \big(\omega_{ij}\big) \in H^2_{M_{(n-m)\times n}}$. Since $\Theta = I_\theta$, it follows from (4.7) that for each $p = 1, 2, \cdots$, $(\overline{\theta}^p \omega_{ij})H^2_{\mathbb{C}^n} = \Theta^{*p}\Omega H^2_{\mathbb{C}^n} = \Omega H^2_{\mathbb{C}^n} \in H^2_{\mathbb{C}^{n-m}}$, which implies that for each $1 \le i \le n - m$ and $1 \le j \le n$,

$$\omega_{ij} \in \bigcap_{p=1}^{\infty} \theta^p H^2.$$

Since θ is not constant, it follows from Lemma 4.13 that $\omega_{ij} = 0$ for all i, j, and hence $\Omega = 0$, a contradiction. If Θ and A are right coprime then $\widetilde{\Theta}$ and $\widetilde{A}$ are left coprime. Thus by what we showed just before, $\det\widetilde{A} \neq 0$, and hence $\det A \neq 0$. This completes the proof. □

The following theorem plays a key role in the theory of coprime factorizations.

THEOREM 4.16. *For $A \in H^2_{M_n}$ and $\Theta := I_\theta$ for some scalar inner function θ, then Θ and A are right (or left) coprime if and only if θ and* $\det A$ *are coprime.*

PROOF. We first prove the theorem when $A \equiv \Delta \in H^\infty_{M_n}$ is inner.

If Δ is a diagonal-constant inner function then this is trivial. Thus we may assume that Δ is not diagonal-constant. Suppose that θ and $\det\Delta$ are not coprime. Write $m_\Delta := \delta$ and $I_\delta = \Delta\Delta_0 = \Delta_0\Delta$ for a nonconstant inner function Δ_0. Thus $(\det\Delta)(\det\Delta_0) = \delta^n$, and hence θ and δ^n (and hence δ) are not coprime. Put $\omega := \text{g.c.d.}\,(\theta,\, \delta)$. Then we can write $\Theta \equiv I_\theta = I_\omega I_{\theta_1}$ and

$$I_\delta = \Delta\Delta_0 = I_\omega I_{\delta_1} \tag{4.8}$$

for some inner functions θ_1 and δ_1. If $\delta = \omega$, then I_δ is an inner divisor of I_θ, so that evidently, Θ and Δ are not right coprime. We now suppose $\delta \neq \omega$. We then claim that

$$\Delta \text{ and } I_\omega \text{ are not right coprime.} \tag{4.9}$$

Towards a proof of (4.9), we assume to the contrary that Δ and I_ω are right coprime. Since by (4.8), $\overline{\omega}\Delta\Delta_0 = I_{\delta_1} \in H^2_{M_n}$, it follows from (2.10) that

$$0 = H_{\overline{\omega}\Delta\Delta_0} = H_{\overline{\omega}\Delta} T_{\Delta_0}. \tag{4.10}$$

But since Δ and I_ω are right coprime, it follows from (4.3) together with (4.10) that

$$\Delta_0 H^2_{\mathbb{C}^n} = \operatorname{ran} T_{\Delta_0} \subseteq \ker H_{\overline{\omega}\Delta} = \omega H^2_{\mathbb{C}^n},$$

which implies that I_ω is a left inner divisor of Δ_0 (cf. [**FF**, Corollary IX.2.2]), so that $\overline{\omega}\Delta_0$ should be inner. Consequently, $I_{\delta_1} = I_{\delta\overline{\omega}} = \Delta(\overline{\omega}\Delta_0)$, which contradicts the definition of m_Δ. This proves (4.9). Since $\Theta = I_{\theta_1} I_\omega$, it follows that Θ and Δ are not right coprime.

Conversely, suppose that Θ and Δ are not right coprime. Thus $\Theta = \Theta_1 G$ and $\Delta = \Delta_1 G$, where $G \in H^\infty_{M_n}$ is not unitary constant. Thus $\det G$ is a common inner divisor of $\det\Theta$ and $\det\Delta$. Since G is not constant, it follows from the Fredholm theory of block Toeplitz operators (cf. [**Do1**]) that

$$0 < \dim \mathcal{H}(G) = \dim\ker T_{G^*} = -\operatorname{ind} T_G = -\operatorname{ind} T_{\det G},$$

(where $\operatorname{ind}(\cdot)$ denotes the Fredholm index) which implies that $\det G$ is not constant, and hence $\det\Delta$ and θ are not coprime.

Now we prove the general case of $A \in H^2_{M_n}$. In view of Lemma 4.15, we may assume that $\det A \neq 0$. Then by Lemma 4.10, the left and the right inner divisors of A are square. Thus A has the following inner-outer factorizations of the form:

$$A = A_e\, A_i,$$

where $A_i \in H^\infty_{M_n}$ is inner and $A_e \in H^2_{M_n}$ is outer. Hence by what we proved just before and Helson-Lowdenslager Theorem (p.23), we can see that Θ and A are right coprime if and only if Θ and A_i are right coprime if and only if θ and $\det A_i$ are coprime if and only if θ and $\det A$ are coprime.

For the case of left coprime-ness, we use $\widetilde{\Phi}(z) := \Phi^*(\overline{z})$. By the case of the right coprime-ness and the fact that $\widetilde{\det A} = \det\widetilde{A}$, it follows that θ and $\det A$ are coprime if and only if $\widetilde{\theta}$ and $\det\widetilde{A}$ are coprime if and only if $\widetilde{\Theta}$ and $\widetilde{A}$ are right coprime if and only if Θ and A are left coprime. This completes the proof. $\square$

Theorem 4.16 shows that if $A \in H^2_{M_n}$ and $\Theta := I_\theta$ for some scalar inner function θ, then "left" coprime-ness and "right" coprime-ness coincide for A and Θ. Thus if θ is an inner function then we shall say that $A \in H^2_{M_n}$ and I_θ are *coprime* whenever they are right or left coprime. Hence if in the representations (4.1) and (4.2), Ω_ℓ or Ω_r is of the form I_θ with an inner function θ, then we shall write

$$\Phi = \theta A^* \quad \text{(coprime)}. \tag{4.11}$$

Lemma 4.17. *Let $\Omega, \Delta \in H^\infty_{M_n}$ be inner and $\Theta \equiv I_\theta$ for an inner function θ. If Ω and Θ are coprime and Δ and Θ are coprime, then* left-l.c.m.(Ω, Δ) *and Θ are coprime.*

PROOF. Suppose that Ω and Θ are coprime and Δ and Θ are coprime. Then by Theorem 4.16, $m_\Omega \equiv \omega$ and θ are coprime and $m_\Delta \equiv \delta$ and θ are coprime, so that by Corollary 4.8, l.c.m. (ω, δ) and θ are coprime. Thus

$$\text{l.c.m.}\,(\omega,\delta) H^2_{\mathbb{C}^n} \bigvee \theta H^2_{\mathbb{C}^n} = H^2_{\mathbb{C}^n}. \tag{4.12}$$

Observe that

$$\text{left-l.c.m.}\,(\Omega,\Delta) H^2_{\mathbb{C}^n} = \Omega H^2_{\mathbb{C}^n} \cap \Delta H^2_{\mathbb{C}^n} \supseteq \omega H^2_{\mathbb{C}^n} \cap \delta H^2_{\mathbb{C}^n} = \text{l.c.m.}\,(\omega,\delta) H^2_{\mathbb{C}^n}.$$

It thus follows from (4.12) that

$$\text{left-l.c.m.}\,(\Omega,\Delta) H^2_{\mathbb{C}^n} \bigvee \Theta H^2_{\mathbb{C}^n} \supseteq \text{l.c.m.}\,(\omega,\delta) H^2_{\mathbb{C}^n} \bigvee \theta H^2_{\mathbb{C}^n} = H^2_{\mathbb{C}^n},$$

which implies that left-l.c.m. (Ω, Δ) and Θ are left coprime and hence coprime. $\square$

LEMMA 4.18. *Let $\Delta_1, \Delta_2 \in H^\infty_{M_n}$ be inner. Then there exist inner matrix functions $\Omega_1, \Omega_2 \in H^\infty_{M_n}$ such that* left-l.c.m. $(\Delta_1, \Delta_2) = \Delta_1\Omega_1 = \Delta_2\Omega_2$.

PROOF. Observe that

$$\text{left-l.c.m.}\,(\Delta_1,\Delta_2) H^2_{\mathbb{C}^n} = \Delta_1 H^2_{\mathbb{C}^n} \cap \Delta_2 H^2_{\mathbb{C}^n} \subseteq \Delta_i H^2_{\mathbb{C}^n}\ (i=1,2),$$

which gives the result (cf. [**FF**, Corollary IX.2.2]). $\square$

PROPOSITION 4.19. *Let $\Phi \in H^2_{M_n}$ be of the form $\Phi = \Theta A^*$* (right coprime). *If Δ is an inner matrix function in $H^2_{M_n}$, put $\Phi_\Delta := P_{H^2_0}(\Delta^*\Phi)$* (cf. p.12). *Then the following hold:*

(a) *If Δ is a left inner divisor of Θ, then*
$$\Phi_\Delta = \Theta_1 A_1^* \quad \text{(right coprime)},$$
where $\Theta_1 = \Delta^*\Theta$ and $A_1 := P_{\mathcal{K}_{\Theta_1}}(A)$.

(b) *If $\Omega :=$ left-g.c.d.(Δ, Θ) and $\Omega^*\Delta = I_{\delta_1}$ for an inner function δ_1, then*
$$\Phi_\Delta = \Theta_1 A_1^* \quad \text{(right coprime)},$$
where $\Theta_1 = \Omega^*\Theta$ and $A_1 := P_{\mathcal{K}_{\Theta_1}}(\delta_1 A)$;

(c) *If $\Omega :=$ left-g.c.d.(Δ, Θ) and $\Omega^*\Theta = I_{\theta_1}$ for an inner function θ_1, then*
$$\Phi_\Delta = \theta_1 A_1^* \quad \text{(coprime)},$$
where $A_1 := P_{\mathcal{K}_{\theta_1}}(A\Delta_1)$ with $\Delta_1 = \Omega^*\Delta$.

PROOF. Let $\Theta = \Delta\Theta_1$ for an inner function Θ_1. Then it follows from Lemma 4.4 that

$$\Phi_\Delta = P_{H^2_0}(\Delta^*\Phi) = P_{H^2_0}(\Delta^*\Theta A^*) = \Theta_1\Big(P_{\mathcal{K}_{\Theta_1}}(A)\Big)^*.$$

Suppose Θ_1 and $A_1 := P_{\mathcal{K}_{\Theta_1}}(A)$ are not right coprime. Put $\Theta_2 :=$ right-g.c.d.(Θ_1, A_1). Then Θ_2 is not unitary constant and we may write

$$\Theta_1 = \Theta_3\Theta_2 \text{ and } A_1 = A_2\Theta_2 \quad (\text{for some } \Theta_3, A_2 \in H^2_{M_n}). \tag{4.13}$$

Thus Θ_2 is a common right inner divisor of A and Θ. This is a contradiction. This proves (a).

For (b), we write $\Omega \equiv$ left-g.c.d. (Δ, Θ). Then by assumption, we may write $\Delta = \delta_1\Omega$. Put $\Theta_1 := \Omega^*\Theta$. Then I_{δ_1} and Θ_1 are (left) coprime. It follows from Lemma 4.4 that

$$\begin{aligned}\Phi_\Delta &= P_{H^2_0}(\Delta^*\Phi) = P_{H^2_0}(I^*_{\delta_1}\Theta_1 A^*)\\ &= P_{H^2_0}\big(\Theta_1(\delta_1 A)^*\big) = \Theta_1\Big(P_{\mathcal{K}_{\Theta_1}}(\delta_1 A)\Big)^*.\end{aligned}$$

Note that Θ_1 and $P_{\mathcal{K}_{\Theta_1}}(\delta_1 A)$ are right coprime if and only if Θ_1 and $\delta_1 A$ are right coprime. Now we will show that Θ_1 and $\delta_1 A$ are right coprime. Since Θ_1 and I_{δ_1} are left coprime, it follows from Theorem 4.16 that Θ_1 and I_{δ_1} are right coprime. Since Θ and A are right coprime, it follows that Θ_1 and A are right coprime. Thus it follows from Corollary 4.9 that Θ_1 and $\delta_1 A$ are right coprime. This proves (b). For (c), write $\Omega \equiv$ left-g.c.d.(Δ, Θ). Then by assumption, we may write

$$\Delta = \Omega \Delta_1 \quad \text{and} \quad \Theta = \theta_1 \Omega,$$

where Δ_1 and I_{θ_1} is (left) coprime. Then it follows from Lemma 4.4 that

$$\begin{aligned}\Phi_\Delta &= P_{H_0^2}(\Delta^* \Phi) = P_{H_0^2}(\Delta_1^* I_{\theta_1} A^*)\\ &= P_{H_0^2}(I_{\theta_1}(A\Delta_1)^*) = I_{\theta_1}\Big(P_{\mathcal{K}_{\theta_1}}(A\Delta_1)\Big)^*.\end{aligned}$$

Since I_{θ_1} and Δ_1 are coprime, and I_{θ_1} and A are coprime, it follows from Lemma 4.7 that I_{θ_1} and $A\Delta_1$ are coprime so that I_{θ_1} and $P_{\mathcal{K}_{\theta_1}}(A\Delta_1)$ are coprime. This completes the proof. □

Corollary 4.20. *Let θ and δ be finite Blaschke products. If $\Phi \in H^2_{M_n}$ is of the form*

$$\Phi = \theta A^* \quad (\text{coprime})\,,$$

then

$$\Phi_\delta \equiv P_{H_0^2}(\overline{\delta}\Phi) = \theta_1 A_1^* \quad (\text{coprime})\,, \tag{4.14}$$

where $\theta_1 = \overline{\text{g.c.d.}(\delta,\theta)}\,\theta$ and $A_1 := P_{\mathcal{K}_{\theta_1}}(\delta_1 A)$ with $\delta_1 = \overline{\text{g.c.d.}(\delta,\theta)}\,\delta$. Moreover, $A_1(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta_1)$.

Proof. The assertion (4.14) follows from Proposition 4.19. Put $A = (a_{ij})_{ij=1}^n$. Since θ is a finite Blaschke product, θ_1 is also a finite Blaschke product. For each $\alpha \in \mathcal{Z}(\theta_1)$, we have $\frac{1}{1-\overline{\alpha}z} \in \mathcal{H}(\theta_1)$, and hence

$$\begin{aligned}A_1(\alpha) &= \big(P_{\mathcal{K}_{\theta_1}}(\delta_1 A)\big)(\alpha)\\ &= \Big((P_{\mathcal{H}(\theta_1)}(\delta_1 a_{ij}))(\alpha)\Big)_{i,j=1}^n\\ &= \left(\left\langle P_{\mathcal{H}(\theta_1)}(\delta_1 a_{ij}),\ \frac{1}{1-\overline{\alpha}z}\right\rangle\right)_{i,j=1}^n\\ &= \left(\left\langle \delta_1 a_{ij},\ \frac{1}{1-\overline{\alpha}z}\right\rangle\right)_{i,j=1}^n\\ &= \big(\ \delta_1(\alpha) a_{ij}(\alpha)\ \big)_{i,j=1}^n\\ &= \delta_1(\alpha) A(\alpha).\end{aligned} \tag{4.15}$$

Since θ_1 and δ_1 are coprime and I_θ and A are coprime, it follows from Theorem 4.16 that $\delta_1(\alpha) \neq 0$ and $A(\alpha)$ is invertible. Thus, by (4.15), $A_1(\alpha)$ is invertible. □

Corollary 4.21. *Let $\Phi \in H^\infty_{M_n}$ be of the form $\Phi = B^*\Theta$* (left coprime). *If Ω is a right inner divisor of Θ and $\Phi^\Omega := P_{H_0^2}(\Phi\Omega^*)$, then*

$$\Phi^\Omega = B_1^* \Delta_1 \quad (\text{left coprime}),$$

where $\Delta_1 = \Theta\Omega^*$ *and* $B_1 := P_{\mathcal{H}_{\Delta_1}} B$.

PROOF. Since $\widetilde{\Phi} = \widetilde{\Theta}\widetilde{B}^*$ (right coprime) and $\widetilde{\Omega}$ is a left inner divisor of $\widetilde{\Theta}$, it follows from Proposition 4.19 that

$$\widetilde{\Phi}_{\widetilde{\Omega}} := P_{H_0^2}(\widetilde{\Omega}^*\widetilde{\Phi}) = \Theta_1 A_1^* \quad \text{(right coprime)},$$

where $\Theta_1 = \widetilde{\Omega}^*\widetilde{\Theta}$ and $A_1 := P_{\mathcal{K}_{\Theta_1}} \widetilde{B}$. It follows from Lemma 4.5 that

$$\widetilde{\Phi}_{\widetilde{\Omega}} = \Theta_1 \left(\widetilde{P_{\mathcal{H}_{\widetilde{\Theta}_1}} B}\right)^* . \tag{4.16}$$

Since $\widetilde{\Phi}_{\widetilde{\Omega}} = P_{H_0^2}\widetilde{(\Phi\Omega^*)} = \widetilde{\Phi^{\Omega}}$, it follows from (4.16) that

$$\Phi^{\Omega} = \left(P_{\mathcal{H}_{\widetilde{\Theta}_1}} B\right)^* \widetilde{\Theta}_1 = B_1^*\Delta_1 \quad \text{(right coprime)},$$

where $B_1 = P_{\mathcal{H}_{\Delta_1}} B$ and $\Delta_1 = \widetilde{\Theta}_1 = \Theta\Omega^*$. This completes the proof. $\square$

Recall that the composition of two inner functions is again an inner function. In general, we cannot guarantee that the composition of two Blaschke products is again a Blaschke product. However, by Frostman's Theorem ([**Ga**], [**MAR**]), if θ is an inner function in H^∞, then for almost all $\alpha \in \mathbb{D}$ (with respect to area measure on $\mathbb{D}$), the function $b_\alpha \circ \theta$ is a Blaschke product.

We are now interested in the following:

QUESTION 4.22. *Let θ, δ and ω be inner functions in H^∞. Suppose θ and δ are coprime.*

(a) *If ω is a finite Blaschke product, are $\omega \circ \theta$ and $\omega \circ \delta$ coprime?*
(b) *If θ and δ are finite Blaschke products, are $\theta \circ \omega$ and $\delta \circ \omega$ coprime?*

The following example shows that the answer to Question 4.22 (a) is negative.

EXAMPLE 4.23. Let $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n \in \mathbb{D}$ satisfy the following properties

(a) $\alpha_i \neq \beta_j$ for each i, j;
(b) $\eta := \prod_{i=1}^m \alpha_i = \prod_{j=1}^n \beta_j$.

Put

$$\theta := \prod_{i=1}^m b_{-\alpha_i}, \quad \delta := \prod_{j=1}^n b_{-\beta_j}, \quad \text{and} \quad \omega := b_\eta.$$

Then θ and δ are coprime. However, we have

$$(\omega \circ \theta)(0) = \omega(\eta) = 0 = (\omega \circ \delta)(0),$$

which implies that $\omega \circ \theta$ and $\omega \circ \delta$ are not coprime. $\square$

To examine Question 4.22 (b), we recall the notion of "capacity of zero" for a set in $\mathbb{D}$. Recall (cf. [**Ga**, p.78]) that a set $L \subseteq \mathbb{D}$ is said to have *positive capacity* if for some compact subset $K \subseteq L$, there exists a nonzero positive measure μ on K such that

$$G_\mu(z) \equiv \int_K \log \frac{1}{|b_\alpha(z)|} d\mu(\alpha) \text{ is bounded on } \mathbb{D}.$$

The function $G_\mu(z)$ is called the *Green's potential*. It is known that every set of positive area has positive capacity, so that every set of capacity zero in $\mathbb{D}$ has area measure zero.

On the other hand, if $\theta \in H^\infty$ is an inner function, then $z_0 \in \mathbb{T}$ is called a *singularity* of θ if θ does not extend analytically from $\mathbb{D}$ to z_0. Write

$$\theta \equiv b \cdot s,$$

where b is a Blaschke product and s is a singular inner function with singular measure μ. It is known ([**Ga**, Theorems 6.1 and 6.2]) that z_0 is a singularity of θ if and only if z_0 is an accumulation point of $\mathcal{Z}(b)$ or $z_0 \in \operatorname{supp}(\mu)$.

Now let $f \in H^\infty$ and let $z_0 \in \mathbb{T}$. Then the *range set* of f at z_0 is defined as

$$\mathcal{R}(f, z_0) := \bigcap_{r>0} f\big(\mathbb{D} \cap \Delta(z_0, r)\big),$$

where $\Delta(z_0, r) := \{z : \ |z - z_0| < r\}$. Thus we can see that the range set is the set of values assumed infinitely often in each neighborhood of z_0. If f is analytic across z_0, and non-constant then $\mathcal{R}(f, z_0) = \emptyset$.

We recall:

Lemma 4.24. ([**Ga**, Theorem 6.6]) *Let θ be an inner function and let z_0 be a singularity of θ. Then*

$$\mathcal{R}(\theta, z_0) = \mathbb{D} \setminus L,$$

where L is a set of capacity zero. Moreover, if $\alpha \in \mathcal{R}(\theta, z_0)$, then $b_\alpha \circ \theta$ is a Blaschke product.

As we have remarked, a set of capacity zero has area measure zero, so that Lemma 4.24 shows that $\mathbb{D}$ is the range set of an inner function θ at its singularity, except possibly for a set of measure zero.

We then have:

Theorem 4.25. *Suppose that θ and δ are finite Blaschke products. If ω is an inner function satisfying one of the following:*

(i) *ω is a finite Blaschke product;*

(ii) $\mathcal{Z}(\theta) \bigcup \mathcal{Z}(\delta) \subseteq \bigcup \Big\{\mathcal{R}(\omega, z_0) : z_0$ *is a singularity of* $\omega\Big\}$,

then we have

$$\theta \text{ and } \delta \text{ are coprime} \iff \theta \circ \omega \text{ and } \delta \circ \omega \text{ are coprime.}$$

Proof. If ω is a finite Blaschke product, then obviously, $\theta \circ \omega$ and $\delta \circ \omega$ are also finite Blaschke products. Suppose that (ii) holds. It then follows from Lemma 4.24 that $\theta \circ \omega$ and $\delta \circ \omega$ are Blaschke products. Observe that

$$\mathcal{Z}(\theta \circ \omega) \cap \mathcal{Z}(\delta \circ \omega) = \Big\{\alpha \in \mathbb{D} : \omega(\alpha) \in \mathcal{Z}(\theta) \cap \mathcal{Z}(\delta)\Big\}. \tag{4.17}$$

If (i) or (ii) holds, then it is clear that $\omega(\mathbb{D}) \supseteq \mathcal{Z}(\theta) \cap \mathcal{Z}(\delta)$. It thus follows from (4.17) that

$$\begin{aligned}\theta \text{ and } \delta \text{ are coprime} &\iff \mathcal{Z}(\theta) \cap \mathcal{Z}(\delta) = \emptyset \\ &\iff \mathcal{Z}(\theta \circ \omega) \cap \mathcal{Z}(\delta \circ \omega) = \emptyset \\ &\iff \theta \circ \omega \text{ and } \delta \circ \omega \text{ are coprime,}\end{aligned}$$

which gives the result. $\square$

Corollary 4.26. *Suppose that* $\{\theta_1, \theta_2, \cdots, \theta_N\}$ *is the set of finite Blaschke products. If* ω *is an inner function satisfying one of the following:*

(i) ω *is a finite Blaschke product;*

(ii) $\bigcup_{i=1}^{N} \mathcal{Z}(\theta_i) \subseteq \bigcup \Big\{ \mathcal{R}(\omega, z_0) : z_0 \text{ is a singularity of } \omega \Big\}$,

then

$$\text{g.c.d.}\{\theta_k : 1 \le k \le N\} = 1 \iff \text{g.c.d.}\{\theta_k \circ \omega : 1 \le k \le N\} = 1. \tag{4.18}$$

Proof. Immediate from Theorem 4.25. □

Theorem 4.27. *If* $f \in H^2$ *is an outer function and* $\omega \in H^\infty$ *is an inner function, then* $f \circ \omega$ *is an outer function.*

Proof. Let f be an outer function and ω be an inner function. Since f is outer, there exists a sequence of polynomials p_n such that $f \cdot p_n \to 1$ as $n \to \infty$. Since $p_n \circ \omega \in H^2$, there exists a sequence of polynomials $q_n^{(m)}$ such that

$$q_n^{(m)} \to p_n \circ \omega \quad \text{as } m \to \infty \quad (\text{for each } n \in \mathbb{N}).$$

We thus have

$$(f \circ \omega) \cdot q_n^{(m)} \longrightarrow (f \cdot p_n) \circ \omega \quad \text{as } m \to \infty.$$

On the other hand, since

$$(f \cdot p_n) \circ \omega \longrightarrow 1 \circ \omega = 1 \quad \text{as } n \to \infty,$$

it follows that $1 \in \text{cl}\,(f \circ \omega) \cdot \mathcal{P}_{\mathbb{C}}$. Thus we can conclude that $f \circ \omega$ is outer. This completes the proof. □

Corollary 4.28. *Suppose that* $\varphi \in H^\infty$. *Then we can write*

$$\varphi = \varphi_i \varphi_e \quad (\text{inner-outer factorization}),$$

then, for an inner function ω, *we have*

$$\varphi \circ \omega = (\varphi_i \circ \omega) \cdot (\varphi_e \circ \omega) \quad (\text{inner-outer factorization}).$$

Proof. Since $\varphi_i \circ \omega$ is an inner function, it follow from Theorem 4.27. □

On the other hand, we may ask what is the relation between the inner parts of the inner-outer factorization and the coprime factorization for bounded analytic functions whose conjugates are of bounded type. Let us consider a simple example: take $\varphi(z) = z - 2$. Then

$$\varphi(z) = \begin{cases} 1 \cdot (z-2) & (\text{inner-outer factoriztion}) \\ z \cdot \overline{(1-2z)} & (\text{coprime}), \end{cases}$$

which shows that the degree of the inner part of the inner-outer factorization is less than the degree of the inner part of the coprime factorization. Indeed, this is not unexpected. In what follows, we will show that this phenomenon holds even for matrix-valued cases.

Lemma 4.29. ([**Ni1**, p. 21]) *Let* $F \in H^2_{M_{m \times n}}$ $(n \ge m)$. *The following re equivalent:*

(a) *F is an outer function;*
(b) g.c.d.$\Big\{\big(\det(F|_{\mathbb{C}^m})\big)_i\Big\} = 1,$

where $F|_{\mathbb{C}^m}$ run through all minors of F in $H^2_{M_m}$ with respect to the canonical bases.

THEOREM 4.30. *Suppose $F \in H^2_{m\times n}$ $(n \geq m)$ is a rational outer function. Then, for a finite Blaschke product $\omega \in H^\infty$, $F \circ \omega$ is an outer function.*

PROOF. Suppose F is a rational outer function. Then by Lemma 4.29,

$$\text{g.c.d.}\Big\{\Big(\det(F|_{\mathbb{C}^m})\Big)_i\Big\} = 1. \tag{4.19}$$

Since $\det\big((F\circ\omega)|_{\mathbb{C}^m}\big) = \det\big((F|_{\mathbb{C}^m})\circ\omega\big) = \Big(\det\big(F|_{\mathbb{C}^m}\big)\Big)\circ\omega$, it follows that

$$\Big(\det\big((F\circ\omega)|_{\mathbb{C}^m}\big)\Big)_i = \Big(\big(\det(F|_{\mathbb{C}^m})\big)\circ\omega\Big)_i. \tag{4.20}$$

Thus it follows from Corollary 4.28 and (4.20) that

$$\Big(\det\big((F\circ\omega)|_{\mathbb{C}^m}\big)\Big)_i = \Big(\det\big(F|_{\mathbb{C}^m}\big)\Big)_i\circ\omega. \tag{4.21}$$

Since F is rational, $\det\big(F|_{\mathbb{C}^m}\big)$ is also rational. It thus follows from (4.19), (4.21) and Corollary 4.26 that

$$\text{g.c.d.}\Big\{\Big(\det\big(F\circ\omega|_{\mathbb{C}^m}\big)\Big)_i\Big\} = \text{g.c.d.}\Big\{\Big(\det\big(F|_{\mathbb{C}^m}\big)\Big)_i\circ\omega\Big\} = 1.$$

Thus by Lemma 4.29, $F\circ\omega$ is an outer function. $\square$

Suppose $\Phi \in H^2_{M_n}$ is such that Φ^* is of bounded type. Then in view of (4.2), we may write $\Phi = \Theta A^*$ (right coprime). Now we define the degree of Φ by

$$\deg(\Phi) := \dim \mathcal{H}(\Theta)\,. \tag{4.22}$$

Since by the Beurling-Lax-Halmos Theorem, the right coprime factorization is unique (up to a unitary constant right factor), it follows that $\deg(\Phi)$ is well-defined. Moreover, by using the well-known Fredholm theory of block Toeplitz operators [**Do2**] we can see that

$$\deg(\Phi) = \dim\mathcal{H}(\Theta) = \deg(\det\Theta). \tag{4.23}$$

We observe that if $\Theta \in H^2_{M_n}$ is an inner matrix function then

$$\deg(\Theta) < \infty \Longleftrightarrow \Theta \text{ is a finite Blaschke-Potapov product.} \tag{4.24}$$

Thus if $\Phi \in H^2_{M_n}$, then Φ is rational if and only if $\deg(\Phi) < \infty$.

On the other hand, the following lemma shows that the notion of degree given in (4.22) is also well defined in the sense that it is independent of the left or the right coprime factorization.

LEMMA 4.31. *Let $\Phi \in H^2_{M_n}$ be such that Φ^* is of bounded type. If*

$$\Phi = \Theta_r A_r^* \ \ \text{(right coprime)} = A_l^*\Theta_l \ \ \text{(left coprime)}\,,$$

then $\dim\mathcal{H}(\Theta_r) = \dim\mathcal{H}(\Theta_l)$.

Proof. Observe that

$$\begin{aligned}\dim \mathcal{H}(\Theta_r) &= \dim \Big(\ker H_{A_r\Theta_r^*}\Big)^{\perp}\\ &= \operatorname{rank} H_{A_r\Theta_r^*} = \operatorname{rank} H_{\Theta_l^* A_l}\\ &= \dim \Big(\ker H_{\widetilde{A}_l\widetilde{\Theta}_l^*}\Big)^{\perp}\\ &= \dim \mathcal{H}(\widetilde{\Theta}_l) = \dim \mathcal{H}(\Theta_l)\,,\end{aligned}$$

where the last equality follows from (4.23). □

We should not expect that $\deg(\Phi) = \deg\big(\det \Phi\big)$ for a rational function $\Phi \in H^2_{M_n}$. To see this, let

$$\Phi := \begin{pmatrix} z & -b_\alpha z\\ 0 & 1\end{pmatrix} \quad (b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}).$$

Then $\det \Phi = z$, so that $\deg(\det \Phi) = 1$. But Φ can be written as

$$\Phi = \begin{pmatrix} b_\alpha z & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} b_\alpha & 0\\ -1 & 1\end{pmatrix}^* \equiv \Theta A^* \quad \text{(right coprime)}\,,$$

so that $\deg(\Phi) = \deg(\det \Theta) = \deg(b_\alpha z) = 2$.

We recall that if $\Phi \in H^2_{M_n}$, define the *local rank* of Φ by

$$\operatorname{Rank} \Phi := \max_{\zeta\in\mathbb{D}} \operatorname{rank} \Phi(\zeta),$$

where $\operatorname{rank} \Phi(\zeta) := \dim \Phi(\zeta)(\mathbb{C}^n)$.

The following lemma provides an information on the local rank.

Lemma 4.32. (*cf.* [**Ni1**, p.44]) *Suppose $\Phi \in H^2_{M_n}$ has the inner-outer factorization*

$$\Phi = \Phi_i \Phi_e.$$

Then $\operatorname{cl} \Phi \mathcal{P}_{\mathbb{C}^n} = \Phi_i H^2_{\mathbb{C}^m}$, *where* $m = \operatorname{Rank} \Phi = \operatorname{Rank} \Phi_i$.

We then have:

Theorem 4.33. *Suppose $\Phi \in H^\infty_{M_n}$. Then we have:*

(a) *If* $\det \Phi = 0$, *then* $\operatorname{codim}\big(\operatorname{ran} T_\Phi\big) = \infty$*;*
(b) *If* $\det \Phi \neq 0$, *then* $\operatorname{codim}\big(\operatorname{ran} T_\Phi\big) \le \operatorname{rank} H_{\Phi^*}$*.*

Proof. Suppose that Φ has the following inner-outer factorization.

$$\Phi = \Phi_i \Phi_e \qquad (\Phi_i \in H^\infty_{M_{n\times m}},\ \Phi_e \in H^\infty_{M_{m\times n}}\ (m \le n))$$

Since Φ_e is an outer function, it follows from Lemma 4.32 that

$$\operatorname{cl} \operatorname{ran} T_\Phi = \operatorname{cl} \Phi H^2_{\mathbb{C}^n} = \operatorname{cl} \Phi \mathcal{P}_{\mathbb{C}^n} = \Phi_i H^2_{\mathbb{C}^m} \quad (\operatorname{Rank} \Phi = m). \tag{4.25}$$

If $\det \Phi = 0$, then by Lemma 4.32, $\operatorname{Rank} \Phi_i = \operatorname{Rank} \Phi = m < n$, so that $\Phi_i \in H^\infty_{M_{n\times m}}$. Thus $\mathcal{H}(\Phi_i) \cong H^2_{\mathbb{C}^{n-m}}$, so that by (4.25), $\operatorname{codim}\big(\operatorname{ran} T_\Phi\big) = \dim\, \mathcal{H}(\Phi_i) = \infty$, which gives (a). Towards (b), suppose that $\det \Phi \neq 0$. If Φ is not rational, then $\operatorname{rank} H_{\Phi^*} = \infty$, which gives the result. If instead $\Phi \in H^\infty_{M_n}$ is rational, then we may write

$$\Phi = A^* \Theta \quad \text{(left coprime)},$$

where $\Theta \in H^\infty_{M_n}$. Then $\det A \neq 0$, and hence T_A is injective. Thus

$$\begin{aligned}\ker T_\Phi^* &= \ker T_{\Theta^*} T_A = \{f \in H^2_{\mathbb{C}^n} : T_{\Theta^*} T_A f = 0\}\\ &\cong T_A \Big\{ f : T_{\Theta^*} T_A f = 0 \Big\}\\ &= (\operatorname{ran} T_A) \bigcap (\ker T_{\Theta^*}) \subseteq \mathcal{H}(\Theta).\end{aligned}$$

Therefore

$$\operatorname{codim}\big(\operatorname{ran} T_\Phi\big) = \dim \ker T_\Phi^* \leq \dim \mathcal{H}(\Theta) = \dim \left(\ker H^*_{\Phi^*}\right)^\perp = \operatorname{rank} H_{\Phi^*},$$

which gives the result. □

Corollary 4.34. *Suppose* $\Phi \in H^\infty_{M_n}$ *be such that* Φ^* *is of bounded type. Thus we may write*

$$\Phi = \begin{cases} \Phi_i \Phi_e & \text{(inner-outer factorization)}\\ A^* \Theta & \text{(left coprime)}.\end{cases}$$

If $\det \Phi \neq 0$, *then* $\deg(\Phi_i) \leq \deg(\Theta)$. *In particular, if* Φ *is a rational function, then* Φ_i *is a finite Blaschke-Potapov product.*

Proof. Since $\det \Phi \neq 0$, it follows from Theorem 4.33 that

$$\deg(\Phi_i) = \operatorname{codim}\big(\operatorname{ran} T_\Phi\big) \leq \operatorname{rank} H_{\Phi^*} = \dim \mathcal{H}(\widetilde{\Theta}) = \deg(\Theta).$$

The last assertion is obvious. □

The following example illustrates Corollary 4.34.

Example 4.35. (i) Let

$$\Phi := \begin{pmatrix} z & z\\ 0 & 0\end{pmatrix}.$$

Then Φ has the following inner-outer factorization:

$$\Phi = \begin{pmatrix} z\\ 0\end{pmatrix} \begin{pmatrix} 1 & 1\end{pmatrix} \equiv \Phi_i \Phi_e.$$

Thus we have

$$\deg(\Phi_i) = \dim \begin{pmatrix} zH^2\\ 0\end{pmatrix}^\perp = \infty,$$

On the other hand,

$$\Phi = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0\\ 0 & 0\end{pmatrix}\right)^* \left(\frac{1}{\sqrt{2}} \begin{pmatrix} z & z\\ 1 & -1\end{pmatrix}\right) \equiv A^* \Theta \quad \text{(left coprime)}.$$

Thus $\deg(\Theta) = 1 < \deg(\Phi_i)$. Note that $\det \Phi = 0$.

(ii) Let

$$\Phi := \begin{pmatrix} z & 0\\ 0 & z+2\end{pmatrix}.$$

Then we have

$$\Phi = \begin{cases} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z+2 \end{pmatrix} \equiv \Phi_i \Phi_e \quad \text{(inner-outer factorization)} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1+2z \end{pmatrix}^* \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \equiv A^* \Theta \quad \text{(left coprime).} \end{cases}$$

Thus by (4.23), we have $\deg(\Theta) = \deg(\det \Theta) = \deg(z^2) = 2$, but $\deg(\Phi_i) = \deg(\det \Phi_i) = \deg(z) = 1$.

Corollary 4.36. *Suppose $\Phi \in H^\infty_{M_n}$ is a rational function with $\det \Phi \neq 0$. Then for a finite Blaschke product ω, we have*

$$\Phi \circ \omega = (\Phi_i \circ \omega) \cdot (\Phi_e \circ \omega) \quad \text{(inner-outer factorization).}$$

Proof. Suppose that Φ is a rational function and $\det \Phi \neq 0$. Then by Theorem 4.34, Φ_i is a finite Blaschke-Potapov product. Thus $\Phi_i \circ \omega$ is a finite Blaschke-Potapov product, so that by Theorem 4.30 we have

$$\Phi \circ \omega = (\Phi_i \circ \omega) \cdot (\Phi_e \circ \omega) \quad \text{(inner-outer factorization).}$$

This completes the proof. □

To proceed, we need:

Lemma 4.37. *If $\Theta_1, \Theta_2 \in H^\infty_{M_n}$ are rational inner functions then the following are equivalent:*

(a) Θ_1 *and* Θ_2 *are right coprime;*
(b) $\ker \Theta_1(\alpha) \cap \ker \Theta_2(\alpha) = \{0\}$ *for any* $\alpha \in \mathbb{D}$.

Proof. Immediate from [**CHKL**, Corollary 3.2]. □

Theorem 4.38. *Let $\Phi \in H^\infty_{M_n}$ be a matrix rational function with $\det \Phi \neq 0$. Thus we may write*

$$\Phi = A^* \Theta \quad \text{(left coprime),}$$

where Θ is a finite Blaschke-Potapov product. Then, for a finite Blaschke product ω, we have

$$\Phi \circ \omega = \big(A \circ \omega\big)^* \big(\Theta \circ \omega\big) \quad \text{(left coprime).}$$

Proof. Note that A is rational and $\det A \neq 0$. Write

$$A = A_i A_e \quad \text{(inner-outer factorization).}$$

By Corollary 4.36, $A \circ \omega$ can be written as:

$$A \circ \omega = (A_i \circ \omega) \cdot (A_e \circ \omega) \quad \text{(inner-outer factorization).} \tag{4.26}$$

Since Θ and A_i are left coprime and hence $\widetilde{\Theta}$ and $\widetilde{A_i}$ are right coprime it follows from Lemma 4.37 that $\widetilde{\Theta \circ \omega} = \widetilde{\Theta} \circ \widetilde{\omega}$ and $\widetilde{A_i \circ \omega} = \widetilde{A_i} \circ \widetilde{\omega}$ are also right coprime. Thus $\Theta \circ \omega$ and $A_i \circ \omega$ are left coprime. It thus follows from (4.26) that $\Theta \circ \omega$ and $A \circ \omega$ are also left coprime. This completes the proof. □

COROLLARY 4.39. *Let $\Phi \in H^\infty_{M_n}$ be a matrix rational function of the form*

$$\Phi = \theta A^* \quad \text{(coprime)},$$

where θ is a finite Blaschke product. Then, for a finite Blaschke product ω, we have

$$\Phi \circ \omega = \big(\theta \circ \omega\big) \cdot \big(A \circ \omega\big)^* \quad \text{(coprime)}.$$

PROOF. Since I_θ and A are coprime, we have, by Lemma 4.15, $\det A \neq 0$, and hence, $\det \Phi \neq 0$. Thus the result follows at once from Theorem 4.38. □

THEOREM 4.40. *Let $\Phi \in H^2_{M_n}$ be such that Φ^* is of bounded type. Thus we may write*

$$\Phi = \Theta A^* \quad \text{(right coprime)}.$$

Then, for a Blaschke factor θ, we have

$$\Phi \circ \theta = \big(\Theta \circ \theta\big)\big(A \circ \theta\big)^* \quad \text{(right coprime)}.$$

PROOF. Let $\theta \equiv b_\alpha$ be arbitrary Blaschke factor. Suppose that Θ and A are right coprime, and assume, to the contrary, that $\Theta \circ \theta$ and $A \circ \theta$ are not right coprime. Write

$$\Theta_1 := \Theta \circ \theta \quad \text{and} \quad A_1 := A \circ \theta.$$

Since Θ_1 and A_1 are not right coprime, there exists a nonconstant inner matrix function $\Delta \in H^\infty_{M_n}$ such that

$$\Theta_1 = \Theta_2 \Delta \quad \text{and} \quad A_1 = A_2 \Delta \qquad (\Theta_2, A_2 \in H^\infty_{M_n}).$$

Note that $\theta^{-1} = b_{-\alpha}$. Put $\Omega := \Delta \circ b_{-\alpha}$. Then Ω is a nonconstant inner matrix function, and

$$\Theta = \Theta_1 \circ b_{-\alpha} = (\Theta_2 \circ b_{-\alpha})\,\Omega$$

and

$$A = A_1 \circ b_{-\alpha} = (A_2 \circ b_{-\alpha})\,\Omega.$$

Since $\Theta_2 \circ b_{-\alpha}$ and $A_2 \circ b_{-\alpha}$ are in $H^\infty_{M_n}$, Ω is a common right inner divisor of Θ and A, so that Θ and A are not right coprime, a contradiction. □

CHAPTER 5

Tensored-scalar singularity

In this chapter we ask the following question: How does one define a singularity for a matrix function of bounded type? Conventionally, the singularity (or the pole) of matrix L^∞-functions is defined by a singularity (or a pole) of some entry of the matrix functions (cf. [**BGR**], [**BR**]). However, when the singularity corresponds to a pole, we shall use the opposite convention and say that the matrix L^∞-function has a pole of order m at the point $\alpha \in \mathbb{D}$ if every nonzero entry has a pole at α of order at least m and some entry has a pole of order exactly m. This notion of singularity (for the pole case) is more suitable for our study of Toeplitz and Hankel operators.

In this chapter, we give a new notion of "tensored-scalar singularity." This new definition takes advantage of the Hankel operator, as if we were using it to characterize functions of bounded type (cf. (2.2)). This notion contributes to give an answer to the question: Under what conditions does it follow that if the product of two Hankel operators with matrix-valued bounded type symbols is zero then either of the operators is zero?

By our conventions to be defined here, $\Phi := \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix}$ does not have a singularity at all but $\Psi := \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{z^2} \end{pmatrix}$ has a pole of order 1 at zero.

To motivate our new idea, let us carefully consider the scalar-valued case.

If $\varphi \in L^\infty$ is of bounded type then we may write, in view of (2.3),

$$\varphi_- = \omega \overline{a} \quad \text{(coprime)}. \tag{5.1}$$

Since $\varphi = \frac{a}{\omega} + \varphi_+$, the singularities of φ come from ω. Thus if θ is a nonconstant inner divisor of ω then θ leads to singularities of φ. Consequently, we can say that φ has a singularity (with respect to θ) if and only if θ is an inner divisor of ω if and only if $\omega H^2 \subseteq \theta H^2$; in other words,

$$\ker H_\varphi = \omega H^2 \subseteq \theta H^2. \tag{5.2}$$

As a new notion of a singularity for a matrix function, we now adopt the matrix-valued version of the scalar-valued case (5.2).

Definition 5.1. Let $\Phi \in L^\infty_{M_n}$ be of bounded type. We say that Φ has a *tensored-scalar singularity* (*with respect to* θ) if there exists a nonconstant inner function θ such that

$$\ker H_\Phi \subseteq \theta H^2_{\mathbb{C}^n}.$$

If φ is a rational function, then ω in (5.2) is a finite Blaschke product, so a tensored-scalar singularity reduces to a matrix pole (cf. [**CHKL**, Definition 3.5]) –

hereafter (to avoid confusion with the convention), we shall call it a *tensored-scalar pole*. In particular, we say that Φ has a *tensored-scalar singularity of order* p ($p \in \mathbb{Z}_+$) if p is the degree of θ_0, where

$$\theta_0 := \text{l.c.m.}\{\theta : \ \Phi \text{ has a tensored-scalar singularity with respect to } \theta\}.$$

Moreover, if $\theta_0 = b_\alpha^p$, then we say that Φ has a tensored-scalar pole of order p at α.

On the other hand, we note that every non-analytic bounded type function $\varphi \in L^\infty$ has, trivially, a tensored-scalar singularity; for, if $\varphi \notin H^\infty$, then by Beurling's Theorem, $\ker H_\varphi = \theta H^2$ for a nonconstant inner function $\theta \in H^\infty$.

We also observe:

Lemma 5.2. *Let $\Phi \in L^\infty_{M_n}$ be of bounded type, and write $\Phi = A\Theta^*$* (right coprime). *Then Φ has a tensored-scalar singularity if and only if Θ has a nonconstant diagonal-constant inner divisor.*

Proof. Observe that $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$, so that $\ker H_\Phi \subseteq \theta H^2_{\mathbb{C}^n}$ ($\theta \in H^\infty$ is inner) if and only if $\Theta H^2_{\mathbb{C}^n} \subseteq \theta H^2_{\mathbb{C}^n}$ if and only if I_θ is an inner divisor of Θ. $\square$

In view of Lemma 5.2, if $\Phi \in L^\infty_{M_n}$ has the following coprime factorization:

$$\Phi_- = \theta B^* \quad \text{(coprime)};$$

then, clearly, Φ has a tensored-scalar singularity.

On the other hand, it is well-known that if $\varphi, \psi \in L^\infty$, then

$$H_\varphi H_\psi = 0 \implies H_\varphi = 0 \ \text{ or } \ H_\psi = 0. \tag{5.3}$$

But (5.3) may fail for matrix-valued functions. For example, if we take

$$\Phi := \begin{pmatrix} 1 & 0 \\ 0 & \overline{z} \end{pmatrix} \quad \text{and} \quad \Psi := \begin{pmatrix} \overline{z} & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.4}$$

then $H_\Phi H_\Psi = 0$, but $H_\Phi \neq 0$ and $H_\Psi \neq 0$.

We will now try to find a general condition for (5.3) to hold in the matrix-valued case. To do so, we need:

Proposition 5.3. *Let $\Phi \in L^\infty_{M_n}$ be of bounded type. Then Φ has a tensored-scalar singularity with respect to θ if and only if $\widetilde{\Phi}$ has a tensored-scalar singularity with respect to $\widetilde{\theta}$.*

Proof. Write

$$\Phi = A\Theta^* \quad \text{(right coprime)}.$$

In view of Lemma 5.2, we may write $\Theta = I_\theta \Theta_1$ (Θ_1 inner). Since A and I_θ are right coprime, by Theorem 4.16 A and I_θ are left coprime, so that $\widetilde{A}$ and $I_{\widetilde{\theta}}$ are right coprime. It thus follows from (4.3) that

$$\ker H_{\widetilde{\Phi}} = \ker H_{\widetilde{\Theta}_1^* \widetilde{A} \overline{I_{\widetilde{\theta}}}} \subseteq \ker H_{\widetilde{A}\overline{I_{\widetilde{\theta}}}} = \widetilde{\theta} H^2_{\mathbb{C}^n},$$

which implies that $\widetilde{\Phi}$ has a tensored-scalar singularity with respect to $\widetilde{\theta}$. For the converse, we use the fact $\Phi = \widetilde{\widetilde{\Phi}}$ and $\theta = \widetilde{\widetilde{\theta}}$. This completes the proof. $\square$

We now have:

THEOREM 5.4. *Let $\Phi, \Psi \in L^\infty_{M_n}$ be of bounded type. If Φ or Ψ has a tensored-scalar singularity then*

$$H_\Psi H_\Phi = 0 \implies H_\Phi = 0 \text{ or } H_\Psi = 0.$$

PROOF. Write

$$\Phi = \Delta^* B \quad \text{(left coprime)} \quad \text{and} \quad \Psi = A\Theta^* \quad \text{(right coprime)}.$$

Suppose that Φ has a tensored-scalar singularity. By Lemma 5.2, there exists a nonconstant inner function δ such that

$$\Delta = \delta\Delta_1 \quad (\Delta_1 \text{ inner}).$$

Suppose $H_\Psi H_\Phi = 0$. Since $\widetilde{\Delta}$ and $\widetilde{B}$ are right coprime, it follows from (4.3) that

$$\mathcal{H}(I_{\widetilde{\delta}}) \subseteq \mathcal{H}(\widetilde{\Delta}) = \Big(\ker H^*_\Phi\Big)^\perp \subseteq \ker H_\Psi = \Theta H^2_{\mathbb{C}^n}. \tag{5.5}$$

Write $\Theta \equiv \big(\theta_{ij}\big)_{i,j=1}^n$. Since $\widetilde{\delta}$ is not constant, $\mathcal{H}_{\widetilde{\delta}}$ has at least an outer function ξ that is invertible in H^∞ (cf. [**CHL1**, Lemma 3.4]). It thus follows from (5.5) that $\Theta^*(\xi, 0, 0, \cdots)^t \in H^2_{\mathbb{C}^n}$, which implies $\overline{\theta}_{1j}\xi \in H^2$ for each $j = 1, \cdots, n$. Similarly, we can show that

$$\overline{\theta}_{ij}\,\xi \in H^2 \quad \text{for each } i, j = 1, 2, \cdots, n,$$

so that $\overline{\theta}_{ij} \in \frac{1}{\xi}H^2 \subseteq H^2$ for each $i, j = 1, 2, \cdots, n$. Therefore each θ_{ij} is constant and hence, Θ is a constant unitary. Therefore $\Psi \in H^2_{M_n}$, which gives $H_\Psi = 0$.

If instead Ψ has a tensored-scalar singularity, then by Proposition 5.3, $\widetilde{\Psi}$ has a tensored-scalar singularity. Thus if $H_\Psi H_\Phi = 0$, then $H_{\widetilde{\Phi}} H_{\widetilde{\Psi}} = 0$, so that $H_\Phi = H^*_{\widetilde{\Phi}} = 0$ by what we proved just above. $\square$

We observe that if $\Phi \in L^\infty_{M_n}$ is normal then by (2.9),

$$[T^*_\Phi, T_\Phi] = H^*_{\Phi^*}H_{\Phi^*} - H^*_\Phi H_\Phi.$$

On the other hand, we recall ([**GHR**, Theorem 3.3], [**Gu2**, Corollary 2]) that if $\Phi \in L^\infty_{M_n}$ then

$$H^*_{\Phi^*}H_{\Phi^*} \geq H^*_\Phi H_\Phi \iff \exists\, K \in H^\infty_{M_n} \text{ such that } ||K||_\infty \leq 1 \text{ and } \Phi - K\Phi^* \in H^\infty_{M_n}. \tag{5.6}$$

The following theorem is of independent interest.

THEOREM 5.5. *Let $\Phi, \Psi \in L^\infty_{M_n}$ be of bounded type. If Φ or Ψ has a tensored-scalar singularity then*

$$H^*_\Phi H_\Phi = H^*_\Psi H_\Psi \iff \Phi - U\Psi \in H^\infty_{M_n}$$

for some unitary constant matrix $U \in M_n$.

PROOF. Write

$$\Phi = A\Theta^* \quad \text{(right coprime)}.$$

If Φ has a tensored-scalar singularity then by Lemma 5.2, Θ has an inner divisor of the form I_θ, with θ a nonconstant inner function. Thus we may write $\Theta = \theta\Theta_1$ for an inner matrix function $\Theta_1 \in H^\infty_{M_n}$. It follows that A and I_θ are right coprime, so that by Theorem 4.16, A and I_θ are left coprime, and hence $\widetilde{A}$ and $I_{\widetilde{\theta}}$ are right coprime. Thus

$$\ker H^*_\Phi = \ker H_{\widetilde{\Phi}} = \ker H_{\overline{\widetilde{\theta}}\widetilde{\Theta}_1^* \widetilde{A}} \subseteq \ker H_{\overline{\widetilde{\theta}}\widetilde{A}} = \widetilde{\theta} H^2_{\mathbb{C}^n}. \tag{5.7}$$

If $H^*_\Phi H_\Phi = H^*_\Psi H_\Psi$, then $\ker H_\Psi = \ker H_\Phi \subseteq \theta H^2_{\mathbb{C}^n}$, which implies that Ψ also has a tensored-scalar singularity. Therefore without loss of generality we may assume that Φ has a tensored-scalar singularity. Since $H^*_\Phi H_\Phi = H^*_\Psi H_\Psi$, we have $H^*_{\Phi^*_-} H_{\Phi^*_-} = H^*_{\Psi^*_-} H_{\Psi^*_-}$. Then by (5.6), there exist $U, U' \in H^\infty_{M_n}$ with $||U||_\infty \le 1$ and $||U'||_\infty \le 1$ such that

$$\Phi^*_- - U\Psi^*_- \in H^2_{M_n} \quad \text{and} \quad \Psi^*_- - U'\Phi^*_- \in H^2_{M_n}. \tag{5.8}$$

It follows from (5.8) that $\Phi^*_- - UU'\Phi^*_- \in H^2_{M_n}$, so that $H_{\Phi^*_-} - H_{UU'\Phi^*_-} = 0$, and hence $(I - T^*_{\widetilde{UU'}})H_\Phi = 0$, which together with (5.7) implies

$$\mathcal{H}(I_{\widetilde{\theta}}) \subseteq \operatorname{ran} H_\Phi \subseteq \ker (I - T^*_{\widetilde{UU'}}). \tag{5.9}$$

Thus, we have $F = T^*_{\widetilde{UU'}} F$ for each $F \in \mathcal{H}(I_{\widetilde{\theta}})$. But since $||\widetilde{UU'}^*||_\infty = ||UU'||_\infty \le 1$, we should have $\widetilde{UU'}^* F \in H^2_{\mathbb{C}^n}$, so that $F = \widetilde{UU'}^* F$ for each $F \in \mathcal{H}(I_{\widetilde{\theta}})$. Thus UU' is constant, and actually $UU' = I_n$. Therefore U is a unitary constant and by (5.8), $\Phi - U\Psi \in H^\infty_{M_n}$. The converse is evident. $\square$

For $A \in L^2_{M_n}$ and an inner matrix function $\Theta \in H^2_{M_n}$, we define

$$(T_A)_\Theta := P_{\mathcal{H}(\Theta)} T_A|_{\mathcal{H}(\Theta)},$$

that is, $(T_A)_\Theta f := P_{\mathcal{H}(\Theta)} Af$ for each $f \in \mathcal{H}(\Theta)$. Note that T_A is densely defined and possibly unbounded. However $(T_A)_\Theta$ may be bounded under certain conditions. To see this, let

$$\overline{(BMO)_{M_n}} := \Big\{ \Phi \equiv (\varphi_{ij}) \in L^2_{M_n} : \overline{\varphi_{ij}} \in BMO \Big\}.$$

We then have:

Lemma 5.6. *Let $A \in H^2_{M_n}$ and $\Theta \in H^\infty_{M_n}$ be an inner matrix function. If $G \equiv A^*\Theta \in H^2_{M_n} \cap \overline{(BMO)_{M_n}}$, then $(T_A)_\Theta$ is bounded.*

Proof. Since $G^* \in (BMO)_{M_n}$, we can find a matrix function $F \in H^2_{M_n}$ such that

$$\Phi \equiv G^* + F \in L^\infty_{M_n}.$$

(cf. [**Pe**, Theorem 1.3]). Thus $\Theta\Phi \in L^\infty_{M_n}$. Observe that

$$\begin{aligned} P_{\mathcal{H}(\Theta)} T_{\Theta\Phi}|_{\mathcal{H}(\Theta)} &= P_{\mathcal{H}(\Theta)} T_{\Theta(\Theta^* A + F)}|_{\mathcal{H}(\Theta)} \\ &= P_{\mathcal{H}(\Theta)} T_{A + \Theta F}|_{\mathcal{H}(\Theta)} \\ &= P_{\mathcal{H}(\Theta)} T_A|_{\mathcal{H}(\Theta)} \\ &= (T_A)_\Theta, \end{aligned}$$

which implies $(T_A)_\Theta$ is bounded. $\square$

In view of Lemma 5.6, $(T_A)_\Theta$ is understood in the sense that the compression $(T_A)_\Theta$ is bounded even though T_A is possibly unbounded if $A^*\Theta \in H^2_{M_n} \cap \overline{(BMO)_{M_n}}$. (In particular, we are interested in the case $A^*\Theta \equiv \Phi_-$ for a matrix function $\Phi \in L^\infty_{M_n}$.)

Here we pause to ask, when is $(T_A)_\Theta$ injective. First of all, we take a look at possible cases when $n = 2$.

(i) Let
$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Theta := \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$
Put $f := \begin{pmatrix} 1 & 0 \end{pmatrix}^t$. Then $f \in \mathcal{H}(\Theta)$, $Af = 0$, and $(T_A)_\Theta f = 0$. Thus $(T_A)_\Theta$ is not injective.

(ii) Let
$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Theta := \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}.$$
Put $f := \begin{pmatrix} 0 & 1 \end{pmatrix}^t$. Then $f \in \mathcal{H}(\Theta)$, $Af = \begin{pmatrix} 1 & 0 \end{pmatrix}^t \neq 0$, and $(T_A)_\Theta f = 0$. Thus $(T_A)_\Theta$ is not injective.

(iii) Let
$$A := \begin{pmatrix} z\delta & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Theta := \begin{pmatrix} b_\alpha z & 0 \\ 0 & z \end{pmatrix} \quad (\text{where } \delta(z) := \frac{\sqrt{1-|\alpha|^2}}{1-\overline{\alpha} z}).$$
Put $f := \begin{pmatrix} b_\alpha & 0 \end{pmatrix}^t$. Then $f \in \mathcal{H}(\Theta)$, $Af = \begin{pmatrix} zb_\alpha\delta & 0 \end{pmatrix}^t \neq 0$, and $(T_A)_\Theta f = 0$. Thus $(T_A)_\Theta$ is not injective. Note that Θ has a tensored-scalar singularity and
$$\Theta A^* = \begin{pmatrix} b_\alpha\overline{\delta} & 0 \\ z & z \end{pmatrix} \in H^2_{M_2}.$$
Note that A and Θ are not right coprime.

We observe:

Lemma 5.7. *Let $A \in H^2_{M_n}$ and let $\Theta \in H^\infty_{M_n}$ be an inner function such that*
$$\Phi \equiv A^*\Theta = \Delta B^* \in H^2_{M_n} \cap \overline{(BMO)_{M_n}},$$
where Δ and B are right coprime. Then $\ker (T_A)_\Theta = \mathcal{H}(\Theta) \cap \Delta H^2_{\mathbb{C}^n}$. *In particular, if Φ^* has a tensored-scalar singularity with respect to θ, then* $\ker (T_A)_\Theta \subseteq I_\theta \mathcal{H}(\Theta_1)$ *with $\Theta_1 := \overline{\theta}\Theta$.*

Proof. Let $f \in \mathcal{H}(\Theta)$. Then we have
$$\begin{aligned} (T_A)_\Theta f = 0 &\Longleftrightarrow B\Delta^* f = \Theta^* A f \in H^2_{\mathbb{C}^n} \\ &\Longleftrightarrow f \in \ker H_{B\Delta^*} = \Delta H^2_{\mathbb{C}^n}, \end{aligned}$$
which implies $\ker (T_A)_\Theta = \mathcal{H}(\Theta) \cap \Delta H^2_{\mathbb{C}^n}$. Suppose that Φ^* has a tensored-scalar singularity with respect to θ. Then, by Lemma 5.2, we may write
$$\Theta = I_\theta \Theta_1 \text{ and } \Delta = I_\theta \Delta_1 \quad (\text{where } \Theta_1 \text{ and } \Delta_1 \text{ are inner}).$$
We thus have
$$\begin{aligned} \ker (T_A)_\Theta &= \mathcal{H}(\Theta) \cap \Delta H^2_{\mathbb{C}^n} \subseteq \mathcal{H}(\Theta) \cap I_\theta H^2_{\mathbb{C}^n} \\ &= \Big[\mathcal{H}(I_\theta) \bigoplus I_\theta \mathcal{H}(\Theta_1)\Big] \cap I_\theta H^2_{\mathbb{C}^n} \\ &= I_\theta \mathcal{H}(\Theta_1), \end{aligned}$$
which gives the result. $\square$

Corollary 5.8. *Let $A \in H^2_{M_n}$ and $\Theta \equiv I_\theta \in H^\infty_{M_n}$ for an inner function θ such that $A^*\Theta \in H^2_{M_n} \cap \overline{(BMO)_{M_n}}$. If A and Θ are coprime then $(T_A)_\Theta$ is injective. In particular, if θ is a finite Blaschke product, then $(T_A)_\Theta$ is invertible.*

Proof. Since A and $\Theta \equiv I_\theta$ are coprime it follows that

$$A^*\Theta = \Theta A^* \quad \text{(coprime)}.$$

It thus follows from Lemma 5.7 that $\ker(T_A)_\Theta = \mathcal{H}(\Theta) \cap \Theta H^2_{\mathbb{C}^n} = \{0\}$, which implies that $(T_A)_\Theta$ is injective. If θ is a finite Blaschke product, then $(T_A)_\Theta$ is a finite dimensional operator, and it follows that $(T_A)_\Theta$ is invertible. □

Theorem 5.9. *Let $A \in H^2_{M_n}$ and let $\Theta = I_\theta \in H^\infty_{M_n}$ for an inner function θ such that $A^*\Theta \in H^2_{M_n} \cap \overline{(BMO)_{M_n}}$. If A and Θ are coprime, then $(T_A)_\Theta$ has a linear inverse (possibly unbounded).*

Proof. By Corollary 5.8, $(T_A)_\Theta$ is injective. Thus it suffices to show that $(T_A)^*_\Theta$ is injective. Observe that for $U, V \in \mathcal{H}(\Theta)$,

$$\big\langle (T_A)_\Theta U,\ V \big\rangle = \big\langle AU,\ V \big\rangle = \int_{\mathbb{T}} \mathrm{tr}\Big((A^*V)^*U\Big)\, d\mu = \big\langle U, A^*V \big\rangle,$$

which implies

$$(T_A)^*_\Theta = (T_{A^*})_\Theta. \tag{5.10}$$

On the contrary, assume that $(T_A)^*_\Theta$ is not injective. Then there exists a nonzero vector $f \in \mathcal{H}(\Theta)$ such that

$$0 = (T_A)^*_\Theta f = P_{\mathcal{H}(\Theta)}(A^* f),$$

which gives $A^* f \in \Theta H^2_{\mathbb{C}^n}$. Since $\Theta = I_\theta$ and $f \in \mathcal{H}(\Theta)$, it follows that

$$\Theta^*(A^* f) = A^*(\Theta^* f) \in H^2_{\mathbb{C}^n} \cap (H^2_{\mathbb{C}^n})^\perp = \{0\},$$

which implies that $A^* f = 0$. Since A and Θ are coprime it follows from Lemma 4.15 that A^* is invertible and hence $f = 0$, a contradiction. This proves $(T_A)^*_\Theta$ is injective. □

Remark 5.10. From Lemma 5.7, we can see that if $A \in H^2_{M_n}$ and $\Theta \in H^\infty_{M_n}$ is an inner matrix function such that

$$A^*\Theta = \Delta B^* \in H^2_{M_n} \cap \overline{(BMO)_{M_n}},$$

where Δ and B are right coprime, then

$$\Theta = \Delta \implies (T_A)_\Theta \text{ is injective.} \tag{5.11}$$

But (5.11) does not hold in general if $\Theta \neq \Delta$. For example, let

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Theta := \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A^*\Theta = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^* \equiv \Delta B^*.$$

Note that Δ and B are right coprime. It thus follows from Lemma 5.7 that

$$\ker(T_A)_\Theta = \mathcal{H}(\Theta) \cap \Delta H^2_{\mathbb{C}^n} = \mathbb{C} \oplus \{0\} \neq \{0\},$$

which implies that $(T_A)_\Theta$ is not injective. Here we note that Θ and A are not right coprime and $\det A = 0$. Thus we may expect that if

(1) Θ and A are right coprime;
(2) $\det A \neq 0$,

then $(T_A)_\Theta$ is injective although $\Theta \neq \Delta$. However this is not the case. For example, let

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \Theta := \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A^*\Theta = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^* \equiv \Delta A^*.$$

Note that Δ and A are right coprime. It thus follows from Lemma 5.7 that

$$\ker(T_A)_\Theta = \mathcal{H}(\Theta) \cap \Delta H^2_{\mathbb{C}^n} = \mathbb{C} \oplus \{0\} \neq \{0\},$$

which implies that $(T_A)_\Theta$ is not injective. Note that Θ and A are right coprime, and $\det A \neq 0$.

CHAPTER 6

An interpolation problem and a functional calculus

In this chapter, we consider both an interpolation problem for matrix functions of bounded type and a functional calculus for compressions of the shift operator.

We first review the classical Hermite-Fejér interpolation problem, following [**FF**]. Let θ be a finite Blaschke product of degree d:

$$\theta = e^{i\xi} \prod_{i=1}^{N} (\widetilde{b}_i)^{m_i} \qquad \left(\widetilde{b}_i(z) := \frac{z-\alpha_i}{1-\overline{\alpha}_i z}, \text{ where } \alpha_i \in \mathbb{D} \right),$$

where $d = \sum_{i=1}^N m_i$. For our purposes, rewrite θ in the form

$$\theta = e^{i\xi} \prod_{j=1}^{d} b_j,$$

where

$$b_j := \widetilde{b}_k \quad \text{if } \sum_{l=0}^{k-1} m_l < j \le \sum_{l=0}^{k} m_l$$

and, for notational convenience, set $m_0 := 0$. Let

$$\varphi_j := \frac{q_j}{1-\overline{\alpha}_j z} b_{j-1} b_{j-2} \cdots b_1 \quad (1 \le j \le d), \tag{6.1}$$

where $\varphi_1 := q_1(1-\overline{\alpha}_1 z)^{-1}$ and $q_j := (1-|\alpha_j|^2)^{\frac{1}{2}}$ $(1 \le j \le d)$. It is well known (cf. [**Ta**]) that $\{\varphi_j\}_{j=1}^d$ is an orthonormal basis for $\mathcal{H}(\theta)$.

For our purposes we concentrate on the data given by sequences of $n \times n$ complex matrices. Given the sequence $\{K_{ij} : \ 1 \le i \le N, \ 0 \le j < m_i\}$ of $n \times n$ complex matrices and a set of distinct complex numbers $\alpha_1, \ldots, \alpha_N$ in $\mathbb{D}$, the classical Hermite-Fejér interpolation problem entails finding necessary and sufficient conditions for the existence of a contractive analytic matrix function K in $H^\infty_{M_n}$ satisfying

$$\frac{K^{(j)}(\alpha_i)}{j!} = K_{i,j} \qquad (1 \le i \le N, \ 0 \le j < m_i). \tag{6.2}$$

To construct a matrix polynomial $K(z) \equiv P(z)$ satisfying (6.2), let $p_i(z)$ be the polynomial of order $d - m_i$ defined by

$$p_i(z) := \prod_{k=1, k\ne i}^{N} \left(\frac{z-\alpha_k}{\alpha_i - \alpha_k} \right)^{m_k}.$$

Consider the matrix polynomial $P(z)$ of degree $d-1$ defined by

(6.3)
$$P(z) := \sum_{i=1}^{N}\left(K'_{i,0}+K'_{i,1}(z-\alpha_i)+K'_{i,2}(z-\alpha_i)^2+\cdots+K'_{i,m_i-1}(z-\alpha_i)^{m_i-1}\right)p_i(z),$$

where the $K'_{i,j}$ are obtained by the following equations:

$$K'_{i,j} = K_{i,j} - \sum_{k=0}^{j-1}\frac{K'_{i,k}\,p_i^{(j-k)}(\alpha_i)}{(j-k)!} \quad (1\le i\le N;\ 0\le j< m_i)$$

and $K'_{i,0} = K_{i,0}$ $(1\le i\le N)$. Then $P(z)$ satisfies (6.2).

Let W be the unitary operator from $\bigoplus_1^d \mathbb{C}^n$ onto $\mathcal{H}(I_\theta)$ defined by

$$W := (I_{\varphi_1}, I_{\varphi_2}, \cdots, I_{\varphi_d}), \tag{6.4}$$

where the φ_j are the functions in (6.1). It is known [**FF**, Theorem X.1.5] that if θ is the finite Blaschke product of order d, then U_θ is unitarily equivalent to the lower triangular matrix M on $\mathbb{C}^d$ defined by

$$M := \begin{pmatrix} \alpha_1 & 0 & 0 & 0 & \cdots & \cdots \\ q_1q_2 & \alpha_2 & 0 & 0 & \cdots & \cdots \\ -q_1\overline{\alpha}_1q_3 & q_2q_3 & \alpha_3 & 0 & \cdots & \cdots \\ q_1\overline{\alpha}_2\overline{\alpha}_3q_4 & -q_2\overline{\alpha}_3q_4 & q_3q_4 & \alpha_4 & \cdots & \cdots \\ -q_1\overline{\alpha}_2\overline{\alpha}_3\overline{\alpha}_4q_5 & q_2\overline{\alpha}_3\overline{\alpha}_4q_5 & -q_3\overline{\alpha}_4q_5 & q_4q_5 & \alpha_5 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{6.5}$$

Now let $P(z)\in H^\infty_{M_n}$ be a matrix polynomial of degree k. Then the matrix $P(M)$ on $\mathbb{C}^{n\times d}$ is defined by

$$P(M) := \sum_{i=0}^{k} P_i\otimes M^i, \quad \text{where } P(z)=\sum_{i=0}^{k}P_iz^i. \tag{6.6}$$

If M is given by (6.5) and P is the matrix polynomial defined by (6.3) then the matrix $P(M)$ is called the *Hermite-Fejér matrix* determined by (6.6). In particular, if $(T_P)_\Theta := P_{\mathcal{H}(\Theta)}T_P|_{\mathcal{H}(\Theta)}$ is the compression of T_P to $\mathcal{H}(\Theta)$ (where $\Theta \equiv:= I_\theta$ for an inner function θ) (cf. p.12), then it is known [**FF**, Theorem X.5.6] that

$$W^*(T_P)_\Theta W = P(M), \tag{6.7}$$

which says that $P(M)$ is a matrix representation for $(T_P)_\Theta$.

We now consider an interpolation problem for matrix functions of bounded type. Our interpolation problem involves a certain matrix-valued functional equation: $\Phi - K\Phi^*\in H^\infty_{M_n}$ (where $\Phi\in L^\infty_{M_n}$ and $K\in H^\infty_{M_n}$ is unknown). We may ask, when does there exist such a matrix H^∞-function K? If such a function K exists, how do we find it? If Φ is a matrix-valued *rational* function, this interpolation problem reduces to the classical Hermite-Fejér Interpolation Problem.

More concretely, we consider the following question: For $\Phi\in L^\infty_{M_n}$,

when does there exist a function $K\in H^\infty_{M_n}$ satisfying $\Phi - K\Phi^*\in H^\infty_{M_n}$?

For notational convenience we write, for $\Phi \in L^\infty_{M_n}$,

$$\mathcal{C}(\Phi) := \Big\{ K \in H^\infty_{M_n} :\ \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

Thus question can be rewritten as: For $\Phi \in L^\infty_{M_n}$,

$$\text{when is } \mathcal{C}(\Phi) \text{ nonempty ?} \tag{6.8}$$

Question (6.8) resembles an interpolation problem, as we will see below. In this chapter we consider Question (6.8) for matrix functions of bounded type.

If $C(\Phi) \neq \emptyset$, i.e., there exists a function $K \in H^\infty_{M_n}$ such that $\Phi_-^* - K\Phi_+^* \in H^2_{M_n}$, then $H_{\Phi_-^*} = T^*_{\widetilde{K}} H_{\Phi_+^*}$ for some $K \in H^\infty_{M_n}$, which implies $\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}$. Thus we have

$$\mathcal{C}(\Phi)\ \neq\ \emptyset \Longrightarrow \ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}. \tag{6.9}$$

Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. In view of (4.2), we may write

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^* \quad \text{(right coprime).} \tag{6.10}$$

Thus, if $\mathcal{C}(\Phi) \neq \emptyset$, then by (6.9) and (4.3), $\Theta_1 H^2_{\mathbb{C}^n} \subseteq \Theta_2 H^2_{\mathbb{C}^n}$. It then follows (cf. [**FF**, Corollary IX.2.2]) that Θ_2 is a left inner divisor of Θ_1. Therefore, whenever we consider the interpolation problem (6.8) for a function $\Phi \in L^\infty_{M_n}$ such that Φ and Φ^* are of bounded type, we may assume that $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is of the form

$$\Phi_+ = \Theta_0\Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_0 B^* \quad \text{(right coprime).} \tag{6.11}$$

On the other hand, if $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type, then in view of (2.7), we may also write

$$\Phi_+ = \theta_1 A^* \quad \text{and} \quad \Phi_- = \theta_2 B^*, \tag{6.12}$$

where $\theta_1, \theta_2 \in H^\infty$ are inner functions. If $\mathcal{C}(\Phi) \neq \emptyset$, then we can also show that θ_2 is an inner divisor of θ_1 even though the factorizations in (6.12) are not right coprime (cf. [**CHL2**, Proposition 3.2]). Thus if $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type then we may, without loss of generality, assume that Φ is of the form

$$\Phi_+ = \theta_0\theta_1 A^* \quad \text{and} \quad \Phi_- = \theta_0 B^*, \tag{6.13}$$

where $\theta_0, \theta_1 \in H^\infty_{M_n}$ are inner functions and $A, B \in H^2_{M_n}$. In view of Lemma 5.2, we may also assume that the pairs $\{I_{\overline{\theta}_0\overline{\theta}_1}, A^*\}$ and $\{I_{\overline{\theta}_0}, B^*\}$ have no common tensored-scalar singularity.

First of all, we consider the case of matrix-valued rational functions $\Phi \in L^\infty_{M_n}$. In this case we may write

$$\Phi_+ = \theta_1\theta_0 A^* \quad \text{and} \quad \Phi_- = \theta_1 B^*,$$

where $\theta_0, \theta_1 \in H^\infty$ are finite Blaschke products. Observe that

$$K \in \mathcal{C}(\Phi) \iff \Phi - K\Phi^* \in H^\infty_{M_n} \iff \theta_0 B - KA \in \theta_1\theta_0 H^2_{M_n}. \tag{6.14}$$

Write

$$(\theta_1\theta_0)(z) = \prod_{i=1}^N \left(\frac{z-\alpha_i}{1-\overline{\alpha}_i z}\right)^{m_i} \qquad (d := \sum_{i=1}^N m_i),$$

i.e., $\theta_1\theta_0$ is a finite Blaschke product of degree d. Then the last assertion in (6.14) holds if and only if the following equations hold: for each $i = 1, \dots, N$,

$$\begin{pmatrix} B_{i,0} \\ B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,m_i-1} \end{pmatrix} = \begin{pmatrix} K_{i,0} & 0 & 0 & \cdots & 0 \\ K_{i,1} & K_{i,0} & 0 & \cdots & 0 \\ K_{i,2} & K_{i,1} & K_{i,0} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ K_{i,m_i-1} & K_{i,m_i-2} & \dots & K_{i,1} & K_{i,0} \end{pmatrix} \begin{pmatrix} A_{i,0} \\ A_{i,1} \\ A_{i,2} \\ \vdots \\ A_{i,m_i-1} \end{pmatrix}, \tag{6.15}$$

where

$$K_{i,j} := \frac{K^{(j)}(\alpha_i)}{j!}, \quad A_{i,j} := \frac{A^{(j)}(\alpha_i)}{j!} \quad \text{and} \quad B_{i,j} := \frac{(\theta_0 B)^{(j)}(\alpha_i)}{j!}.$$

Thus K is a function in $H^\infty_{M_n}$ for which

$$\frac{K^{(j)}(\alpha_i)}{j!} = K_{i,j} \qquad (1 \le i \le N,\ 0 \le j < m_i), \tag{6.16}$$

where the $K_{i,j}$ are determined by the equation (6.15). This is exactly the classical Hermite-Fejér interpolation problem. Therefore, the solution (6.3) for the classical Hermite-Fejér interpolation problem provides a polynomial $K \in \mathcal{C}(\Phi)$.

We turn our attention to the case of matrix functions of bounded type.

To proceed, we need:

Proposition 6.1. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. In view of* (6.11)*, we may write*

$$\Phi_+ = \Theta_0\Theta_1 A^* \quad \textit{and} \quad \Phi_- = \Theta_0 B^* \quad \text{(right coprime)}.$$

Suppose $\Theta_1 A^* = A_1^*\Theta$ (where A_1 and Θ are left coprime). Then the following hold:

(a) *If $K \in \mathcal{C}(\Phi)$, then $K = K'\Theta$ for some $K' \in H^\infty_{M_n}$;*

(b) *If I_ω is an inner divisor of Θ_1, then I_ω is an inner divisor of Θ and*

$$\mathcal{C}(\Phi^{1,\omega}) = \left\{\overline{\omega}K : K \in \mathcal{C}(\Phi)\right\}, \tag{6.17}$$

where $\Phi^{1,\omega} := \Phi_-^* + P_{H_0^2}(\overline{\omega}\Phi_+)$ (cf. p.12)*.*

Proof. Observe that

$$K \in \mathcal{C}(\Phi) \Longleftrightarrow B\Theta_0^* - KA\Theta_1^*\Theta_0^* \in H^2_{M_n} \Longleftrightarrow B\Theta_1 - KA \in H^2_{M_n}\Theta_0\Theta_1\,,$$

which implies that $KA \in H^2_{M_n}\Theta_1$, and hence $KA\Theta_1^* \in H^2_{M_n}$. Let $\Theta_1 A^* = A_1^*\Theta$, where A_1 and Θ are left coprime. Then

$$0 = H_{KA\Theta_1^*} = H_{K\Theta^* A_1} = T_{\widetilde{K}^*} H_{\Theta^* A_1}. \tag{6.18}$$

Since A_1 and Θ are left coprime, $\widetilde{A}_1$ and $\widetilde{\Theta}$ are right coprime, so that $\text{ran} H_{\Theta^* A_1} = \mathcal{H}(\widetilde{\Theta})$. Thus by (6.18), $\widetilde{K}^*\mathcal{H}(\widetilde{\Theta}) \subseteq (H^2_{\mathbb{C}^n})^\perp$, so that $\widetilde{K} = \widetilde{\Theta}\widetilde{K'}$ for some $K' \in H^\infty_{M_n}$, which implies $K = K'\Theta$. This proves (a).

For (b), suppose I_ω is an inner divisor of Θ_1. Thus we may write $\Theta_1 = \Omega_1 I_\omega$ for some inner function Ω_1. Now we will show that I_ω is an inner divisor of Θ. Since $\Theta_1 A^* = A_1^*\Theta$, we have $H_{AI_\omega{}^*} = H_{\Theta^* A_1 \Omega_1}$, and hence $\ker H_{\Theta^* A_1\Omega_1} = \omega H^2_{\mathbb{C}^n}$. Thus we may write

$$\Theta^* A_1 \Omega_1 = D(I_\omega)^* = (I_\omega)^* D \quad \text{(coprime)},$$

which implies that I_ω is an inner divisor of Θ. Write $\Theta = \Omega I_\omega$. Thus if $K \in \mathcal{C}(\Phi)$, then by (a), $\Phi_-^* - K'\Theta\Phi_+^* \in H^2_{M_n}$, which implies

$$(\Phi^{1,\omega})_-^* - (K'\Omega)(\Phi^{1,\omega})_+^* \in H^2_{M_n}.$$

Thus we have $\overline{\omega}K = K'\Omega \in \mathcal{C}(\Phi^{1,\omega})$, which implies that

$$\Big\{\overline{\omega}K : K \in \mathcal{C}(\Phi)\Big\} \subseteq \mathcal{C}(\Phi^{1,\omega}).$$

The above argument is reversible, and this proves (6.17). □

Corollary 6.2. *Let $\Phi \in L^\infty_{M_n}$ be of bounded type of the form*

$$\Phi_+ = \theta_0\theta_1 A^* \quad \textit{and} \quad \Phi_- = \theta_0 B^* \quad \text{(coprime)},$$

where θ_0 and θ_1 are finite Blaschke products and $A, B \in H^2_{M_n}$. If ω is an inner divisor of θ_1, then

$$\mathcal{C}(\Phi) = \Big\{\omega K : K \in \mathcal{C}(\Phi^{1,\omega})\Big\}. \tag{6.19}$$

Proof. Immediate from Proposition 6.1. □

Now let $\Theta \in H^\infty_{M_n}$ be an inner matrix function and suppose

$$\Theta := \prod_{i=1}^N \Delta_i \quad (\Delta_i \text{ is an inner matrix function}).$$

Write

$$\Omega_j := \prod_{i=1}^j \Delta_i \quad (j = 1, \cdots, N-1).$$

Then by Lemma 4.6 (a), we have

$$\begin{aligned}
\mathcal{K}_\Theta &= \mathcal{K}_{\Delta_N} \oplus \mathcal{K}_{\Omega_{N-1}}\Delta_N\\
&= \mathcal{K}_{\Delta_N} \oplus \big(\mathcal{K}_{\Delta_{N-1}} \oplus \mathcal{K}_{\Omega_{N-2}}\Delta_{N-1}\big)\Delta_N\\
&= \mathcal{K}_{\Delta_N} \oplus \mathcal{K}_{\Delta_{N-1}}\Delta_N \oplus \big(\mathcal{K}_{\Delta_{N-2}} \oplus \mathcal{K}_{\Omega_{N-3}}\Delta_{N-2}\big)\Delta_{N-1}\Delta_N\\
&\ \ \vdots\\
&= \mathcal{K}_{\Delta_N} \oplus \mathcal{K}_{\Delta_{N-1}}(\Delta_N) \oplus \mathcal{K}_{\Delta_{N-2}}(\Delta_{N-1}\Delta_N) + \cdots + \mathcal{K}_{\Delta_1}(\Delta_2 \cdots \Delta_N),
\end{aligned}$$

Thus if $B \in \mathcal{K}_\Theta$, then we may write

$$B = B_1 + \sum_{i=2}^N B_i \left(\prod_{j=N+2-i}^N \Delta_j\right) \quad (B_i \in \mathcal{K}_{\Delta_{N+1-i}}). \tag{6.20}$$

Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. To consider the interpolation problem $\mathcal{C}(\Phi) \neq \emptyset$, we may write, in view of (6.13),

$$\Phi_+ = \theta_0\theta_1 A^* \quad \text{and} \quad \Phi_- = \theta_0 B^*,$$

where θ_0, θ_1 are inner functions and $A, B \in H^2_{M_n}$. We may assume that $\Phi_\pm(0) = 0$. Thus, in view of Lemma 4.4, $A \in \mathcal{K}_{\theta_0\theta_1}$ and $B \in \mathcal{K}_{\theta_0}$. Let

$$\theta_0 := \prod_{i=1}^N \delta_i \qquad (\delta_i \in H^\infty \text{ is inner}).$$

Then by (6.20) and Lemma 4.6 (a), we may write

$$\begin{cases} A = A_1 + \sum_{i=2}^N \left(\prod_{j=N+2-i}^N \delta_j\right) A_i + \theta_0 A_{N+1}; \\ B = B_1 + \sum_{i=2}^N \left(\prod_{j=N+2-i}^N \delta_j\right) B_i, \end{cases} \tag{6.21}$$

where $A_i, B_i \in \mathcal{K}_{\delta_{N+1-i}} (i = 1, 2, \cdots, N)$ and $A_{N+1} \in \mathcal{K}_{\theta_1}$.

We then have:

LEMMA 6.3. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then in view of* (6.13)*, we may write*

$$\Phi_+ = \theta_0\theta_1 A^* \quad \text{and} \quad \Phi_- = \theta_0 B^*.$$

Put $\theta_0 := \prod_{i=1}^N \delta_i$ (δ_i inner). Then in view of (6.21)*, we may write*

$$A = \sum_{i=1}^N \lambda_i A_i + \theta_0 A_{N+1} \quad \text{and} \quad B = \sum_{i=1}^N \lambda_i B_i,$$

where $\lambda_1 = 1$, $\lambda_i \equiv \prod_{j=N+2-i}^N \delta_j$ $(i = 2, 3, \cdots N)$, $A_{N+1} \in \mathcal{K}_{\theta_1}$ and $A_i, B_i \in \mathcal{K}_{\delta_{N+1-i}}$. Suppose that A and I_{θ_1} are coprime. Then for $K \in H^\infty_{M_n}$,

$$K \in \mathcal{C}(\Phi) \Longleftrightarrow P_{\mathcal{K}_{\theta_0\theta_1}} K = \theta_1 \left(\sum_{i=1}^N \lambda_i K_i\right),$$

where $K_i \in \mathcal{K}_{\delta_{N+1-i}}$ $(i = 1, 2, \cdots, N)$ satisfies the equation

$$\sum_{i=1}^N \lambda_i B_i - \left(\sum_{i=1}^N \lambda_i K_i\right)\left(\sum_{i=1}^N \lambda_i A_i\right) \in \theta_0 H^2_{M_n}. \tag{6.22}$$

PROOF. Suppose that A and I_{θ_1} are coprime. Let $K \in \mathcal{C}(\Phi)$. If we put $\Psi := \Phi^{1,\theta_1}$, then by Proposition 6.1(b) we can show that $K = \theta_1 K'$ for some $K' \in \mathcal{C}(\Psi)$. It thus suffices to show that

$$K' \in \mathcal{C}(\Psi) \Longleftrightarrow P_{\mathcal{K}_{\theta_0}} K' = \sum_{i=1}^N \lambda_i K_i, \tag{6.23}$$

where $K_i \in \mathcal{K}_{\delta_{N+1-i}}$ $(i = 1, 2, \cdots, N-1)$ satisfies the equation (6.22). By Lemma 4.4, we have

$$\Psi_+ = P_{H^2_0}(\overline{\theta}_1 \Phi_+) = \theta_0 \big(P_{\mathcal{K}_{\theta_0}} A\big)^* = \theta_0 \left(\sum_{i=1}^N \lambda_i A_i\right)^*.$$

Then

$$\begin{aligned} \Psi_-^* - K'\Psi_+^* \in H^2_{M_n} &\Longleftrightarrow \sum_{i=1}^N \lambda_i B_i - K' \sum_{i=1}^N \lambda_i A_i \in \theta_0 H^2_{M_n} \\ &\Longleftrightarrow P_{\mathcal{K}_{\theta_0}}\left(\sum_{i=1}^N \lambda_i B_i\right) - P_{\mathcal{K}_{\theta_0}}\left(K' \sum_{i=1}^N \lambda_i A_i\right) = 0 \\ &\Longleftrightarrow \sum_{i=1}^N \lambda_i B_i - P_{\mathcal{K}_{\theta_0}}\left(\big(P_{\mathcal{K}_{\theta_0}} K'\big)\left(\sum_{i=1}^N \lambda_i A_i\right)\right) = 0. \end{aligned} \tag{6.24}$$

In view of (6.21), we may write

$$P_{\mathcal{K}_{\theta_0}} K' \equiv \sum_{i=1}^{N} \lambda_i K_i,$$

where $\lambda_1 = 1,\ \lambda_i := \prod_{j=N+2-i}^{N} \delta_j,\quad K_i \in \mathcal{K}_{\delta_{N+1-i}}$. It thus follows from (6.24) that $K' \in \mathcal{C}(\Psi)$ if and only if

$$\sum_{i=1}^{N} \lambda_i B_i - \left(\sum_{i=1}^{N} \lambda_i K_i\right)\left(\sum_{i=1}^{N} \lambda_i A_i\right) \in \theta_0 H^2_{M_n},$$

which gives (6.23). This completes the proof. □

Theorem 6.4. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be of bounded type of the form*

$$\Phi_+ = \theta^N A^* \quad \text{and} \quad \Phi_- = \theta^m B^* \quad (N \geq m),$$

where θ is inner. Then in view of (6.21), we may write

$$A = A_0 + \sum_{i=1}^{m-1} \theta^i A_i + \theta^m A_m \quad \text{and} \quad B = B_0 + \sum_{i=1}^{m-1} \theta^i B_i,$$

where $A_m \in \mathcal{K}_{\theta^{N-m}}$ and $A_i, B_i \in \mathcal{K}_\theta$ $(i = 0, 1, \cdots, m-1)$. Suppose that A and I_θ are coprime. Then for $K \in H^\infty_{M_n}$,

$$K \in \mathcal{C}(\Phi) \Longleftrightarrow P_{\mathcal{K}_{\theta^N}} K = \theta^{N-m}\left(K_0 + \sum_{i=1}^{m-1} \theta^i K_i\right),$$

where the $K_i \in \mathcal{K}_\theta$ satisfy the equation

$$\begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_{m-1} \end{pmatrix} = \begin{pmatrix} K_0 & 0 & \cdots & 0 \\ K_1 & K_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_{m-1} & K_{m-2} & \cdots & K_0 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{m-1} \end{pmatrix} + \begin{pmatrix} G_0 \\ G_1 \\ \vdots \\ G_{m-1} \end{pmatrix}, \tag{6.25}$$

where

$$G_k = P_{H^2_{M_n}}\left(\overline{\theta} \sum_{j=0}^{k-1} K_j A_{k-j}\right) - P_{\theta H^2_{M_n}}\left(\sum_{j=0}^{k} K_j A_{k-j}\right) \quad (k = 0, 1, \cdots, m-1). \tag{6.26}$$

Proof. Suppose that A and I_θ are coprime. It suffices to show that (6.22) hods if and only if (6.25) holds. Suppose that there exists $K_i \in \mathcal{K}_\theta$ $(i = 0, 1, \cdots, m-1)$ such that

$$B_0 + \sum_{i=1}^{m-1} \theta^i B_i - \left(K_0 + \sum_{i=1}^{m-1} \theta^i K_i\right)\left(A_0 + \sum_{i=1}^{m-1} \theta^i A_i\right) \in \theta^m H^2_{M_n},$$

or equivalently,

$$\sum_{k=0}^{m-1} \theta^k \left(B_k - \sum_{j=0}^{k} K_j A_{k-j}\right) \in \theta^m H^2_{M_n}. \tag{6.27}$$

Put $D_k := \sum_{j=0}^k K_j A_{k-j}$ $(k = 0, 1, \cdots m-1)$. Note that for each $i, j = 0, 1, \cdots, m-1$, we have $K_i, A_j \in \mathcal{K}_\theta$, and hence $K_i A_j \in \mathcal{K}_{\theta^2} = \mathcal{K}_\theta \oplus \theta K_\theta$. Thus we may write, for $k = 0, 1, \cdots m-1$,

$$D_k = D_k' + \theta D_k'' \quad (D_k', D_k'' \in \mathcal{K}_\theta).$$

It thus follows from (6.27) that

$$\begin{cases} B_0 &= K_0 A_0 - \theta D_0'' = K_0 A_0 - P_{\theta H^2_{M_n}}(D_0) \\ B_k &= \sum_{j=0}^k K_j A_{k-j} + \Big(P_{H^2_{M_n}}(\overline{\theta} D_{k-1}) - P_{\theta H^2_{M_n}}(D_k)\Big) \ \ (k = 1, \cdots, m-1), \end{cases}$$

which gives

$$\begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_{m-1} \end{pmatrix} = \begin{pmatrix} K_0 & 0 & \cdots & 0 \\ K_1 & K_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_{m-1} & K_{m-2} & \cdots & K_0 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{m-1} \end{pmatrix} + \begin{pmatrix} G_0 \\ G_1 \\ \vdots \\ G_{m-1} \end{pmatrix},$$

where $G_0 = -P_{\theta H^2_{M_n}}(D_0)$ and $G_k = P_{H^2_{M_n}}(\overline{\theta} D_{k-1}) - P_{\theta H^2_{M_n}}(D_k)$. This argument is reversible. This completes the proof. $\square$

THEOREM 6.5. *Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be normal of the form*

$$\Phi_+ = \theta^N A^* \quad \text{and} \quad \Phi_- = \theta^m B^* \quad (N \geq m),$$

where θ is inner and I_θ and B are coprime. If $\mathcal{C}(\Phi) \neq \emptyset$, then we may write

$$A = A_0 + \sum_{i=1}^{m-1} \theta^i A_i + \theta^m A_m \quad \text{and} \quad B = B_0 + \sum_{i=1}^{m-1} \theta^i B_i,$$

where A_0 and B_0 are invertible a.e. on $\mathbb{T}$, $A_m \in \mathcal{K}_{\theta^{N-m}}$ and $A_i, B_i \in \mathcal{K}_\theta$ $(i = 0, 1, \cdots, m-1)$. If $K_i \in \mathcal{K}_\theta$ and $\sum_{j=0}^i K_j A_{i-j} \in \mathcal{K}_\theta$ for all $i = 0, 1, \cdots m-1$, put

$$\begin{pmatrix} \widetilde{K}_0 \\ \widetilde{K}_1 \\ \vdots \\ \widetilde{K}_{m-1} \end{pmatrix} = \begin{pmatrix} \widetilde{A}_0 & 0 & \cdots & 0 \\ \widetilde{A}_1 & \widetilde{A}_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{A}_{m-1} & \widetilde{A}_{m-2} & \cdots & \widetilde{A}_0 \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{B}_0 \\ \widetilde{B}_1 \\ \vdots \\ \widetilde{B}_{m-1} \end{pmatrix}. \tag{6.28}$$

Then for $K \in H^\infty_{M_n}$,

$$K \in \mathcal{C}(\Phi) \Longleftrightarrow P_{\mathcal{K}_{\theta^N}} K = \theta^{N-m} \left(K_0 + \sum_{i=1}^{m-1} \theta^i K_i \right), \tag{6.29}$$

PROOF. Suppose that I_θ and B are coprime and $\mathcal{C}(\Phi) \neq \emptyset$. Then I_θ and B_0 are coprime, so that by Lemma 4.15, B_0 is invertible a.e. on $\mathbb{T}$. Now we will show that

$$A_0 \text{ is invertible a.e. on } \mathbb{T}. \tag{6.30}$$

Since I_{θ^m} and B are coprime, it follows that $\ker H_{\Phi_-^*} = \theta^m H^2_{\mathbb{C}^n}$. Since $\mathcal{C}(\Phi) \neq \emptyset$, it follows from Proposition 6.1(b) that $\mathcal{C}(\Phi^{1,\theta^{N-m}}) \neq \emptyset$. Thus we have

$$\theta^m H^2_{\mathbb{C}^n} = \ker H_{\Phi_-^*} \supseteq \ker H_{(\Phi^{1,\theta^{N-m}})_+^*}. \tag{6.31}$$

Observe that

$$(\Phi^{1,\theta^{N-m}})_+ = \theta^m (P_{\mathcal{K}_{\theta^m}} A)^* = \theta^m \left(A_0 + \sum_{i=1}^{m-1} \theta^i A_i \right)^* .$$

It thus follows from (6.31) that A_0 and I_θ are coprime. Thus by Lemma 4.15, $\det A_0 \neq 0$. This proves (6.30). Suppose (6.28) holds. Then a direct calculation shows that G_k in (6.26) should be zero for each $k = 0, 1, \cdots, m-1$, so that (6.29) follows from Theorem 6.4. $\square$

We now turn our attention to a functional calculus for compressions of the shift operator.

It is well known that the functional calculus for polynomials of compressions of the shift results in the Hermite-Fejér matrix via the classical Hermite-Fejér Interpolation Problem. We now extend the polynomial calculus to an H^∞-functional calculus (so called the Sz.-Nagy-Foiaş functional calculus) via the triangularization theorem, and then extend it further to an $\overline{H^\infty} + H^\infty$-functional calculus for compressions of the shift operator.

First of all, we extend the representation (6.7) to the case of matrix H^∞-functions. We refer to [**AC**] and [**Ni1**] for details on this representation. For an explicit criterion, we need to introduce the Triangularization Theorem concretely. There are three cases to consider.

Case 1 : Let B be a Blaschke product and let $\Lambda := \{\lambda_n : n \geq 1\}$ be the sequence of zeros of B counted with their multiplicities. Write

$$\beta_1 := 1, \quad \beta_k := \prod_{n=1}^{k-1} \frac{\lambda_n - z}{1 - \overline{\lambda}_n z} \cdot \frac{|\lambda_n|}{\lambda_n} \qquad (k \geq 2),$$

and let

$$\delta_j := \frac{d_j}{1 - \overline{\lambda}_j z} \beta_j \qquad (j \geq 1),$$

where $d_j := (1 - |\lambda_j|^2)^{\frac{1}{2}}$. Let μ_B be the measure on $\mathbb{N}$ given by $\mu_B(\{n\}) := \frac{1}{2} d_n^2, (n \in \mathbb{N})$. Then the map $V_B : L^2(\mu_B) \to \mathcal{H}(B)$ defined by

$$V_B(c) := \frac{1}{\sqrt{2}} \sum_{n \geq 1} c(n) d_n \delta_n, \quad c \equiv \{c(n)\}_{n \geq 1}, \tag{6.32}$$

is unitary and U_B is mapped onto the operator

$$V_B^* U_B V_B = (I - J_B) M_B, \tag{6.33}$$

where $(M_B c)(n) := \lambda_n c(n)$ $(n \in \mathbb{N})$ is a multiplication operator and

$$(J_B c)(n) := \sum_{k=1}^{n-1} c(k) |\lambda_k|^{-2} \cdot \frac{\beta_n(0)}{\beta_k(0)} d_k d_n \quad (n \in \mathbb{N})$$

is a lower-triangular Hilbert-Schmidt operator.

Case 2 : Let s be a singular inner function with continuous representing measure $\mu \equiv \mu_s$. Let μ_λ be the projection of μ onto the arc $\{\zeta : \zeta \in \mathbb{T},\ 0 < \arg\zeta \leq \arg\lambda\}$ and let

$$s_\lambda(\zeta) := \exp\Big(- \int_{\mathbb{T}} \frac{t + \zeta}{t - \zeta} d\mu_\lambda(t) \Big) \quad (\zeta \in \mathbb{D}).$$

Then the map $V_s : L^2(\mu) \to \mathcal{H}(s)$ defined by

$$(V_s c)(\zeta) = \sqrt{2} \int_{\mathbb{T}} c(\lambda) s_\lambda(\zeta) \frac{\lambda d\mu(\lambda)}{\lambda - \zeta} \quad (\zeta \in \mathbb{D}) \tag{6.34}$$

is unitary, and U_s is mapped onto the operator

$$V_s^* U_s V_s = (I - J_s) M_s, \tag{6.35}$$

where $(M_s c)(\lambda) := \lambda c(\lambda)$ $(\lambda \in \mathbb{T})$ is a multiplication operator and

$$(J_s c)(\lambda) = 2 \int_{\mathbb{T}} e^{\mu(t) - \mu(\lambda)} c(t) d_{\mu_\lambda}(t) \quad (\lambda \in \mathbb{T})$$

is a lower-triangular Hilbert-Schmidt operator.

Case 3 : Let Δ be a singular inner function with pure point representing measure $\mu \equiv \mu_\Delta$. We enumerate the set $\{t \in \mathbb{T} : \mu(\{t\}) > 0\}$ as a sequence $\{t_k\}_{k \in \mathbb{N}}$. Write $\mu_k := \mu(\{t_k\})$, $k \geq 1$. Further, let μ_Δ be a measure on $\mathbb{R}_+ = [0, \infty)$ such that $d\mu_\Delta(\lambda) = \mu_{[\lambda]+1} d\lambda$ and define a function Δ_λ on the unit disk $\mathbb{D}$ by the formula

$$\Delta_0 := 1, \quad \Delta_\lambda(\zeta) := \exp\left\{ - \sum_{k=1}^{[\lambda]} \mu_k \frac{t_k + \zeta}{t_k - \zeta} - (\lambda - [\lambda]) \mu_{[\lambda]+1} \frac{t_{[\lambda]+1} + \zeta}{t_{[\lambda]+1} - \zeta} \right\},$$

where $[\lambda]$ is the integer part of λ $(\lambda \in \mathbb{R}_+)$. Then the map $V_\Delta : L^2(\mu_\Delta) \to \mathcal{H}(\Delta)$ defined by

$$(V_\Delta c)(\zeta) := \sqrt{2} \int_{\mathbb{R}_+} c(\lambda) \Delta_\lambda(\zeta) (1 - \bar{t}_{[\lambda]+1} \zeta)^{-1} d\mu_\Delta(\lambda) \ (\zeta \in \mathbb{D}) \tag{6.36}$$

is unitary and U_Δ is mapped onto the operator

$$V_\Delta^* U_\Delta V_\Delta = (I - J_\Delta) M_\Delta, \tag{6.37}$$

where $(M_\Delta c)(\lambda) := t_{[\lambda]+1} c(\lambda)$, $(\lambda \in \mathbb{R}_+)$ is a multiplication operator and

$$(J_\Delta c)(\lambda) := 2 \int_0^\lambda c(t) \frac{\Delta_\lambda(0)}{\Delta_t(0)} d\mu_\Delta(t) \quad (\lambda \in \mathbb{R}_+)$$

is a lower-triangular Hilbert-Schmidt operator.

Collecting together the above three cases we get:

Triangularization Theorem. ([**Ni1**, p.123]) Let θ be an inner function with the canonical factorization $\theta = B \cdot s \cdot \Delta$, where B is a Blaschke product, and s and Δ are singular inner functions with representing measures μ_s and μ_Δ respectively, with μ_s continuous and μ_Δ a pure point measure. Then the map $V : L^2(\mu_B) \times L^2(\mu_s) \times L^2(\mu_\Delta) \to \mathcal{H}(\theta)$ defined by

$$V := \begin{pmatrix} V_B & 0 & 0 \\ 0 & BV_s & 0 \\ 0 & 0 & BsV_\Delta \end{pmatrix} \tag{6.38}$$

is unitary, where $V_B, \mu_B, V_S, \mu_S, V_\Delta, \mu_\Delta$ are defined in (6.32) - (6.37) and U_θ is mapped onto the operator

$$M := V^* U_\theta V = \begin{pmatrix} M_B & 0 & 0 \\ 0 & M_s & 0 \\ 0 & 0 & M_\Delta \end{pmatrix} + J, \tag{6.39}$$

where M_B, M_S, M_Δ are defined in (6.33), (6.35) and (6.37) and

$$J := -\begin{pmatrix} J_B M_B & 0 & 0 \\ 0 & J_s M_s & 0 \\ 0 & 0 & J_\Delta M_\Delta \end{pmatrix} + A$$

is a lower-triangular Hilbert-Schmidt operator, with $A^3 = 0$, rank$A \le 3$.

Now we note that every compression of the shift operator is completely non-unitary (cf. [**CHL2**, Proof of Theorem 3.3]). Therefore if M is given by (6.39) then M is an absolutely continuous contraction. Thus if $\Phi \in H^\infty_{M_n}$, then we can define $\Phi(M)$ as a H^∞-functional calculus (the so-called Sz.-Nagy-Foiaş functional calculus). Then we have:

THEOREM 6.6. *Let $\Phi \in H^\infty_{M_n}$ and let $\theta \in H^\infty$ be an inner function. If we write*

$$M := V^* U_\theta V \quad \textit{and} \quad \mathcal{V} := V \otimes I_n, \tag{6.40}$$

where $V : L \equiv L^2(\mu_B) \times L^2(\mu_s) \times L^2(\mu_\Delta) \to \mathcal{H}(\theta)$ is unitary as in (6.38)*, then*

$$\mathcal{V}^*(T_\Phi)_\Theta \mathcal{V} = \Phi(M) \quad (\Theta := I_\theta). \tag{6.41}$$

REMARK 6.7. $\Phi(M)$ is called a *matrix representation* for $(T_\Phi)_\Theta$.

PROOF OF THEOREM 6.6. If $\Phi(z) \equiv \big(\phi_{rs}(z)\big)_{1\le r,s\le n} \in H^\infty_{M_n}$, we may write

$$\Phi(z) = \sum_{i=0}^\infty A_i z^i \qquad (A_i \in M_n).$$

We also write $\phi_{rs}(z) := \sum_0^\infty a_i^{(rs)} z^i$ and then $A_i = \Big(a_i^{(rs)}\Big)_{1\le r,s\le n}$. We thus have

$$(T_\Phi)_\Theta = P_{\mathcal{H}(\Theta)} T_\Phi|_{\mathcal{H}(\Theta)} = \sum_{i=0}^\infty \left(U_\theta^i \otimes \Big(a_i^{(rs)}\Big)_{1\le r,s\le n} \right) = \sum_{i=0}^\infty \left(U_\theta^i \otimes A_i \right).$$

Let $\{\psi_j\}$ be an orthonormal basis for $\mathcal{H}(\theta)$ and put $e_j := V^*\psi_j$. Then $\{e_j\}$ forms an orthonormal basis for $L^2(\mu_B) \times L^2(\mu_s) \times L^2(\mu_\Delta)$. Thus for each $f \in \mathbb{C}^n$, we have $\mathcal{V}(e_j \otimes f) = \phi_j \otimes f$. It thus follows that

$$\begin{aligned} \Big\langle (T_\Phi)_\Theta(\psi_j \otimes f),\ \psi_k \otimes g \Big\rangle &= \sum_{i=0}^\infty \Big\langle (M^i \otimes A_i)(e_j \otimes f),\ e_k \otimes g \Big\rangle \\ &= \Big\langle \Phi(M)(e_j \otimes f),\ e_k \otimes g \Big\rangle, \end{aligned}$$

which gives $\mathcal{V}^*(T_\Phi)_\Theta \mathcal{V} = \Phi(M)$. □

We can now extend the representation (6.41) to $\overline{H^\infty_{M_n}} + H^\infty_{M_n}$ (where $\overline{H^\infty_{M_n}}$ denotes the set of $n \times n$ matrix functions whose entries belong to $\overline{H^\infty} := \{g : \overline{g} \in H^\infty\}$). Let $Q \in \overline{H^\infty_{M_n}} + H^\infty_{M_n}$ be of the form $Q = Q_-^* + Q_+$. If $\Theta := I_\theta$ for an inner function θ, then we define

$$(T_Q)_\Theta := P_{\mathcal{H}(\Theta)} T_Q|_{\mathcal{H}(\Theta)}\,.$$

Then

$$\begin{aligned}(T_Q)_\Theta &= (T_{Q_-^*})_\Theta + (T_{Q_+})_\Theta \\ &= (T_{Q_-})_\Theta^* + (T_{Q_+})_\Theta \quad \text{(by (5.10))}.\end{aligned}$$

If $M := V^* U_\theta V$, where $V : L \equiv L^2(\mu_B) \times L^2(\mu_s) \times L^2(\mu_\Delta) \to \mathcal{H}(\theta)$ is unitary as in (6.38), we also define $Q(M)$ by

$$Q(M) := (Q_-(M))^* + Q_+(M), \tag{6.42}$$

where $Q_\pm(M)$ is defined by the Sz.-Nagy-Foiaş functional calculus.

We then have:

LEMMA 6.8. *Let $Q \in \overline{H^\infty_{M_n}} + H^\infty_{M_n}$ and $\Theta := I_\theta$ for an inner function θ. Then*

(a) *$\mathcal{V}^*(T_Q)_\Theta \mathcal{V} = Q(M)$;*

(b) *$Q(M)^* = Q^*(M)$,*

where $\mathcal{V}$ and M are given by (6.40).

PROOF. It follows from Theorem 6.6 that

$$\mathcal{V}^*(T_Q)_\Theta\mathcal{V} = \mathcal{V}^*\Big((T_{Q_-})_\Theta^* + (T_{Q_+})_\Theta\Big)\mathcal{V} = (Q_-(M))^* + Q_+(M) = Q(M),$$

which gives (a). By definition of $Q(M)$, we have

$$\begin{aligned}Q(M)^* = \Big(Q_-(M)^* + Q_+(M)\Big)^* &= Q_-(M) + Q_+(M)^* \\ &= Q_-(M) + Q_+^*(M) = Q^*(M),\end{aligned}$$

which gives (b). □

We are tempted to guess that $Q(M)^*Q(M) = (Q^*Q)(M)$. But this is not the case. To see this, let $\theta = z^3$ and let $Q(z) := \left(\begin{smallmatrix} z^2 & 0 \\ 0 & z^2\end{smallmatrix}\right)$. Then we have

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix}$$

and

$$Q(M) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \otimes M^2 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix}.$$

Thus

$$Q(M)^*Q(M) = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \quad \text{and} \quad (Q^*Q)(M) = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix},$$

which gives $Q^*(M)Q(M) \neq (Q^*Q)(M)$.

In some sense, the following lemma shows that the operator induced by Q in (6.42) is closed under constant matrix multiplication.

LEMMA 6.9. *Let* $Q \in \overline{H^\infty_{M_n}} + H^\infty_{M_n}$ *and and* $\Theta := I_\theta$ *for an inner function* θ. *Then for any constant matrix* $\Lambda \in M_n$,

(a) $(\Lambda Q)(M) = (\Lambda \otimes I) Q(M)$;
(b) $(Q\Lambda)(M) = Q(M)(\Lambda \otimes I)$,

where M *is given by* (6.40).

PROOF. Let us write $Q(z) := \sum_{j=-\infty}^{\infty} Q_j z^j$. Since Λ is a constant matrix, $(\Lambda Q)(z) = \sum_{j=-\infty}^{\infty} \Lambda Q_j z^j$. Thus for each $U, W \in \mathcal{H}(\Theta)$,

$$\begin{aligned}
\Big\langle \big((\Lambda \otimes I) Q(M)\big)\, \mathcal{V}^* U,\ \mathcal{V}^* W \Big\rangle &= \sum_{j=-\infty}^{\infty} \Big\langle \big((\Lambda \otimes I)(Q_j \otimes M^j)\big)\, \mathcal{V}^* U, \mathcal{V}^* W \Big\rangle \\
&= \sum_{j=-\infty}^{\infty} \Big\langle (\Lambda Q_j \otimes M^j)\, \mathcal{V}^* U,\ \mathcal{V}^* W \Big\rangle \\
&= \Big\langle \big((\Lambda Q)(M)\big)\, \mathcal{V}^* U,\ \mathcal{V}^* W \Big\rangle,
\end{aligned}$$

where $\mathcal{V}$ is given by (6.40). This proves (a). Observe that

$$\begin{aligned}
(Q\Lambda)(M)^* &= (Q\Lambda)^*(M) \qquad \text{(by Lemma 6.8(b))} \\
&= (\Lambda^* Q^*)(M) \\
&= (\Lambda^* \otimes I)(Q^*(M)) \qquad \text{(by (a))} \\
&= (\Lambda \otimes I)^* (Q(M))^* \qquad \text{(by Lemma 6.8(b))} \\
&= \Big(Q(M)(\Lambda \otimes I)\Big)^*,
\end{aligned}$$

which gives (b). $\square$

CHAPTER 7

Abrahamse's Theorem for matrix-valued symbols

In 1970, P.R. Halmos posed the following problem, listed as Problem 5 in his lecture "Ten problems in Hilbert space" [**Hal1**], [**Hal2**]:

Is every subnormal Toeplitz operator either normal or analytic ?

A Toeplitz operator T_φ is called *analytic* if $\varphi \in H^\infty$. Any analytic Toeplitz operator is easily seen to be subnormal: indeed, $T_\varphi h = P(\varphi h) = \varphi h = M_\varphi h$ for $h \in H^2$, where M_φ is the normal operator of multiplication by φ on L^2. The question is natural because normal and analytic Toeplitz operators are fairly well understood, and they are both subnormal. In 1984, Halmos' Problem 5 was answered in the negative by C. Cowen and J. Long [**CoL**]. However, Cowen and Long's construction does not provide an intrinsic connection between subnormality and the theory of Toeplitz operators. Until now researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. In this sense, we reformulate Halmos' Problem 5:

Which Toeplitz operators are subnormal ?

The most interesting partial answer to Halmos' Problem 5 was given by M.B. Abrahamse [**Ab**], who gave a general sufficient condition for the answer to Halmos' Problem 5 to be affirmative.

Abrahamse's Theorem ([**Ab**, Theorem]). Let $\varphi \in L^\infty$ be such that φ or $\overline{\varphi}$ is of bounded type. If

(i) T_φ is hyponormal;
(ii) $\ker [T_\varphi^*, T_\varphi]$ is invariant under T_φ,

then T_φ is normal or analytic.

Consequently, if $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type, then every subnormal Toeplitz operator must be either normal or analytic, since $\ker [S^*, S]$ is invariant under S for every subnormal operator S. It is actually sufficient to assume that S is 2-hyponormal. We say that a block Toeplitz operator T_Φ is *analytic* if $\Phi \in H^\infty_{M_n}$. Evidently, any analytic block Toeplitz operator with a normal symbol is subnormal because the multiplication operator M_Φ is a normal extension of T_Φ. As a first inquiry in the above reformulation of Halmos' Problem 5 the following question can be raised:

Is Abrahamse's Theorem valid for Toeplitz operators with matrix-valued symbols ?

In general, a straightforward matrix-valued version of Abrahamse's Theorem is doomed to fail: for instance, if

$$\Phi := \begin{pmatrix} z + \overline{z} & 0 \\ 0 & z \end{pmatrix},$$

then clearly, both Φ and Φ^* are of bounded type and

$$T_\Phi = \begin{pmatrix} U+U^* & 0 \\ 0 & U \end{pmatrix} \quad (\text{where } U \text{ is the shift on } H^2)$$

is subnormal, but neither normal nor analytic.

In this chapter we extend the above result to the case of bounded type symbols: we shall say that T_Φ has a *bounded type symbol* if both Φ and Φ^* are of bounded type.

Recently, it was shown in [**CHL1**] that if $\Phi \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type of the form

$$\Phi_- = \theta B^* \quad (\text{coprime}) \tag{7.1}$$

and if T_Φ is hyponormal and $\ker [T^*_\varphi, T_\varphi]$ is invariant under T_φ, then T_Φ is normal or analytic. However, the condition (7.1) forces the inner part of the right coprime factorization (4.2) of Φ_- to be diagonal-constant. Also, it was shown in [**CHKL**] that if Φ is a matrix-valued *rational* function then the condition (7.1) can be weakened to the condition that the inner part of the right coprime factorization (4.2) of Φ_- has a nonconstant diagonal-constant inner divisor. We note that in view of Lemma 5.2, those conditions of [**CHL1**] and [**CHKL**] are special cases of the condition of "having a tensored-scalar singularity." Indeed, in this chapter, we will show that for a bounded type symbols $\Phi \in L^\infty_{M_n}$, if Φ has a tensored-scalar singularity then we get a full-fledged matrix-valued version of Abrahamse's Theorem.

To proceed, we need the following result.

Lemma 7.1. *Let $\Phi \in L^\infty_{M_n}$ be such that Φ^* is of bounded type. If T_Φ is hyponormal, then there exists an inner function θ such that $\operatorname{ran}[T^*_\Phi, T_\Phi] \subseteq \mathcal{H}(I_\theta)$.*

Proof. If Φ^* is of bounded type then, in view of (2.6),

$$\Phi^* = A\Theta^* \quad (\Theta \equiv I_\theta,\ A \in H^2_{M_n}).$$

If T_Φ is hyponormal then by Lemma 2.3, Φ is normal. Thus by (2.9), $[T^*_\Phi, T_\Phi] = H^*_{\Phi^*}H_{\Phi^*} - H^*_\Phi H_\Phi$. Thus if T_Φ is hyponormal then $\ker H_{\Phi^*} \subseteq \ker H_\Phi$, and hence $\ker H_{\Phi^*} \subseteq \ker [T^*_\Phi, T_\Phi]$, which gives $\Theta H^2_{\mathbb{C}^n} \subseteq \ker [T^*_\Phi, T_\Phi]$, or $\operatorname{ran} [T^*_\Phi, T_\Phi] \subseteq \mathcal{H}(\Theta)$. $\square$

The following lemma shows what tensored-scalar singularities do in the passage from hyponormality to subnormality.

Lemma 7.2. *Let $\Phi \in L^\infty_{M_n}$ be such that Φ^* is of bounded type. Suppose T_Φ is hyponormal; thus, in view of Lemma 7.1, there exists an inner function $\Omega \equiv I_\omega \in H^\infty_{M_n}$ (ω a nonconstant scalar inner function) such that* $\operatorname{ran}[T^*_\Phi, T_\Phi] \subseteq \mathcal{H}(\Omega)$. *If $\Phi^*\Omega$ has a tensored-scalar singularity then T_Φ is normal.*

Proof. By assumption, $\Omega H^2_{\mathbb{C}^n} \subseteq \ker[T^*_\Phi, T_\Phi]$. If T_Φ is hyponormal then by Lemma 2.3, Φ is normal. Thus by (2.9), we have

$$\begin{aligned} 0 = T^*_\Omega [T^*_\Phi, T_\Phi] T_\Omega &= T_{\Omega^*\Phi^*} T_{\Phi\Omega} - T_{\Omega^*\Phi} T_{\Phi^*\Omega} \\ &= \Big(T_{\Omega^*\Phi^*\Phi\Omega} - H^*_{\Phi\Omega} H_{\Phi\Omega} \Big) - \Big(T_{\Omega^*\Phi\Phi^*\Omega} - H^*_{\Phi^*\Omega} H_{\Phi^*\Omega} \Big) \\ &= H^*_{\Phi^*\Omega} H_{\Phi^*\Omega} - H^*_{\Phi\Omega} H_{\Phi\Omega}. \end{aligned}$$

Thus if $\Phi^*\Omega$ has a tensored-scalar singularity, then it follows from Theorem 5.5 that

$$\Phi^*\Omega - U\Phi\Omega \in H^\infty_{M_n} \quad \text{for some unitary constant } U \in M_n. \tag{7.2}$$

Since T_Φ is hyponormal, by Lemma 2.3, there exists a matrix function $K \in H^\infty_{M_n}$ such that $\Phi - K\Phi^* \in H^\infty_{M_n}$. Thus it follows from (7.2) that $\Phi^*\Omega - UK\Phi^*\Omega \in H^\infty_{M_n}$, so that $(I - T^*_{\widetilde{UK}})H_{\Phi^*\Omega} = 0$, which gives

$$\text{ran}\, H_{\Phi^*\Omega} \subseteq \ker\,(I - T^*_{\widetilde{UK}}). \tag{7.3}$$

On the other hand, since $\Phi^*\Omega$ has a tensored-scalar singularity, it follows that $\ker H_{\Phi^*\Omega} \subseteq \zeta H^2_{\mathbb{C}^n}$ (ζ nonconstant inner). Thus by Lemma 5.2, we can write

$$\Phi^*\Omega = B(\zeta\Delta)^* \quad \text{(right coprime)}.$$

We thus have

$$\widetilde{\Phi^*\Omega} = \overline{\widetilde{\zeta}}\widetilde{\Delta}^*\widetilde{B}.$$

Since I_ζ and B are right coprime, it follows from Theorem 4.16 that I_ζ and B are left coprime, so that $I_{\widetilde{\zeta}}$ and $\widetilde{B}$ are right coprime. We then have $\ker H^*_{\Phi^*\Omega} = \ker H_{\widetilde{\Phi^*\Omega}} \subseteq \widetilde{\zeta} H^2_{\mathbb{C}^n}$, which together with (7.3) implies

$$\mathcal{H}(I_{\widetilde{\zeta}}) \subseteq \text{cl ran}\, H_{\Phi^*\Omega} \subseteq \ker\,(I - T^*_{\widetilde{UK}}).$$

The same argument as the one used right after (5.9) in Theorem 5.5 shows that $UK = I$, and hence $K = U^*$ is a constant unitary. Therefore $[T^*_\Phi, T_\Phi] = H^*_{\Phi^*}(I - T_{\widetilde{K}}T^*_{\widetilde{K}})H_{\Phi^*} = 0$. This completes the proof. $\square$

The main theorem of this chapter now follows.

THEOREM 7.3. (Abrahamse's Theorem for matrix-valued symbols) *Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Assume Φ has a tensored-scalar singularity. If*

(i) *T_Φ is hyponormal;*
(ii) *$\ker\,[T^*_\Phi, T_\Phi]$ is invariant under T_Φ,*

then T_Φ is normal. Hence, in particular, if T_Φ is subnormal then T_Φ is normal.

PROOF. Suppose T_Φ is hyponormal and $\ker\,[T^*_\Phi, T_\Phi]$ is invariant under T_Φ. Let Φ and Φ^* be of bounded type. Thus in view of (4.2), we may write

$$\Phi_- = \Theta B^* \quad \text{(right coprime)}.$$

Suppose Φ has a tensored-scalar singularity, so that by Lemma 5.2, Θ has a nonconstant diagonal-constant inner function I_θ. Thus $\Theta = \theta\Theta_1$ for some inner matrix function Θ_1. Since T_Φ is hyponormal we may write, in view of (6.11),

$$\Phi_+ = \Theta\Theta_2 A^* \quad \text{(right coprime)},$$

where Θ_2 is an inner matrix function. Since by Lemma 2.3, Φ is normal it follows from (2.9) that

$$[T^*_\Phi, T_\Phi] = H^*_{A\Theta_2^*\Theta^*} H_{A\Theta_2^*\Theta^*} - H^*_{B\Theta^*} H_{B\Theta^*}, \tag{7.4}$$

which implies $\Theta\Theta_2 H^2_{\mathbb{C}^n} \subseteq \ker\,[T^*_\Phi, T_\Phi] = \big(\text{ran}\,[T^*_\Phi, T_\Phi]\big)^\perp$, so that

$$\text{ran}\,[T^*_\Phi, T_\Phi] \subseteq \mathcal{H}(\Theta\Theta_2)\,. \tag{7.5}$$

We first suppose $\operatorname{ran}[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(\Theta_1\Theta_2)$. Since $\ker H_{\Phi^*\Theta_1\Theta_2} = \ker H_{\overline{\theta}A} = \theta H^2_{\mathbb{C}^n}$, so that $\Phi^*\Theta_1\Theta_2$ has a tensored-scalar singularity it follows from Lemma 7.2 with $\Omega \equiv \Theta_1\Theta_2$ that T_Φ is normal. We thus suppose that $\operatorname{ran}[T_\Phi^*, T_\Phi]$ is not contained in $\mathcal{H}(\Theta_1\Theta_2)$. In this case, we assume to the contrary that T_Φ is not normal. In view of Lemma 7.1, there exists a diagonal-constant inner function $\Omega \equiv I_\omega$ such that

$$\operatorname{ran}[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(\Omega), \quad \text{or equivalently,} \quad \Omega^*\left(\operatorname{ran}[T_\Phi^*, T_\Phi]\right) \subseteq (H^2_{\mathbb{C}^n})^\perp. \tag{7.6}$$

By Lemma 7.2, $\Phi^*\Omega$ has no tensored-scalar singularity. But since $\Phi_+^*\Omega = (\overline{\theta}\omega)A\Theta_2^*\Theta_1^*$, it follows from Lemma 5.2 that θ is an inner divisor of ω. Let $h \in \operatorname{ran}[T_\Phi^*, T_\Phi]$ be an arbitrary vector. Since by assumption, $\ker[T_\Phi^*, T_\Phi]$ is invariant for T_Φ, we have $T_\Phi^*\left(\operatorname{ran}[T_\Phi^*, T_\Phi]\right) \subseteq \operatorname{ran}[T_\Phi^*, T_\Phi]$, and hence, $T_\Phi^* h \in \operatorname{ran}[T_\Phi^*, T_\Phi]$. From (7.6), $\Omega^*\Phi^*h \in (H^2_{\mathbb{C}^n})^\perp$. Thus we have

$$\Omega^*\Phi^*h = \Omega^*(\Phi_+^* + \theta\Theta_1 B^*)h = \Phi_+^*(\Omega^*h) + \theta\Theta_1\Omega^*B^*h \in (H^2_{\mathbb{C}^n})^\perp.$$

Since by (7.6), $\Omega^*h \in (H^2_{\mathbb{C}^n})^\perp$, it follows that

$$\theta B^*\Omega^*h \in (H^2_{\mathbb{C}^n})^\perp. \tag{7.7}$$

Since $\Phi_- = \Theta B^* \in zH^2_{M_n}$, we can choose $B \in \mathcal{K}_\Theta$. Thus by Lemma 4.6 we may write

$$B = \theta B_1 + B_2, \tag{7.8}$$

where $B_1 \in \mathcal{K}_{\Theta_1}$ and $B_2 \in \mathcal{K}_\theta$. Then it follows from (7.7) and (7.8) that

$$B_1^*\Omega^*h + (\overline{\omega}\theta)B_2^*h \in (H^2_{\mathbb{C}^n})^\perp, \quad \text{so that} \quad (\overline{\omega}\theta)B_2^*h \in (H^2_{\mathbb{C}^n})^\perp.$$

We thus have

$$\big(\overline{z}\overline{\overline{\widetilde{\theta}}}\widetilde{B}_2\big)\big(\widetilde{\omega}\overline{\overline{\widetilde{h}}}\big) \in H^2_{\mathbb{C}^n}, \quad \text{or equivalently,} \quad \widetilde{\omega}\overline{\overline{\widetilde{h}}} \in \ker H_{\overline{z}\overline{\overline{\widetilde{\theta}}}\widetilde{B}_2}. \tag{7.9}$$

We now claim that

$$\widetilde{\omega}\overline{\overline{\widetilde{h}}} \in z\widetilde{\theta}H^2_{\mathbb{C}^n}. \tag{7.10}$$

To see this we first suppose that θ and z are not coprime. Then z is an inner divisor of θ. Since B and Θ are right coprime, it follows from Theorem 4.16 that B and I_θ are coprime, and hence B and $I_{z\theta}$ are coprime. Thus by (7.8), B_2 and $I_{z\theta}$ are coprime. Therefore $\widetilde{B}_2$ and $I_{z\widetilde{\theta}}$ are coprime, which implies that, by (4.3) together with (7.9), $\widetilde{\omega}\overline{\overline{\widetilde{h}}} \in \ker H_{\overline{z}\overline{\overline{\widetilde{\theta}}}\widetilde{B}_2} = z\widetilde{\theta}H^2_{\mathbb{C}^n}$, giving (7.10).

Suppose now that θ and z are coprime. Then by (4.3) and (7.9) we have

$$\widetilde{\omega}\overline{\overline{\widetilde{h}}} \in \ker H_{\overline{z}\overline{\overline{\widetilde{\theta}}}\widetilde{B}_2} \subseteq \ker H_{\overline{\overline{\widetilde{\theta}}}\widetilde{B}_2} = \widetilde{\theta}H^2_{\mathbb{C}^n}. \tag{7.11}$$

But since $\Omega^*h \in (H^2_{\mathbb{C}^n})^\perp$, and hence $\widetilde{\omega}\overline{\overline{\widetilde{h}}} \in zH^2_{\mathbb{C}^n}$, we have

$$\widetilde{\omega}\overline{\overline{\widetilde{h}}} \in \widetilde{\theta}H^2_{\mathbb{C}^n} \bigcap zH^2_{\mathbb{C}^n} = z\widetilde{\theta}H^2_{\mathbb{C}^n},$$

giving (7.10). Now by (7.10), we have

$$(\overline{z}\overline{\overline{\widetilde{\theta}}}\widetilde{\omega})\overline{\overline{\widetilde{h}}} \in H^2_{\mathbb{C}^n}, \quad \text{so that} \quad (\theta\overline{\omega})h \in (H^2_{\mathbb{C}^n})^\perp.$$

Therefore we can see that

$$(\overline{\theta}\Omega)^*h \in (H^2_{\mathbb{C}^n})^\perp \text{ for each } h \in \operatorname{ran}[T_\Phi^*, T_\Phi],$$

which implies that $\text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(\overline{\theta}\Omega) = \mathcal{H}(I_{\omega\overline{\theta}})$. Thus we can repeat the above argument with $\overline{\theta}\Omega$ in place of Ω in (7.6). Then induction on p shows that

$$\text{ran}\,[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(I_{\omega\overline{\theta}^p}) \text{ for all } p \in \mathbb{Z}_+.$$

In particular, $\omega\overline{\theta}^p \in H^2$, and hence θ^p is an inner divisor of ω for all $p \in \mathbb{Z}_+$. Thus it follows from Lemma 4.13 that

$$\omega H^2 \subseteq \bigcap_{p=1}^{\infty} \theta^p H^2 = \{0\},$$

a contradiction. Therefore T_Φ should be normal. This proves the first assertion.

The second assertion follows at once from the fact that if T_Φ is subnormal then $\ker\,[T_\Phi^*, T_\Phi]$ is invariant for T_Φ. This completes the proof of Theorem 7.3. $\square$

Remark 7.4. (a) We note that the assumption "Φ has a tensored-scalar singularity" is essential in Theorem 7.3. As we saw before, if $\Phi := \left(\begin{smallmatrix} \overline{z}+z & 0 \\ 0 & z \end{smallmatrix}\right)$, then T_Φ is neither normal nor analytic. But since $\ker H_\Phi = \ker H_{\left(\begin{smallmatrix} \overline{z} & 0 \\ 0 & 0 \end{smallmatrix}\right)} = \left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right) H^2_{\mathbb{C}^n}$, it follows that $\Theta \equiv \left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right)$ does not have any nonconstant diagonal-constant inner divisor, so that Φ does not have a tensored-scalar singularity.

(b) If $n = 1$, then $\Theta \equiv \theta \in H^\infty$ is vacuously diagonal-constant, so that Theorem 7.3 reduces to the original Abrahamse's Theorem. $\square$

We will next give an example that illustrates Theorem 7.3. To do so we need some auxiliary observations. If $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type and for which T_Φ is hyponormal, we will, in view of (4.1) and (6.13), assume that

$$\Phi_+ = A^*\Omega_1\Omega_2 \quad \text{and} \quad \Phi_- = B_\ell^*\Omega_2 \text{ (left coprime)}, \tag{7.12}$$

where $\Omega_1\Omega_2 = \Theta = I_\theta$.

Lemma 7.5. *Let $\Phi = \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then in view of (7.12), we may write*

$$\Phi_+ = A^*\Theta_0\Theta_2 \quad \textit{and} \quad \Phi_- = B^*\Theta_2 \textit{ (left coprime)},$$

where $\Theta_0\Theta_2 = I_\theta$, with θ an inner function. If T_Φ is hyponormal and $\ker\,[T_\Phi^*, T_\Phi]$ *is invariant under T_Φ , then*

$$\Theta_0 H^2_{\mathbb{C}^n} \;\subseteq\; \ker\,[T_\Phi^*, T_\Phi]. \tag{7.13}$$

Moreover if $K \in \mathcal{C}(\Phi)$, then

$$\text{cl ran}\, H_{A\Theta_2^*} \subseteq \ker\,(I - T_{\widetilde{K}}T_{\widetilde{K}}^*). \tag{7.14}$$

Proof. The inclusion (7.13) follows from a slight extension of [**CHL2**, Theorem 3.5] , in which Θ_2 is a diagonal-constant inner function of the form $\Theta_2 = I_{\theta_2}$. However, a careful analysis of [**CHL2**, Proof of Theorem 3.5, Step 1] shows that the proof does not employ the diagonal-*constancy* of Θ_2, but uses only the diagonal-constancy of $\Theta_0\Theta_2$. Towards (7.14), we observe that if $K \in \mathcal{C}(\Phi)$, then by (2.10),

$$[T_\Phi^*, T_\Phi] = H^*_{\Phi_+^*}H_{\Phi_+^*} - H^*_{K\Phi_+^*}H_{K\Phi_+^*} = H^*_{\Phi_+^*}(I - T_{\widetilde{K}}T^*_{\widetilde{K}})H_{\Phi_+^*}\,, \tag{7.15}$$

so that $\ker\,[T_\Phi^*, T_\Phi] = \ker\,(I - T_{\widetilde{K}}T_{\widetilde{K}}^*)H_{\Phi_+^*}$. Thus by (7.13),

$$\{0\} = (I - T_{\widetilde{K}}T_{\widetilde{K}}^*)H_{A\Theta_2^*\Theta_0^*}(\Theta_0 H^2_{\mathbb{C}^n}) = (I - T_{\widetilde{K}}T_{\widetilde{K}}^*)H_{A\Theta_2^*}(H^2_{\mathbb{C}^n}),$$

giving (7.14). □

In [**GHR**], the normality of block Toeplitz operator T_Φ was also characterized in terms of the symbol Φ under a "determinant" condition on the symbol Φ.

Lemma 7.6. (Normality of Block Toeplitz Operators) ([**GHR**]) *Let* $\Phi \equiv \Phi_+ + \Phi_-^*$ *be normal and* $\Phi_+^0 := \Phi_+ - \Phi_+(0)$. *If* $\det\,\Phi_+ \neq 0$, *then*

$$T_\Phi \text{ is normal} \Longleftrightarrow \Phi_+^0 = \Phi_- U \quad \text{for some constant unitary matrix } U. \tag{7.16}$$

We then have:

Example 7.7. Let $\varphi, \psi \in L^\infty$ be of bounded type and consider

$$\Phi := \begin{pmatrix} \overline{b}_\alpha & \varphi \\ \psi & \overline{b}_\alpha \end{pmatrix} \quad (\alpha \in \mathbb{D}),$$

where b_α is a Blaschke factor of the form $b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}$. In view of (2.3) we may write

$$\varphi_- := \theta_0 \overline{a} \quad \text{and} \quad \psi_- := \theta_1 \overline{b} \quad \text{(coprime)}.$$

If $\theta_0(\alpha) = 0$ and $\theta_1(\alpha) \neq 0$, then T_Φ is never subnormal.

Proof. Write

$$\Phi \equiv \begin{pmatrix} \overline{b}_\alpha & \varphi \\ \psi & \overline{b}_\alpha \end{pmatrix} \equiv \Phi_-^* + \Phi_+ = \begin{pmatrix} b_\alpha & \psi_- \\ \varphi_- & b_\alpha \end{pmatrix}^* + \begin{pmatrix} 0 & \varphi_+ \\ \psi_+ & 0 \end{pmatrix}$$

and assume that T_Φ is subnormal. Since by Lemma 2.3 Φ is normal, a straightforward calculation shows that

$$|\varphi| = |\psi|. \tag{7.17}$$

Also there exists a matrix function $K \equiv \left(\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right) \in \mathcal{E}(\Phi)$, i.e., $||K||_\infty \leq 1$ such that $\Phi - K\Phi^* \in H^\infty_{M_2}$. Thus since $\Phi_-^* - K\Phi_+^* \in H^2_{M_2}$, and hence

$$\begin{pmatrix} \overline{b}_\alpha & \overline{\theta}_0 a \\ \overline{\theta}_1 b & \overline{b}_\alpha \end{pmatrix} - \begin{pmatrix} k_2\overline{\varphi_+} & k_1\overline{\psi_+} \\ k_4\overline{\varphi_+} & k_3\overline{\psi_+} \end{pmatrix} \in H^2_{M_2},$$

we have

$$\begin{cases} \overline{b}_\alpha - k_2\overline{\varphi_+} \in H^2, & \overline{\theta}_1 b - k_4\overline{\varphi_+} \in H^2 \\ \overline{b}_\alpha - k_3\overline{\psi_+} \in H^2, & \overline{\theta}_0 a - k_1\overline{\psi_+} \in H^2, \end{cases} \tag{7.18}$$

which via Cowen's Theorem gives that the following Toeplitz operators are all hyponormal:

$$T_{\overline{b}_\alpha + \varphi_+}, \quad T_{\overline{\theta}_1 b + \varphi_+}, \quad T_{\overline{b}_\alpha + \psi_+}, \quad T_{\overline{\theta}_0 a + \psi_+}. \tag{7.19}$$

Note that $\varphi_+\psi_+$ is not identically zero, so that $\det\Phi_+$ is not. Put

$$\theta_0 = b_\alpha^m \theta_0' \quad (m \geq 1;\ \theta_0'(\alpha) \neq 0).$$

Note $a(\alpha) \neq 0$ because $\theta_0(\alpha) = 0$ and θ_0 and a are coprime. Then a straightforward calculation together with (7.14) shows that

$$m = 1. \tag{7.20}$$

Thus we may write, by a direct calculation,

$$\Phi_- = \begin{pmatrix} \theta_0' & a \\ b_\alpha b & \theta_1 \end{pmatrix}^* \begin{pmatrix} \theta_0 & 0 \\ 0 & b_\alpha \theta_1 \end{pmatrix} \equiv B^* \Omega_2 \quad \text{(left coprime)}.$$

But since $\Omega_2 \equiv \begin{pmatrix} \theta_0 & 0 \\ 0 & b_\alpha \theta_1 \end{pmatrix}$ has a diagonal inner divisor I_{b_α}, it follows from Theorem 7.3 that T_Φ is normal. Since $\det \Phi_+ \neq 0$, it follows form Lemma 7.6 that $\Phi_+^0 = \Phi_- U$ for some constant unitary matrix $U \equiv \left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$. We observe

$$\begin{aligned} \Phi_+^0 = \Phi_- U &\Longleftrightarrow \begin{pmatrix} 0 & b_\alpha \theta_1 \theta_3 \overline{d} \\ \theta_0 \theta_2 \overline{c} & 0 \end{pmatrix} = \begin{pmatrix} b_\alpha & \theta_1 \overline{b} \\ \theta_0 \overline{a} & b_\alpha \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \\ &\Longrightarrow \begin{cases} 0 = c_1 b_\alpha + c_3 \theta_1 \overline{b} \\ 0 = c_4 b_\alpha + c_2 \theta_0 \overline{a} \end{cases} \\ &\Longrightarrow \begin{cases} c_1 = 0, \ \theta_1 \overline{b} = 0 \ (\text{i.e., } \theta_1 = 1) \\ c_4 \neq 0, \ \theta_0 = b_\alpha \end{cases} \\ &\Longrightarrow U = \begin{pmatrix} 0 & c_2 \\ c_3 & c_4 \end{pmatrix} \qquad (c_4 \neq 0), \end{aligned}$$

which contradicts the fact that U is unitary. Therefore T_Φ is never subnormal. $\square$

CHAPTER 8

A subnormal Toeplitz completion

In this chapter we consider a subnormal "Toeplitz" completion problem.

Given a partially specified operator matrix with some known entries, the problem of finding suitable operators to complete the given partial operator matrix so that the resulting matrix satisfies certain given properties is called a *completion problem.* A *subnormal completion* of a partial operator matrix is a particular specification of the unspecified entries resulting in a subnormal operator. A *partial block Toeplitz matrix* is simply an $n \times n$ matrix some of whose entries are specified Toeplitz operators and whose remaining entries are unspecified. A *subnormal Toeplitz completion* of a partial block Toeplitz matrix is a subnormal completion whose unspecified entries must be Toeplitz operators.

We now consider:

PROBLEM 8.1. *Let b_λ be a Blaschke factor of the form $b_\lambda(z) := \frac{z-\lambda}{1-\overline{\lambda}z}$ $(\lambda \in \mathbb{D})$. Complete the unspecified Toeplitz entries of the partial block Toeplitz matrix*

$$A := \begin{pmatrix} T_{\overline{b}_\alpha} & ? \\ ? & T_{\overline{b}_\beta} \end{pmatrix} \quad (\alpha, \beta \in \mathbb{D})$$

to make A subnormal.

Recently, in [**CHL2**], we have considered Problem 8.1 for the cases $\alpha = \beta = 0$. The solution given in [**CHL2**, Theorem 5.1] relies upon very intricate and long computations using the symbol involved. However our solution in this chapter provides a shorter and more insightful proof by employing the results of the previous chapter.

We now give an answer to Problem 8.1:

THEOREM 8.2. *Let $\varphi, \psi \in L^\infty$ and consider*

$$A := \begin{pmatrix} T_{\overline{b}_\alpha} & T_\varphi \\ T_\psi & T_{\overline{b}_\beta} \end{pmatrix} \quad (\alpha, \beta \in \mathbb{D})\,.$$

The following statements are equivalent.

(a) *A is normal;*
(b) *A is subnormal;*
(c) *A is 2-hyponormal,*

except in the following special case:

$$\varphi_- = b_\alpha \theta_0' \overline{a} \text{ and } \psi_- = b_\alpha \theta_1' \overline{b} \text{ (coprime) with } (ab)(\alpha) = (\theta_0'\theta_1')(\alpha) \neq 0\,. \tag{8.1}$$

However, unless only one of θ_0' and θ_1' is constant, the exceptional case (8.1) implies that if A is subnormal then either A is normal or $A - \beta$ is quasinormal for some $\beta \in \mathbb{C}$.

Proof. Clearly (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a): Write

$$\Phi \equiv \begin{pmatrix} \overline{b}_\alpha & \varphi \\ \psi & \overline{b}_\beta \end{pmatrix} \equiv \Phi_-^* + \Phi_+ = \begin{pmatrix} b_\alpha & \psi_- \\ \varphi_- & b_\beta \end{pmatrix}^* + \begin{pmatrix} 0 & \varphi_+ \\ \psi_+ & 0 \end{pmatrix}$$

and assume that T_Φ is 2-hyponormal. Since $\ker [T^*, T]$ is invariant under T for every 2-hyponormal operator $T \in \mathcal{B}(\mathcal{H})$ (cf. [**CuL2**]), we note that Theorem 7.3 holds for 2-hyponormal operators T_Φ under the same assumption on the symbol. If T_Φ is hyponormal then by Lemma 2.3 there exists a matrix function $K \equiv \left(\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right) \in H^\infty_{M_2}$ such that $||K||_\infty \le 1$ and $\Phi - K\Phi^* \in H^\infty_{M_2}$, i.e.,

$$\begin{pmatrix} \overline{b}_\alpha & \overline{\varphi_-} \\ \overline{\psi_-} & \overline{b}_\beta \end{pmatrix} - \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \begin{pmatrix} 0 & \overline{\psi_+} \\ \overline{\varphi_+} & 0 \end{pmatrix} \in H^2_{M_2}, \tag{8.2}$$

which implies that

$$H_{\overline{b}_\alpha} = H_{k_2\overline{\varphi_+}} = H_{\overline{\varphi_+}} T_{k_2} \quad \text{and} \quad H_{\overline{b}_\beta} = H_{k_3\overline{\psi_+}} = H_{\overline{\psi_+}} T_{k_3}. \tag{8.3}$$

If $\overline{\varphi_+}$ is not of bounded type then $\ker H_{\overline{\varphi_+}} = \{0\}$, so that $k_2 = 0$, a contradiction; and if $\overline{\psi_+}$ is not of bounded type then $\ker H_{\overline{\psi_+}} = \{0\}$, so that $k_3 = 0$, a contradiction. Thus since $\overline{\varphi_+}$ and $\overline{\psi_+}$ are of bounded type, it follows that Φ^* of bounded type. Since T_Φ is hyponormal, it follows from (2.13) that Φ is also of bounded type. Thus we can write

$$\varphi_- \equiv \theta_0 \overline{a} \quad \text{and} \quad \psi_- \equiv \theta_1 \overline{b} \quad \text{(coprime)}.$$

Since Φ is normal, i.e., $\Phi\Phi^* = \Phi^*\Phi$, a straightforward calculation shows that $\alpha = \beta$. Thus by (8.2), we have

$$\begin{cases} \overline{b}_\alpha - k_2\overline{\varphi_+} \in H^2, & \overline{\theta}_0 a - k_1 \overline{\psi_+} \in H^2 \\ \overline{b}_\alpha - k_3\overline{\psi_+} \in H^2, & \overline{\theta}_1 b - k_4 \overline{\varphi_+} \in H^2\,, \end{cases} \tag{8.4}$$

which implies that the following Toeplitz operators are all hyponormal (via Cowen's Theorem):

$$T_{\overline{b}_\alpha + \varphi_+}, \quad T_{\overline{\theta}_1 b + \varphi_+}, \quad T_{\overline{b}_\alpha + \psi_+}, \quad T_{\overline{\theta}_0 a + \psi_+}. \tag{8.5}$$

Put

$$\theta_0 = b_\alpha^m \theta_0' \quad \text{and} \quad \theta_1 = b_\alpha^n \theta_1' \qquad (m, n \ge 0;\ \theta_0'(\alpha) \ne 0,\ \theta_1'(\alpha) \ne 0).$$

By Example 7.7, if $m \ne 0$ and $n = 0$ then we get a contradiction. Also a similar argument to Example 7.7 shows that

$$\text{either } m = n = 0 \text{ or } m = n = 1.$$

Thus we have to consider the case $m = n = 0$ and the case $m = n = 1$.

Case A ($m = n = 0$): A straightforward calculation shows that $\ker H_\Phi = \Theta H^2_{\mathbb{C}^2}$, where

$$\Theta \equiv \begin{pmatrix} b_\alpha \theta_1 & 0 \\ 0 & b_\alpha \theta_0 \end{pmatrix}.$$

Since Θ has a diagonal-constant inner divisor I_{b_α}, it follows from Lemma 5.2 and Theorem 7.3 that T_Φ is normal.

Case B-1 $(m = n = 1;\ (ab)(\alpha) \neq (\theta_0'\theta_1')(\alpha))$: A straightforward calculation shows that $\ker H_\Phi = \Theta H^2_{\mathbb{C}^2}$, where

$$\Theta \equiv \begin{pmatrix} b_\alpha\theta_1' & 0 \\ 0 & b_\alpha\theta_0' \end{pmatrix}.$$

Since Θ has a diagonal-constant inner divisor I_{b_α}, it follows from Lemma 5.2 and Theorem 7.3 that T_Φ is normal.

Case B-2 $(m = n = 1;\ (ab)(\alpha) = (\theta_0'\theta_1')(\alpha))$: A straightforward calculation shows that

$$\ker H_{\widetilde{\Phi}} = \widetilde{\Theta}_2 H^2_{\mathbb{C}^2},$$

where

$$\Theta_2 := \nu \begin{pmatrix} \theta_0 & -\overline{\gamma}\theta_1 \\ \gamma\theta_0' & \theta_1' \end{pmatrix} \quad (\gamma = -\frac{b(\alpha)}{\theta_0'(\alpha)} = -\frac{\theta_1'(\alpha)}{a(\alpha)};\ \nu := \frac{1}{\sqrt{|\gamma|^2+1}})$$

Thus we can write

$$\widetilde{\Phi}_- = \begin{pmatrix} \widetilde{b}_\alpha & \widetilde{\theta}_0\widetilde{\overline{a}} \\ \widetilde{\theta}_1\widetilde{\overline{b}} & \widetilde{b}_\alpha \end{pmatrix} = \widetilde{\Theta}_2\widetilde{B}^* \quad \text{(right coprime)},$$

so that

$$\Phi_- = \begin{pmatrix} b_\alpha & \theta_1\overline{b} \\ \theta_0\overline{a} & b_\alpha \end{pmatrix} = B^*\Theta_2 \quad \text{(left coprime)}. \tag{8.6}$$

By a scalar-valued version of (6.13) and (8.5), we can see that $\varphi_+ = \theta_1\theta_3\overline{d}$ and $\psi_+ = \theta_0\theta_2\overline{c}$ for some inner functions θ_2, θ_3, where $d \in \mathcal{H}(z\theta_1\theta_3)$ and $c \in \mathcal{H}(z\theta_0\theta_2)$. Thus, in particular, $c(\alpha) \neq 0$ and $d(\alpha) \neq 0$. Let

$$\theta_2 = b_\alpha^q\theta_2' \ \text{ and } \ \theta_3 = b_\alpha^p\theta_3' \quad (\text{where } \theta_2'(\alpha) \neq 0 \neq \theta_3'(\alpha)).$$

If we write $\Phi_+ = \Theta_0\Theta_2A^*$ ($\Theta_0\Theta_2 = I_\theta$ for an inner function θ), then a straightforward calculation shows that

$$A = \begin{pmatrix} 0 & \theta_1'\theta_3'c \\ \theta_0'\theta_2'd & 0 \end{pmatrix}.$$

We thus have

$$H_{A\Theta_2^*}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \nu \begin{pmatrix} -\gamma H_{\overline{b}_\alpha}(\theta_3'c) \\ H_{\overline{b}_\alpha}(\theta_2'd) \end{pmatrix}. \tag{8.7}$$

It was known that $\operatorname{ran} H_{\overline{b}_\alpha} = \mathcal{H}(b_{\overline{\alpha}}) = \bigvee\{\delta_1\}$ (where $\delta_1 := \frac{\sqrt{1-|\alpha|^2}}{1-\alpha z}$). It thus follows from (8.7) that

$$\begin{pmatrix} \delta_1 \\ \beta\delta_1 \end{pmatrix} \in \operatorname{cl}\operatorname{ran} H_{A\Theta_2^*} \subseteq \ker(I - T_{\widetilde{K}}T_{\widetilde{K}}^*), \tag{8.8}$$

where $\beta \neq 0$. We observe that if $k \in H^2$, then

$$T_{k(\overline{z})}\delta_1 = k(\alpha)\delta_1 : \tag{8.9}$$

indeed, if $k \in H^2$ and $n \geq 0$, then

$$\langle k(\overline{z})\delta_1,\ z^n\rangle = \langle \delta_1,\ \overline{k(\overline{z})}z^n\rangle = \overline{\langle \widetilde{k}z^n,\ \delta_1\rangle} = \sqrt{1-|\alpha|^2}\,\overline{\widetilde{k}(\overline{\alpha})\overline{\alpha}^n} = \sqrt{1-|\alpha|^2}\,k(\alpha)\alpha^n,$$

so that

$$T_{k(\overline{z})}\delta_1 = P(k(\overline{z})\delta_1) = \sqrt{1-|\alpha|^2}k(\alpha)\sum_{n=0}^{\infty}\alpha^n z^n = k(\alpha)\frac{\sqrt{1-|\alpha|^2}}{1-\alpha z} = k(\alpha)\delta_1\,,$$

which proves (8.9). Thus a straightforward calculation together with (8.8) shows that

$$\alpha_1 k_1 + \alpha_2 k_3 = 1 \quad \text{and} \quad \alpha_1 k_2 + \alpha_2 k_4 = \overline{\beta}\,, \tag{8.10}$$

where $\alpha_1 = \overline{k_1(\alpha)+\beta k_2(\alpha)}$ and $\alpha_2 = \overline{k_3(\alpha)+\beta k_4(\alpha)}$. We also have, from (8.8),

(8.11)

$$\left|\left|\begin{pmatrix}\delta_1\\ \beta\delta_1\end{pmatrix}\right|\right|_2 = \left|\left|T^*_{\widetilde{K}}\begin{pmatrix}\delta_1\\ \beta\delta_1\end{pmatrix}\right|\right|_2 = \left|\left|\begin{pmatrix}(k_1(\alpha)+\beta k_2(\alpha))\delta_1\\ (k_3(\alpha)+\beta k_4(\alpha))\delta_1\end{pmatrix}\right|\right|_2 = \left|\left|\begin{pmatrix}\overline{\alpha_1}\delta_1\\ \overline{\alpha_2}\delta_1\end{pmatrix}\right|\right|_2\,,$$

which implies

$$1+|\beta|^2 = |\alpha_1|^2 + |\alpha_2|^2. \tag{8.12}$$

From (8.4), we can see that

$$k_1 = \theta_2 k_1',\ k_2 = \theta_1'\theta_3 k_2',\ k_3 = \theta_0'\theta_2 k_3',\ k_4 = \theta_3 k_4', \tag{8.13}$$

where $k_i' \in H^\infty$ for $i = 1, \cdots, 4$. Thus by (8.10) and (8.13), we can see that θ_2 and θ_3 are both constant. Without loss of generality, we may assume that $\theta_2 = \theta_3 = 1$. We next claim that

$$\theta_0 = \theta_1 = b_\alpha,\ \ \text{i.e.,}\ \ \theta_0' \text{ and } \theta_1' \text{ are both constant.} \tag{8.14}$$

By our assumption, if θ_0' or θ_1' is constant then both θ_0' and θ_1' are constant. First of all, suppose that both θ_0' and θ_1' have nonconstant Blaschke factors. Thus there exist $v, w \in \mathbb{D}$ such that $\theta_0'(v) = 0 = \theta_1'(w)$. But since $k_3 = \theta_0' k_3'$ and $k_2 = \theta_1' k_2'$, it follows from (8.10) that

$$k_1(v) = \frac{1}{\alpha_1} \quad \text{and} \quad k_4(w) = \frac{\overline{\beta}}{\alpha_2} \tag{8.15}$$

(where we note that $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$). Observe that $|k_1(v)| = 1 = |k_4(w)|$: indeed, if $|k_1(v)| < 1$, then $|\alpha_1| > 1$, so that by (8.12), $|\alpha_2| < |\beta|$, which implies $|k_4(w)| > 1$, which contradicts the fact $||K||_\infty \leq 1$. If instead $|k_4(w)| < 1$, we similarly get a contradiction. Since $||k_1||_\infty \leq 1$ and $||k_4||_\infty \leq 1$, it follows from the Maximum Modulus Theorem that k_1 and k_4 are both constant, i.e.,

$$k_1 = \frac{1}{\alpha_1} \quad \text{and} \quad k_4 = \frac{\overline{\beta}}{\alpha_2}. \tag{8.16}$$

Then from (8.10), we should have $k_2 = k_3 = 0$, which leads to a contradiction, using (8.4).

Next, we assume that θ_0' has a nonconstant Blaschke factor and θ_1' has a nonconstant singular inner factor. Since θ_0' has a nonconstant Blaschke factor,

$$\exists\ w \in \mathbb{D} \text{ such that } \theta_0'(w) = 0, \text{ so that by (8.13), } k_3(w) = 0.$$

Thus by (8.10), $k_1(w) = \frac{1}{\alpha_1}$. But since $|k_1(w)| < 1$ (otherwise k_1 would be constant, so that $k_3 \equiv 0$, a contradiction from (8.4)), it follows that $1 < |\alpha_1|$. Thus by (8.12),

$$|\alpha_2| < |\beta|. \tag{8.17}$$

On the other hand, since θ_1' has a nonconstant singular inner factor, we can see that there exists $\delta \in [0, 2\pi)$ such that θ_1' has nontangential limit 0 at $e^{i\delta}$ (cf. [**Ga**, Theorem II.6.2]). Thus by (8.13), k_2 has nontangential limit 0 at $e^{i\delta}$ and in turn, by (8.10), k_4 has nontangential limit $\frac{\overline{\beta}}{\alpha_2}$ at $e^{i\delta}$. But since $||k_4||_\infty \le 1$, it follows that $\left|\frac{\overline{\beta}}{\alpha_2}\right| \le 1$, i.e., $|\beta| \le |\alpha_2|$, which contradicts (8.17).

Next, if both θ_0' and θ_1' have nonconstant singular factors then a similar argument gives a contradiction. This proves (8.14).

Now, in view of (8.6) and (8.14), we may write

$$\Phi_- = \begin{pmatrix} b_\alpha & b_\alpha \overline{b} \\ b_\alpha \overline{a} & b_\alpha \end{pmatrix} = B^* \Theta_2 \ \ \text{(left coprime)}, \ \ \text{where } \Theta_2 := \nu \begin{pmatrix} b_\alpha & -\overline{\gamma} b_\alpha \\ \gamma & 1 \end{pmatrix}.$$

We can also write

$$\Phi_+ = \begin{pmatrix} 0 & b_\alpha \overline{d} \\ b_\alpha \overline{c} & 0 \end{pmatrix} = A^* \Theta_0 \Theta_2 = A^* \left(\nu \begin{pmatrix} 1 & \overline{\gamma} b_\alpha \\ -\gamma & b_\alpha \end{pmatrix} \right) \left(\nu \begin{pmatrix} b_\alpha & -\overline{\gamma} b_\alpha \\ \gamma & 1 \end{pmatrix} \right),$$

where $\Theta_0 := \nu \begin{pmatrix} 1 & \overline{\gamma} b_\alpha \\ -\gamma & b_\alpha \end{pmatrix}$ and $\Theta_0 \Theta_2 = b_\alpha I_2$. Then by (7.13),

$$\Theta_0 H^2_{\mathbb{C}^2} \subseteq \ker [T^*_\Phi, T_\Phi], \quad \text{so that} \quad \operatorname{ran} [T^*_\Phi, T_\Phi] \subseteq \mathcal{H}(\Theta_0). \tag{8.18}$$

Since $\dim \mathcal{H}(\Theta_0) = 1$, it follows that

$$\operatorname{ran} [T^*_\Phi, T_\Phi] = \{0\} \ \text{ or } \ \operatorname{ran} [T^*_\Phi, T_\Phi] = \mathcal{H}(\Theta_0).$$

If $\operatorname{ran} [T^*_\Phi, T_\Phi] = \{0\}$, then evidently T_Φ is normal. Suppose $\operatorname{ran} [T^*_\Phi, T_\Phi] = \mathcal{H}(\Theta_0)$. We recall a well-known result of B. Morrel ([**Mo**]; [**Con**, p.162]). If $T \in \mathcal{B}(\mathcal{H})$ satisfies the following properties: (i) T is hyponormal; (ii) $[T^*, T]$ is rank-one; and (iii) $\ker [T^*, T]$ is invariant for T, then $T - \beta$ is quasinormal for some $\beta \in \mathbb{C}$, i.e., $T - \beta$ commutes with $(T - \beta)^*(T - \beta)$. Since T_Φ satisfies the above three properties, we can conclude that $T_{\Phi - \beta}$ is quasinormal for some $\beta \in \mathbb{C}$. □

On the other hand, due to a technical problem, we omitted a detailed proof for the case B-2 from the proof of [**CHL2**, Theorem 5.1]. The proof of the case B-2 (with $\alpha = 0$) in the proof of Theorem 8.2 above provides the portion of the proof that did not appear in [**CHL2**]. In particular, Theorem 8.2 incorporates an extension of a corrected version of [**CHL2**, Theorem 5.1], in which the exceptional case (8.1) was inadvertently omitted.

Corollary 8.3. *If*

$$T_\Phi := \begin{pmatrix} T_{\overline{b}_\alpha} & T_\varphi \\ T_\psi & T_{\overline{b}_\beta} \end{pmatrix} \quad (\alpha, \beta \in \mathbb{D};\ \varphi, \psi \in L^\infty),$$

then T_Φ is subnormal if and only if $\alpha = \beta$ and one of the following holds:

(a) $\varphi = e^{i\theta} b_\alpha + \zeta$ *and* $\psi = e^{i\omega} \varphi$ $(\zeta \in \mathbb{C};\ \theta, \omega \in [0, 2\pi))$;

(b) $\varphi = \mu \overline{b_\alpha} + e^{i\theta} \sqrt{1 + |\mu|^2}\, b_\alpha + \zeta$ *and* $\psi = e^{i(\pi - 2 \arg \mu)} \varphi$ $(\mu, \zeta \in \mathbb{C},\ |\mu| \neq 0, 1,\ \theta \in [0, 2\pi))$,

except the following special case:

$$\varphi_- = b_\alpha \theta_0' \overline{a} \ \textit{and} \ \psi_- = b_\alpha \theta_1' \overline{b} \ \textit{(coprime) with} \ (ab)(\alpha) = (\theta_0' \theta_1')(\alpha) \neq 0. \tag{8.19}$$

Proof. We use the notations of the proof of Theorem 8.2. From the viewpoint of the proof of Theorem 8.2, we should consider Case A and Case B-1.

Case A ($m = n = 0$): Since T_Φ is normal and $\det \Phi_+ \neq 0$, it follows form Lemma 7.6 that $\Phi_+^0 = \Phi_- U$ for some constant unitary matrix $U \equiv \left(\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right)$. We observe

$$
\begin{aligned}
\Phi_+^0 = \Phi_- U &\Longleftrightarrow \begin{pmatrix} 0 & \theta_1\theta_3\overline{d} \\ \theta_0\theta_2\overline{c} & 0 \end{pmatrix} = \begin{pmatrix} b_\alpha & \theta_1\overline{b} \\ \theta_0\overline{a} & b_\alpha \end{pmatrix}\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \\
&\Longleftrightarrow \begin{cases} 0 = c_1 b_\alpha + c_3\theta_1\overline{b} \\ 0 = c_4 b_\alpha + c_2\theta_0\overline{a} \\ \theta_1\theta_3\overline{d} = c_2 b_\alpha + c_4\theta_1\overline{b} \\ \theta_0\theta_2\overline{c} = c_3 b_\alpha + c_1\theta_0\overline{a} \end{cases} \\
&\Longrightarrow \begin{cases} c_1 = 0, \ \theta_1\overline{b} = 0 \\ c_4 = 0, \ \theta_0\overline{a} = 0 \\ \theta_1\theta_3\overline{d} = c_2 b_\alpha \\ \theta_0\theta_2\overline{c} = c_3 b_\alpha\,. \end{cases}
\end{aligned}
\tag{8.20}
$$

Since U is unitary we have $c_2 = e^{i\omega_1}$ and $c_3 = e^{i\omega_2}$ $(\omega_1, \omega_2 \in [0, 2\pi))$. Thus we have

$$\varphi = e^{i\omega_1} b_\alpha + \beta_1 \quad \text{and} \quad \psi = e^{i\omega_2} b_\alpha + \beta_2.$$

But since $|\varphi| = |\psi|$, it follows that

$$\varphi = e^{i\theta} b_\alpha + \zeta \quad \text{and} \quad \psi = e^{i\omega}\varphi \quad (\theta, \omega \in [0, 2\pi, \zeta \in \mathbb{C})).$$

Case B-1 ($m = n = 1$; $(ab)(\alpha) \neq (\theta_0'\theta_1')(\alpha)$): Since T_Φ is normal, it follows from the same argument as in (8.20), we can see that

$$\theta_2 = \theta_3 = 1 \quad \text{and} \quad \theta_0' = \theta_1' = 1,$$

in other words,

$$\varphi_+ = m_1 b_\alpha + \beta_1, \quad \psi_+ = m_2 b_\alpha + \beta_2, \quad \varphi_- = \nu b_\alpha, \quad \psi_- = \mu b_\alpha$$

$(m_1, m_2, \beta_1, \beta_2, \mu, \nu \in \mathbb{C})$. Thus we can write

$$\Phi_+ = \begin{pmatrix} 0 & \varphi_+ \\ \psi_+ & 0 \end{pmatrix} \quad \text{and} \quad \Phi_-^* = \begin{pmatrix} \overline{b}_\alpha & \mu\overline{b}_\alpha \\ \nu\overline{b}_\alpha & \overline{b}_\alpha \end{pmatrix} \quad (\mu \neq 0 \neq \nu).$$

Since T_Φ is normal we have

$$\begin{pmatrix} H^*_{\overline{\varphi_+}} H_{\overline{\varphi_+}} & 0 \\ 0 & H^*_{\overline{\psi_+}} H_{\overline{\psi_+}} \end{pmatrix} = \begin{pmatrix} (1 + |\nu|^2) H_{\overline{b}_\alpha} & (\mu + \overline{\nu}) H_{\overline{b}_\alpha} \\ (\overline{\mu} + \nu) H_{\overline{b}_\alpha} & (1 + |\mu|^2) H_{\overline{b}_\alpha} \end{pmatrix},$$

which implies that

$$\begin{cases} \nu = -\overline{\mu} \\ H^*_{\overline{\varphi_+}} H_{\overline{\varphi_+}} = (1 + |\nu|^2) H_{\overline{b}_\alpha} \\ H^*_{\overline{\psi_+}} H_{\overline{\psi_+}} = (1 + |\mu|^2) H_{\overline{b}_\alpha}. \end{cases} \tag{8.21}$$

By the case assumption, $1 \neq |ab| = |\mu\nu| = |\mu|^2$, i.e., $|\mu| \neq 1$. We thus have

$$\varphi_+ = e^{i\theta_1} \sqrt{1 + |\mu|^2}\, b_\alpha + \beta_1 \quad \text{and} \quad \psi_+ = e^{i\theta_2} \sqrt{1 + |\mu|^2}\, b_\alpha + \beta_2,$$

$(\beta_1, \beta_2 \in \mathbb{C};\ \theta_1, \theta_2 \in [0, 2\pi))$. Since $|\varphi| = |\psi|$, a straightforward calculation shows that

$$\varphi = \mu\, \overline{b}_\alpha + e^{i\theta}\sqrt{1+|\mu|^2}\, b_\alpha + \zeta \quad \text{and} \quad \psi = e^{i\,(\pi - 2\arg\mu)}\varphi, \tag{8.22}$$

where $\mu \neq 0,\ |\mu| \neq 1,\ \zeta \in \mathbb{C}$, and $\theta \in [0, 2\pi)$. $\square$

CHAPTER 9

Hyponormal Toeplitz pairs

In this chapter, we consider (jointly) hyponormal Toeplitz pairs with matrix-valued bounded type symbols. In [**CuL1**], the authors studied hyponormality of pairs of Toeplitz operators (called Toeplitz pairs) when both symbols are trigonometric polynomials. The core of the main result of [**CuL1**] is that the hyponormality of $\mathbf{T} \equiv (T_\varphi, T_\psi)$ (φ, ψ trigonometric polynomials) forces that the co-analytic parts of φ and ψ necessarily coincide up to a constant multiple, i.e.,

$$\varphi - \beta\psi \in H^2 \quad \text{for some } \beta \in \mathbb{C}. \tag{9.1}$$

In [**HL4**], (9.1) was extended for Toeplitz pairs whose symbols are rational functions with some constraint. As a result, the following question arises at once: Does (9.1) still hold for Toeplitz pairs whose symbols are *matrix-valued* trigonometric polynomials or rational functions? This chapter is concerned with this question. More generally, we give a characterization of hyponormal Toeplitz pairs with matrix-valued bounded type symbols by using the theory established in the previous chapters. Consequently, we will show that (9.1) is still true for matrix-valued trigonometric polynomials under some invertibility and commutativity assumptions on the Fourier coefficients of the symbols (those assumptions always hold vacuously for scalar-valued cases). Moreover, we give a characterization of the (joint) hyponormality of Toeplitz pairs with bounded type symbols, consider the self-commutators of the Toeplitz pairs with matrix-valued rational symbols, and derive rank formulae for them.

We first observe that if $\mathbf{T} = (T_\varphi, T_\psi)$ then the self-commutator of $\mathbf{T}$ can be expressed as:

(9.2)

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T_\varphi^*, T_\varphi] & [T_\psi^*, T_\varphi] \\ [T_\varphi^*, T_\psi] & [T_\psi^*, T_\psi] \end{pmatrix} = \begin{pmatrix} H^*_{\overline{\varphi_+}} H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}} H_{\overline{\varphi_-}} & H^*_{\overline{\varphi_+}} H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\varphi_-}} \\ H^*_{\overline{\psi_+}} H_{\overline{\varphi_+}} - H^*_{\overline{\varphi_-}} H_{\overline{\psi_-}} & H^*_{\overline{\psi_+}} H_{\overline{\psi_+}} - H^*_{\overline{\psi_-}} H_{\overline{\psi_-}} \end{pmatrix}.$$

The hyponormality of Toeplitz pairs is also related to the kernels of Hankel operators involved with the analytic and co-analytic parts of the symbol. Indeed it was shown in [**Gu2**, Lemma 6.2] that if neither φ nor ψ is analytic and if (T_φ, T_ψ) is hyponormal, then

$$\ker H_{\overline{\varphi_+}} \subseteq \ker H_{\overline{\psi_-}} \quad \text{and} \quad \ker H_{\overline{\psi_+}} \subseteq \ker H_{\overline{\varphi_-}}\,. \tag{9.3}$$

Tuples (or pairs) of Toeplitz operators will be called *Toeplitz tuples* (or *Toeplitz pairs*).

On a first perusal, one might be tempted to guess that (9.1) still holds for Toeplitz pairs whose symbols are matrix-valued trigonometric polynomials. However this is not the case. To see this we take

$$\Phi := \begin{pmatrix} z^{-1}+2z & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Psi := \begin{pmatrix} 0 & 0 \\ 0 & z^{-1}+2z \end{pmatrix}.$$

Then a straightforward calculation shows that if $\mathbf{T} = (T_\Phi, T_\Psi)$ then $[\mathbf{T}^*, \mathbf{T}] \geq 0$, i.e., $\mathbf{T}$ is hyponormal, but evidently, $\Phi_- \neq \Lambda \Psi_-$ for any constant matrix $\Lambda \in M_2$. However, we note that

$$\Phi_- = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^*$$

and by Theorem 4.16,

$$\Theta \equiv \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \text{ and } A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ are not right coprime.}$$

As we may expect, if the condition "Θ and A are right coprime" is assumed then we might get a matrix-valued version of (9.1). As we will see in the sequel, a corollary to the main result of this chapter follows the spirit of (9.1) (Corollary 9.22). The main theorem of this chapter (Theorem 9.20) gives a complete characterization of the hyponormality of Toeplitz pairs with bounded type symbols. Roughly speaking, this characterization says that the hyponormality of a Toeplitz pair can be determined by the hyponormality of a *single* Toeplitz operator.

To proceed, we consider some basic facts.

Recall (cf. p.11) that for each $\Phi \in L^\infty_{M_n}$, if we put

$$\mathcal{E}(\Phi) := \Big\{ K \in H^\infty_{M_n} : \ ||K||_\infty \leq 1 \ \text{ and } \ \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

If $\Phi \in L^\infty_{M_n}$, then by (2.9),

$$[T^*_\Phi, T_\Phi] = H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi + T_{\Phi^*\Phi - \Phi\Phi^*}\,.$$

Since the normality of Φ is a necessary condition for the hyponormality of T_Φ, the positivity of $H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi$ is an essential condition for the hyponormality of T_Φ. Thus, we isolate this property as a new notion, weaker than hyponormality. The reader will notice at once that this notion is meaningful for non-scalar symbols.

Definition 9.1. For $\Phi, \Psi \in L^\infty_{M_n}$, let

$$[T_\Phi, T_\Psi]_p := H^*_{\Psi^*} H_\Phi - H^*_{\Phi^*} H_\Psi.$$

Then $[T^*_\Phi, T_\Phi]_p$ is called the *pseudo-selfcommutator* of T_Φ. Also T_Φ is said to be *pseudo-hyponormal* if $[T^*_\Phi, T_\Phi]_p$ is positive semidefinite.

As in the case of hyponormality of scalar Toeplitz operators, we can see that the pseudo-hyponormality of T_Φ is independent of the constant matrix term $\Phi(0)$. Thus whenever we consider the pseudo-hyponormality of T_Φ we may assume, without loss of generality, that $\Phi(0) = 0$. Observe that if $\Phi \in L^\infty_{M_n}$ then

$$[T^*_\Phi, T_\Phi] = [T^*_\Phi, T_\Phi]_p + T_{\Phi^*\Phi - \Phi\Phi^*}.$$

Thus T_Φ is hyponormal if and only if T_Φ is pseudo-hyponormal and Φ is normal; also (via Theorem 3.3 of [**GHR**]) T_Φ is pseudo-hyponormal if and only if $\mathcal{E}(\Phi) \neq \emptyset$.

Recall that for $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$, we write

$$\mathcal{C}(\Phi) := \Big\{ K \in H^\infty_{M_n} : \ \Phi - K\Phi^* \in H^\infty_{M_n} \Big\}.$$

Thus if $\Phi \in L^\infty_{M_n}$, then

$$K \in \mathcal{E}(\Phi) \iff K \in \mathcal{C}(\Phi) \text{ and } ||K||_\infty \leq 1.$$

We note (cf. [**GHR**]) that if T_Φ is pseudo-hyponormal, then $||\Phi_-||_2 \leq ||\Phi_+||_2$: indeed, if $K \in \mathcal{E}(\Phi)$, then $\Phi_-^* - K\Phi_+^* \in H^2_{M_n}$, so that

$$||\Phi_-||_2 = ||\Phi_-^*||_2 \leq ||K\Phi_+^*||_2 \leq ||K||_\infty ||\Phi_+^*||_2 \leq ||\Phi_+^*||_2 = ||\Phi_+||_2.$$

In view of (6.11) in Chapter 6, whenever we study the pseudo-hyponormality of Toeplitz operators with symbol Φ such that Φ and Φ^* are of bounded type, we may assume that the symbol $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is of the form

$$\Phi_+ = \Theta_0\Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_0 B^* \quad \text{(right coprime)}. \tag{9.4}$$

We next consider hyponormality of Toeplitz operators with bounded type symbols. To do so, we use an interpolation problem developed in Chapter 6.

Proposition 9.2. (Pull-back symbols) *Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. In view of* (9.4)*, we may write*

$$\Phi_+ = \Theta_0\Theta_1 A^* \quad \textit{and} \quad \Phi_- = \Theta_0 B^* \quad \text{(right coprime)}.$$

Suppose $\Theta_1 A^* = A_1^*\Theta$ (where A_1 and Θ are left coprime). Then the following hold:

(a) *If I_ω is an inner divisor of Θ_1, put $\Phi^{1,\omega} := \Phi_-^* + P_{H_0^2}(\overline{\omega}\Phi_+)$* (cf. p.12). *Then T_Φ is pseudo-hyponormal if and only if $T_{\Phi^{1,\omega}}$ is pseudo-hyponormal;*

(b) *Put $\Upsilon := \Phi_-^* + P_{H^2}(\Theta_0 A_1^*)$. Then T_Φ is pseudo-hyponormal if and only if T_Υ is pseudo-hyponormal.*

Proof. (a) By Proposition 6.1(b), we have

$$\mathcal{C}(\Phi^{1,\omega}) = \big\{ \overline{\omega} K : K \in \mathcal{C}(\Phi) \big\}.$$

Thus the result follows at once from the observation that $||K||_\infty = ||\overline{\omega}K||_\infty$.

(b) By Proposition 6.1(a), we have

$$\begin{aligned} T_\Phi \text{ is pseudo-hyponormal} &\iff \Phi_-^* - K'\Theta\Phi_+^* \in H^2_{M_n} \quad (K' \in H^2 \text{ and } ||K'||_\infty \leq 1) \\ &\iff \Phi_-^* - K'A_1\Theta_0^* \in H^2_{M_n} \\ &\iff \Phi_-^* - K'\big(P_{H^2_{M_n}}(\Theta_0 A_1^*)\big)^* \in H^2_{M_n} \\ &\iff T_\Upsilon \text{ is pseudo-hyponormal.} \end{aligned}$$

□

For an operator $S \in \mathcal{B}(\mathcal{H})$, $S^\sharp \in \mathcal{B}(\mathcal{H})$ is called the Moore-Penrose inverse of S if

$$SS^\sharp S = S, \quad S^\sharp SS^\sharp = S^\sharp, \quad (S^\sharp S)^* = S^\sharp S, \quad \text{and} \quad (SS^\sharp)^* = SS^\sharp.$$

It is known ([**Har**, Theorem 8.7.2]) that if an operator S on a Hilbert space has closed range then S has a Moore-Penrose inverse. Moreover, the Moore-Penrose inverse is unique whenever it exists. On the other hand, it is well-known that if

$$S := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \quad \text{on } \mathcal{H}_1 \oplus \mathcal{H}_2,$$

(where the $\mathcal{H}_j$ are Hilbert spaces, $A \in \mathcal{B}(\mathcal{H}_1)$, $C \in \mathcal{B}(\mathcal{H}_2)$, and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$), then

$$S \geq 0 \iff A \geq 0,\ C \geq 0, \text{ and } B = A^{\frac{1}{2}} D C^{\frac{1}{2}} \text{ for some contraction } D; \tag{9.5}$$

moreover, in ([**CuL1**, Lemma 1.2] and [**Gu2**, Lemma 2.1]) it was shown that if $A \geq 0$, $C \geq 0$ and $\operatorname{ran} A$ is closed then

$$S \geq 0 \iff B^* A^\sharp B \leq C \text{ and } \operatorname{ran} B \subseteq \operatorname{ran} A, \tag{9.6}$$

or equivalently ([**CMX**, Lemma 1.4]),

$$|\langle Bg, f \rangle)|^2 \leq \langle Af, f \rangle \langle Cg, g \rangle \quad \text{for all } f \in \mathcal{H}_1,\ g \in \mathcal{H}_2 \tag{9.7}$$

and furthermore, if both A and C are of finite rank then

$$\operatorname{rank} S = \operatorname{rank} A + \operatorname{rank}(C - B^* A^\sharp B). \tag{9.8}$$

In fact, if $A \geq 0$ and $\operatorname{ran} A$ is closed then we can write

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \operatorname{ran} A \\ \ker A \end{pmatrix} \to \begin{pmatrix} \operatorname{ran} A \\ \ker A \end{pmatrix},$$

so that the Moore-Penrose inverse of A is given by

$$A^\sharp = \begin{pmatrix} (A_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{9.9}$$

We introduce a notion which will help simplify our arguments.

Definition 9.3. Let $\Phi, \Psi \in L^\infty_{M_n}$. For a Toeplitz pair $\mathbf{T} \equiv (T_\Phi, T_\Psi)$, the *pseudo-commutator* of $\mathbf{T}$ is defined by (cf. Definition 9.1)

$$\begin{aligned} [\mathbf{T}^*, \mathbf{T}]_p &:= \begin{pmatrix} [T_\Phi^*, T_\Phi]_p & [T_\Psi^*, T_\Phi]_p \\ [T_\Phi^*, T_\Psi]_p & [T_\Psi^*, T_\Psi]_p \end{pmatrix} \\ &= \begin{pmatrix} H^*_{\Phi^*_+} H_{\Phi^*_+} - H^*_{\Phi^*_-} H_{\Phi^*_-} & H^*_{\Phi^*_+} H_{\Psi^*_+} - H^*_{\Psi^*_-} H_{\Phi^*_-} \\ H^*_{\Psi^*_+} H_{\Phi^*_+} - H^*_{\Phi^*_-} H_{\Psi^*_-} & H^*_{\Psi^*_+} H_{\Psi^*_+} - H^*_{\Psi^*_-} H_{\Psi^*_-} \end{pmatrix}. \end{aligned}$$

Then $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ is said to be *pseudo-(jointly) hyponormal* if $[\mathbf{T}^*, \mathbf{T}]_p \geq 0$. Evidently, if $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ is pseudo-hyponormal then T_Φ and T_Ψ are pseudo-hyponormal.

We begin with:

Lemma 9.4. *Let* $\Phi, \Psi \in L^\infty_{M_n}$. *If* $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ *is hyponormal then* Φ *and* Ψ *commute.*

Proof. Suppose that $\mathbf{T}$ is hyponormal. Then T_Φ and T_Ψ are hyponormal, and hence Φ and Ψ are normal. Thus by (2.9) we have

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [T_\Phi^*, T_\Phi] & [T_\Psi^*, T_\Phi] \\ [T_\Phi^*, T_\Psi] & [T_\Psi^*, T_\Psi] \end{pmatrix}$$
$$= \begin{pmatrix} H^*_{\Phi_+^*} H_{\Phi_+^*} - H^*_{\Phi_-^*} H_{\Phi_-^*} & H^*_{\Phi_+^*} H_{\Psi_+^*} - H^*_{\Psi_-^*} H_{\Phi_-^*} + T_{\Psi^*\Phi - \Phi\Psi^*} \\ H^*_{\Psi_+^*} H_{\Phi_+^*} - H^*_{\Phi_-^*} H_{\Psi_-^*} + T_{\Phi^*\Psi - \Psi\Phi^*} & H^*_{\Psi_+^*} H_{\Psi_+^*} - H^*_{\Psi_-^*} H_{\Psi_-^*} \end{pmatrix}.$$

But since $\mathbf{T}$ is hyponormal it follows from (9.7) that for any $m \geq 0,\ x, y \in H^2_{\mathbb{C}^n}$,

$$\begin{aligned} &\left|\left\langle (H^*_{\Phi_+^*} H_{\Psi_+^*} - H^*_{\Psi_-^*} H_{\Phi_-^*} + T_{\Psi^*\Phi - \Phi\Psi^*})\, I_{z^m} y,\ I_{z^m} x \right\rangle\right|^2 \\ &\leq \left\langle (H^*_{\Phi_+^*} H_{\Phi_+^*} - H^*_{\Phi_-^*} H_{\Phi_-^*})(I_{z^m} x),\ I_{z^m} x \right\rangle \\ &\qquad \cdot \left\langle (H^*_{\Psi_+^*} H_{\Psi_+^*} - H^*_{\Psi_-^*} H_{\Psi_-^*})(I_{z^m} y),\ I_{z^m} y \right\rangle. \end{aligned} \tag{9.10}$$

Note that

$$\left\langle T_{\Psi^*\Phi - \Phi\Psi^*}(I_{z^m} y),\ I_{z^m} x) \right\rangle = \left\langle T^*_{I_{z^m}} T_{\Psi^*\Phi - \Phi\Psi^*} T_{I_{z^m}} y,\ x \right\rangle = \left\langle T_{\Psi^*\Phi - \Phi\Psi^*} y,\ x \right\rangle. \tag{9.11}$$

But since

$$\lim_{m\to\infty} H_C(I_{z^m} \omega) = 0 \quad \text{for any } C \in L^\infty_{M_n} \text{ and } \omega \in H^2_{\mathbb{C}^n},$$

if we take the limits on m in (9.10) and (9.11) then we have

$$\left\langle T_{\Psi^*\Phi - \Phi\Psi^*} y,\ x \right\rangle = \lim_{m\to\infty} \left\langle (T_{\Psi^*\Phi - \Phi\Psi^*})\, I_{z^m} y,\ I_{z^m} x \right\rangle = 0,$$

which implies that $\Psi^*\Phi = \Phi\Psi^*$. Since Ψ is normal it follows from the Fuglede-Putnam Theorem that $\Phi\Psi = \Psi\Phi$. $\square$

We thus have:

Corollary 9.5. *Let $\Phi, \Psi \in L^\infty_{M_n}$ and let $\mathbf{T} := (T_\Phi, T_\Psi)$. Then the following are equivalent:*

(i) $\mathbf{T}$ *is hyponormal;*
(ii) $\mathbf{T}$ *is pseudo-hyponormal, Φ and Ψ are normal, and $\Psi\Phi = \Phi\Psi$.*

Proof. Immediate from Lemma 9.4. $\square$

Lemma 9.6. *Let $\Phi \in H^\infty_{M_n}$ and $\Psi \in L^\infty_{M_n}$. If*

$$\Phi \equiv \Phi_+ = A^*\Theta \quad \text{(left coprime)}$$

then

$$\mathbf{T} = (T_\Phi, T_\Psi) \textit{ is pseudo-hyponormal} \Longleftrightarrow T_{\Psi^{1,\Theta}} \textit{ is pseudo-hyponormal}, \tag{9.12}$$

where $\Psi^{1,\Theta} := \Psi_-^* + P_{H_0^2}(\Psi_+\Theta^*)$ (cf. p.12).

Proof. Since $\Phi \in H^\infty_{M_n}$, $\mathbf{T} = (T_\Phi, T_\Psi)$ is pseudo-hyponormal if and only if

$$[\mathbf{T}^*, \mathbf{T}]_p = \begin{pmatrix} H^*_{\Phi_+^*} H_{\Phi_+^*} & H^*_{\Phi_+^*} H_{\Psi_+^*} \\ H^*_{\Psi_+^*} H_{\Phi_+^*} & [T_\Psi^*,\ T_\Psi]_p \end{pmatrix} \geq 0,$$

or equivalently, by (9.7),

(9.13)

$$\left|\left\langle H^*_{\Psi^*_+} H_{\Phi^*_+} x,\ y\right\rangle\right|^2 \le \left\langle H^*_{\Phi^*_+} H_{\Phi^*_+} x,\ x\right\rangle \left\langle [T^*_\Psi,\ T_\Psi]_p y,\ y\right\rangle \qquad (\text{all } x, y \in H^2_{\mathbb{C}^n}).$$

Since by assumption, $\widetilde{A}$ and $\widetilde{\Theta}$ are right coprime, it follows that

$$\operatorname{cl} \operatorname{ran} H_{\Phi^*_+} = \mathcal{H}(\widetilde{\Theta}) = \operatorname{cl} \operatorname{ran} H_{\Theta^*}.$$

Therefore, the inequality (9.13) becomes

$$\left|\left\langle H_{\Theta^*} x,\ H_{\Psi^*_+} y\right\rangle\right| \le \left\langle H_{\Theta^*} x,\ H_{\Theta^*} x\right\rangle \left\langle [T^*_\Psi,\ T_\Psi]_p y,\ y\right\rangle \qquad (\text{all } x, y \in H^2_{\mathbb{C}^n}).$$

We thus have that $[\mathbf{T}^*, \mathbf{T}]_p \ge 0$ if and only if $[\mathbf{T}^*_\Theta, \mathbf{T}_\Theta]_p \ge 0$, where $\mathbf{T}_\Theta := (T_\Theta, T_\Psi)$. Observe that

$$[\mathbf{T}^*_\Theta, \mathbf{T}_\Theta]_p = \begin{pmatrix} H^*_{\Theta^*} H_{\Theta^*} & H^*_{\Theta^*} H_{\Psi^*_+} \\ H^*_{\Psi^*_+} H_{\Theta^*} & [T^*_\Psi,\ T_\Psi]_p \end{pmatrix}. \tag{9.14}$$

By (2.11) we can see that $H_{\Theta^*} H^*_{\Theta^*}$ is a projection on $\mathcal{H}(\widetilde{\Theta})$. Therefore, it follows from (9.6) that

$$[\mathbf{T}^*_\Theta, \mathbf{T}_\Theta]_p \ge 0 \Longleftrightarrow [T^*_\Psi,\ T_\Psi]_p - H^*_{\Psi^*_+} H_{\Theta^*} H^*_{\Theta^*} H_{\Psi^*_+} \ge 0$$

Observe that by (2.11),

$$[T^*_\Psi,\ T_\Psi]_p - H^*_{\Psi^*_+} H_{\Theta^*} H^*_{\Theta^*} H_{\Psi^*_+} = [T^*_{\Psi^{1,\Theta}},\ T_{\Psi^{1,\Theta}}]_p\,.$$

Thus the inequality (9.13) holds if and only if $T_{\Psi^{1,\Theta}}$ is pseudo-hyponormal. This proves the lemma. □

Remark 9.7. In [**CuL1**, Problem 5.4], the following problem was formulated: *For $n > 1$, find a block Toeplitz operator R, not all of whose diagonals are constant, for which (U^n, R) is hyponormal, where U is the unilateral shift on H^2.* In fact, if (U^n, R) is a hyponormal pair then by (9.5),

$$[R^*, U^n] = [U^{n*}, U^n]^{\frac{1}{2}} D\, [R^*, R]^{\frac{1}{2}} \quad \text{for some contraction } D.$$

But since $P_n := [U^{n*}, U^n]$ is the orthogonal projection of rank n and $P_n U^n = 0$, it follows that

$$U^{n*} R - R U^{n*} = [R^*, R]^{\frac{1}{2}} D^* P_n \implies U^{n*} R U^n - R = [R^*, R]^{\frac{1}{2}} D^* P_n U^n = 0,$$

which implies that R is a block Toeplitz operator T_Φ with symbol $\Phi \in L^\infty_{M_n}$. Thus Lemma 9.6 together with Corollary 9.5 gives an answer to this problem: since $U^n \cong T_{I_z}$, it follows that

$$(U^n, R) \text{ is hyponormal} \iff R \cong T_\Phi \ (\Phi \text{ is normal}) \text{ and } T_{\Phi^{1,I_z}} \text{ is pseudo-hyponormal},$$

where $\cong$ denotes unitary equivalence. For example, if $R = T_\Phi$ with

$$\Phi = \begin{pmatrix} z & 0 \\ 0 & 2z \end{pmatrix},$$

then $(U^2, R) \cong (T_{zI_2}, T_\Phi)$ is a hyponormal pair, but T_Φ is not unitarily equivalent to any Toeplitz operator (indeed, the essential spectrum of T_Φ is $\mathbb{T} \cup 2\,\mathbb{T}$, which is not connected). □

Lemma 9.8. *Let $\Phi, \Psi \in L^\infty_{M_n}$. If $\mathbf{T} = (T_\Phi, T_\Psi)$ is pseudo-hyponormal and $\Theta \in H^\infty_{M_n}$ is an inner matrix function then $\mathbf{T}_\Theta := (T_{\Phi_\Theta}, T_{\Psi_\Theta})$ is pseudo-hyponormal.*

PROOF. Suppose that $\mathbf{T}=(T_\Phi,T_\Psi)$ is pseudo-hyponormal. We then have

$$\begin{aligned}
[(T_{\Phi_\Theta})^*,T_{\Psi_\Theta}]_p &= H^*_{[P_{H_0^2}(\Theta^*\Psi_+)]^*}H_{[P_{H_0^2}(\Theta^*\Phi_+)]^*}-H^*_{[P_{(H^2)^\perp}(\Phi_-^*\Theta)]}H_{[P_{(H^2)^\perp}(\Psi_-^*\Theta)]}\\
&=(H_{\Psi_+^*}T_\Theta)^*H_{\Phi_+^*}T_\Theta-(H_{\Phi_-^*}T_\Theta)^*H_{\Psi_-^*}T_\Theta\\
&=T_\Theta^*(H^*_{\Psi_+^*}H_{\Phi_+^*}-H^*_{\Phi_-^*}H_{\Psi_-^*})T_\Theta\\
&=T_\Theta^*\,[T_\Phi^*,T_\Psi]_p\,T_\Theta,
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
[(\mathbf{T}_\Theta)^*,\mathbf{T}_\Theta]_p &= \begin{pmatrix}[(T_{\Phi_\Theta})^*,T_{\Phi_\Theta}]_p & [(T_{\Psi_\Theta})^*,T_{\Phi_\Theta}]_p\\ [(T_{\Phi_\Theta})^*,T_{\Psi_\Theta}]_p & [(T_{\Psi_\Theta})^*,T_{\Psi_\Theta}]_p\end{pmatrix}\\
&=\begin{pmatrix}T_\Theta & 0\\ 0 & T_\Theta\end{pmatrix}^*[\mathbf{T}^*,\mathbf{T}]_p\begin{pmatrix}T_\Theta & 0\\ 0 & T_\Theta\end{pmatrix},
\end{aligned}$$

which gives the result. □

LEMMA 9.9. *Let $\mathbf{T}\equiv(T_\Phi,T_\Psi)$ and $\mathbf{S}\equiv(T_{\Phi-\Lambda\Psi},T_\Psi)$, where $\Lambda\in M_n$ is a constant normal matrix commuting with Ψ_- and Φ_-. Then*

$$\mathbf{T}\text{ is pseudo-hyponormal}\iff\mathbf{S}\text{ is pseudo-hyponormal.}\tag{9.15}$$

PROOF. Put

$$\mathcal{T}:=\begin{pmatrix}I & -T_\Lambda\\ 0 & I\end{pmatrix}[\mathbf{T}^*,\mathbf{T}]_p\begin{pmatrix}I & 0\\ -T_\Lambda^* & I\end{pmatrix}\quad\text{and}\quad\mathcal{S}:=[\mathbf{S}^*,\ \mathbf{S}]_p.$$

Then

$$\mathcal{S}=\begin{pmatrix}H^*_{(\Phi-\Lambda\Psi)^*}H_{(\Phi-\Lambda\Psi)^*}-H^*_{(\Phi-\Lambda\Psi)}H_{(\Phi-\Lambda\Psi)} & H^*_{(\Phi-\Lambda\Psi)^*}H_{\Psi^*}-H^*_\Psi H_{(\Phi-\Lambda\Psi)}\\ H^*_{\Psi^*}H_{(\Phi-\Lambda\Psi)^*}-H^*_{(\Phi-\Lambda\Psi)}H_\Psi & H^*_{\Psi^*}H_{\Psi^*}-H^*_\Psi H_\Psi\end{pmatrix}.$$

Note that if $\Phi\in L^\infty_{M_n}$ and Λ is a constant matrix such that $\Phi_-\Lambda=\Lambda\Phi_-$, then by the Fuglede-Putnam Theorem,

$$H_\Phi\,T_\Lambda=H_{\Phi\Lambda}=H_{\Lambda\Phi}=T_\Lambda\,H_\Phi.\tag{9.16}$$

Then by a straightforward calculation together with (9.16) and the assumption that $\Lambda\Lambda^*=\Lambda^*\Lambda$ and Λ commutes with Ψ_- and Φ_-, we can show that $\mathcal{S}=\mathcal{T}$. This proves the lemma. □

LEMMA 9.10. *Let $\mathbf{T}:=(T_\Phi,T_\Psi)$ be a pseudo-hyponormal Toeplitz pair with bounded type symbols $\Phi,\Psi\in L^\infty_{M_n}$. If Φ and Ψ are not analytic and if $\Psi_+=\Theta A^*$* (right coprime)*, then*

$$H^*_{\Psi_-^*}H_{\Phi_-^*}T_\Theta=0.\tag{9.17}$$

PROOF. This follows from the positivity test for $[\mathbf{T}^*,\mathbf{T}]_p$ via (9.7). □

We now have

COROLLARY 9.11. *Let $\mathbf{T}:=(T_\Phi,T_\Psi)$ be a pseudo-hyponormal Toeplitz pair with bounded type symbols $\Phi,\Psi\in L^\infty_{M_n}$, which are not analytic. If Ψ has a tensored-scalar singularity then*

$$\ker H_{\Psi_+^*}\subseteq\ker H_{\Phi_-^*}.\tag{9.18}$$

Proof. In view of (9.4), we may write

$$\Phi_+ = \Theta_0\Theta_1 A^*, \quad \Phi_- = \Theta_0 B^*, \quad \Psi_+ = \Theta_2\Theta_3 C^*, \quad \Psi_- = \Theta_2 D^* \quad \text{(right coprime)}.$$

Since Ψ has a tensored-scalar singularity, it follows from (9.17), (2.10), and Theorem 5.4 that $H_{\Phi_-^*\Theta_2\Theta_3} = 0$, and hence $\ker H_{\Psi_+^*} = \Theta_2\Theta_3 H^2_{\mathbb{C}^n} \subseteq \ker H_{\Phi_-^*}$. $\square$

Lemma 9.12. *Let $\mathbf{T} := (T_\Phi, T_\Psi)$ be a pseudo-hyponormal Toeplitz pair with bounded type symbols $\Phi, \Psi \in L^\infty_{M_n}$ of the form*

(9.19)

$$\Phi_+ = \Theta_0\Theta_1 A^*, \quad \Phi_- = \Theta_0 B^*, \quad \Psi_+ = \Theta_2\Theta_3 C^*, \quad \Psi_- = \Theta_2 D^* \quad \textit{(right coprime)}.$$

If Φ^* has a tensored-scalar singularity and $\Theta_0^*\Theta_2 = I_\theta$ for some inner function θ, then I_θ is an inner divisor of Θ_3.

Proof. By assumption, $\Theta_2 = \theta\Theta_0$. Thus we may write

$$\Phi_+ = \Theta_0\Theta_1 A^*, \quad \Phi_- = \Theta_0 B^*, \quad \Psi_+ = \theta\Theta_0\Theta_3 C^*, \quad \Psi_- = \theta\Theta_0 D^* \quad \text{(right coprime)}.$$

By Corollary 9.11, we have $\Theta_0\Theta_1 H^2_{\mathbb{C}^n} \subseteq \theta\Theta_0 H^2_{\mathbb{C}^n}$. It thus follows from [**FF**, Corollary IX.2.2] that $\Theta_1 = \theta\Delta$ for some square inner matrix function Δ. Thus we can write

$$\Phi_+ = \theta\Theta_0\Delta A^*, \quad \Phi_- = \Theta_0 B^*, \quad \Psi_+ = \theta\Theta_0\Theta_3 C^*, \quad \Psi_- = \theta\Theta_0 D^* \quad \text{(right coprime)}.$$

Put

$$\Phi^{(1)} := \Phi_{\Theta_0}, \quad \Psi^{(1)} := \Phi_{\Theta_0}.$$

Then by Lemma 9.8, $(T_{\Phi^{(1)}}, T_{\Psi^{(1)}})$ is pseudo-hyponormal. By Proposition 4.19 we get the following right coprime factorizations:

$$\Phi_+^{(1)} = \theta\Delta A_1^*, \quad \Phi_-^{(1)} = 0, \quad \Psi_+^{(1)} = \theta\Theta_3 C_1^*, \quad \Psi_-^{(1)} = \theta D_1^*,$$

where $A_1 := P_{\mathcal{K}_{\theta\Delta_1}} A$, $C_1 := P_{\mathcal{K}_{\theta\Theta_3}} C$, $D_1 := P_{\mathcal{K}_\theta} D$. It thus follows from Lemma 9.6 that $T_{(\Psi^{(1)})^{1,\Theta_1}}$ is pseudo-hyponormal. Observe that

$$\Big((\Psi^{(1)})^{1,\Theta_1}\Big)_+ = P_{H_0^2}(\Psi_+^{(1)}\Delta^* I_\theta^*) = P_{H_0^2}(\Theta_3(\Delta C_1)^*) = \Theta_3\big(P_{\mathcal{K}_{\Theta_3}}(\Delta C_1)\big)^*,$$

where the last equality follows from Lemma 4.4. Thus we can write

$$\Big((\Psi^{(1)})^{1,\Theta_1}\Big)_+ = \Theta_3' B^* \qquad \text{(right coprime)},$$

where Θ_3' is a left inner divisor of Θ_3. Since $T_{(\Psi^{(1)})^{1,\Theta_1}}$ is pseudo-hyponormal, it follows from (9.4) that I_θ is a (left) inner divisor of Θ_3' and hence is a inner divisor of Θ_3. $\square$

Lemma 9.13. *Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ be a pseudo-hyponormal Toeplitz pair with bounded type symbols $\Phi, \Psi \in L^\infty_{M_n}$ of the form*

$$\Phi_+ = \theta\theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta\theta_3 C^*, \quad \Psi_- = \theta_2 D^* \quad \text{(coprime)},$$

where $\theta := \text{l.c.m.}(\theta_0, \theta_2)$. If we let $\delta := \text{g.c.d.}(\theta_1, \theta_3)$, then

$$\mathbf{T} \ \ \textit{pseudo-hyponormal} \Longleftrightarrow \mathbf{T}_\Delta := (T_{\Phi^{1,\delta}}, T_{\Psi^{1,\delta}}) \ \ \textit{pseudo-hyponormal}.$$

Proof. This follows from a slight variation of [**HL4**, Proof of Theorem 1] for the matrix-valued case by using Theorem 5.9. $\square$

LEMMA 9.14. *Suppose θ is a finite Blaschke product of degree n. Let $\Phi, \Psi \in L^\infty_{M_n}$ be such that $\mathbf{T} := (T_\Phi, T_\Psi)$ and $\mathbf{T}' := (T_{\Phi\circ\theta}, T_{\Psi\circ\theta})$. Then*

$$[\mathbf{T}'^*, \mathbf{T}'] \cong \bigoplus_n [\mathbf{T}^*, \mathbf{T}], \tag{9.20}$$

and

$$[\mathbf{T}'^*, \mathbf{T}']_p \cong \bigoplus_n [\mathbf{T}^*, \mathbf{T}]_p\,, \tag{9.21}$$

where $n = \deg(\theta)$ and $\cong$ means unitary equivalence. In particular, $\mathbf{T}$ is (pseudo) hyponormal if and only if $\mathbf{T}'$ is (pseudo) hyponormal.

Remark. The unitary operator given in Lemma 9.14 depends only on the inner function θ.

PROOF OF LEMMA 9.14. Let $\Phi = (\varphi_{ij}) \in L^\infty_{M_n}$ be arbitrary. Then by a well-known fact due to C. Cowen [**Co1**, Theorem 1], there exists a unitary operator V such that

$$T_{\varphi_{ij}\circ\theta} = V^* \left(\bigoplus_n T_{\varphi_{ij}} \right) V \quad (\text{for } i, j = 1, 2, \cdots, n),$$

where $n = \deg(\theta)$. Put $\mathcal{V} := V \otimes I_n$. Then

$$[T^*_{\Psi\circ\theta},\ T_{\Phi\circ\theta}] = T^*_{\Psi\circ\theta} T_{\Phi\circ\theta} - T_{\Phi\circ\theta} T^*_{\Psi\circ\theta} = \mathcal{V}^* \left(\bigoplus_n [T^*_\Psi, T_\Phi] \right) \mathcal{V}.$$

Therefore we have

$$\begin{aligned}
[\mathbf{T}'^*, \mathbf{T}'] &= \begin{pmatrix} [T^*_{\Phi\circ\theta}, T_{\Phi\circ\theta}] & [T^*_{\Psi\circ\theta}, T_{\Phi\circ\theta}] \\ [T^*_{\Phi\circ\theta}, T_{\Psi\circ\theta}] & [T^*_{\Psi\circ\theta}, T_{\Psi\circ\theta}] \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{V} & 0 \\ 0 & \mathcal{V} \end{pmatrix}^* \begin{pmatrix} \bigoplus_n [T^*_\Phi, T_\Phi] & \bigoplus_n [T^*_\Psi, T_\Phi] \\ \bigoplus_n [T^*_\Phi, T_\Psi] & \bigoplus_n [T^*_\Psi, T_\Psi] \end{pmatrix} \begin{pmatrix} \mathcal{V} & 0 \\ 0 & \mathcal{V} \end{pmatrix} \\
&\cong \bigoplus_n [\mathbf{T}^*, \mathbf{T}]\,,
\end{aligned}$$

giving (9.20). We also observe

$$\begin{aligned}
[T^*_{\Psi\circ\theta},\ T_{\Phi\circ\theta}]_p &= [T^*_{\Psi\circ\theta},\ T_{\Phi\circ\theta}] - T_{(\Psi^*\Phi - \Phi\Psi^*)(\theta)} \\
&= \mathcal{V}^* \left(\bigoplus_n \Big[[T^*_\Psi, T_\Phi] - T_{(\Psi^*\Phi - \Phi\Psi^*)} \Big] \right) \mathcal{V} \\
&\cong \bigoplus_n [T^*_\Psi, T_\Phi]_p\,,
\end{aligned}$$

giving (9.21). The remaining assertions are evident from (9.20) and (9.21). $\square$

LEMMA 9.15. *Let $\mathbf{T} = (T_\Phi, T_\Psi)$ be a pseudo-hyponormal Toeplitz pair with bounded type symbols $\Phi, \Psi \in L^\infty$ of the form*

$$\Phi_+ = \theta^{p+1} A^*, \quad \Phi_- = \theta B^*, \quad \Psi_+ = \theta^{q+1} C^*, \quad \Psi_- = \theta D^* \quad (\text{right coprime}), \tag{9.22}$$

where the θ_i is inner. If $pq = 0$, then $p = q = 0$.

Proof. Without loss of generality we may assume that $q = 0$. Suppose that $p \neq 0$. Since $\mathbf{T}$ is pseudo-hyponormal, it follows from Lemma 9.9 that $(T_{\Phi-\beta\Psi}, T_\Psi)$ is pseudo-hyponormal for all $\beta \in \mathbb{R}$. In particular $T_{\Phi-\beta\Psi}$ is pseudo-hyponormal for all $\beta \in \mathbb{R}$. Observe that

$$\Phi - \beta\Psi = \overline{\theta}(B - \beta D) + \theta^{p+1}(A - \overline{\beta}C\theta^p)^*.$$

Thus by Proposition 9.2, we can see that T_{Υ_β} is pseudo-hyponormal, where

$$\Upsilon_\beta = \overline{\theta}(B - \beta D) + \theta(P_{\mathcal{K}_{I_\theta}}(A - \overline{\beta}\theta^p C))^* = \overline{\theta}(B - \beta D) + \theta\big(P_{\mathcal{K}_{I_\theta}} A\big)^*,$$

which gives a contradiction because $||(\Upsilon_\beta)_-||_2 > ||(\Upsilon_\beta)_+||_2$ if $|\beta|$ is sufficiently large. $\square$

Lemma 9.16. *Let $\mathbf{T} := (T_\Phi, T_\Psi)$ be a pseudo-hyponormal Toeplitz pair with bounded type symbols $\Phi, \Psi \in L^\infty_{M_n}$. Suppose the inner parts of right coprime factorizations of Φ_+ and Ψ_+ commute. If Φ and Ψ have a common tensored-scalar pole, then this pole has the same order.*

Proof. In view of (9.4), we may write

$$\Phi_+ = \Theta_0\Theta_1 A^*, \quad \Phi_- = \Theta_0 B^*, \quad \Psi_+ = \Theta_2\Theta_3 C^*, \quad \Psi_- = \Theta_2 D^* \quad \text{(right coprime)}.$$

Assume that $\Theta_0\Theta_1$ and $\Theta_2\Theta_3$ commute. Also suppose Φ and Ψ have a common tensored-scalar pole of order p and r, respectively, at α . Then we can write

$$\Theta_0 = b_\alpha^p\Delta_0, \quad \Theta_1 = b_\alpha^q\Delta_1, \quad \Theta_2 = b_\alpha^r\Delta_2, \quad \Theta_3 = b_\alpha^s\Delta_3 \quad \left(b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}\right),$$

where $p, r \geq 1, q, s \geq 0$, Δ_i and I_{b_α} are coprime for $i = 0, 1, 2, 3$. By Theorem 4.40 and Lemma 9.14, we may assume that $\alpha = 0$. Assume to the contrary that $r \neq p$. Then without loss of generality we may assume that $p < r$. Let

$$\Delta' := \prod_{i=0}^{3} \Delta_i \quad \text{and} \quad \Delta := z^{p-1}\Delta',$$

and write $\Phi^{(1)} := \Phi_\Delta$ and $\Psi^{(1)} := \Psi_\Delta$. It then follows from Lemma 9.8 that $(T_{\Phi^{(1)}}, T_{\Psi^{(1)}})$ is pseudo-hyponormal. Since by assumption, $\Theta_0\Theta_1$ and $\Theta_2\Theta_3$ commute, it follows that

$$\Theta_2\Theta_3 = z^{r+s}\Delta_2\Delta_3 \quad \text{and} \quad \Delta = z^{p-1}\Delta_2\Delta_3\Delta_0\Delta_1,$$

so that left-g.c.d.$(\Delta, \Theta_2\Theta_3) = z^{p-1}\Delta_2\Delta_3$. Thus we have

$$\big(\text{left-g.c.d.}(\Delta, \Theta_2\Theta_3)\big)^*\Theta_2\Theta_3 = I_z^{r+s-p+1} \equiv I_{\theta_1}.$$

By Proposition 4.19 (c), we can write

$$\Psi_+^{(1)} = z^{r+s-p+1}C_1^* \qquad \text{(coprime)},$$

where $C_1 = P_{\mathcal{K}_{\theta_1}}(C\Delta_0\Delta_1)$. Similarly we also get the following coprime factorizations:

$$\Phi_+^{(1)} = z^{q+1}A_1^*, \quad \Phi_-^{(1)} = zB_1^*, \quad \Psi_-^{(1)} = z^{r-p+1}D_1^*$$

for some $A_1, B_1, D_1 \in H^2_{M_n}$, in particular $B_1 \in M_n$ is invertible a.e. on $\mathbb{T}$. Applying Lemma 9.12 with $\theta = z^{r-p}$ and $\theta_3 = z^s$, we have $0 < r - p \leq s$. Then by using

Proposition 9.2 and Lemmas 9.6, 9.8, and 9.9, we can show that $s \ge q$ and $T_{\Phi^{(\gamma)}}$ is pseudo-hyponormal, where

$$\Phi_+^{(\gamma)} = z^{r-p+1} A_\gamma^* \quad \text{and} \quad \Phi_-^{(\gamma)} = z^{r-p+1}\big(D_1 + \gamma z^{r-p} B_1\big)^*,$$

with

$$A_\gamma := P_{\mathcal{K}_{z^{r-p+1}}}(C_1 + \overline{\gamma} z^{r+s-p-q} A_1).$$

Since $T_{\Phi^{(\gamma)}}$ is pseudo-hyponormal it follows that $s = q$. Thus we have

$$A_\gamma = P_{\mathcal{K}_{z^{r-p+1}}} C_1 + P_{\mathcal{K}_{z^{r-p+1}}}(\overline{\gamma} z^{r-p} A_1).$$

Put

$$P_{\mathcal{K}_{z^{r-p+1}}} C_1 = \sum_{i=0}^{r-p} z^i C_1^{(i)} \quad \text{and} \quad D_1 = \sum_{i=0}^{r-p} z^i D_1^{(i)},$$

where $C_1^{(i)}, D_1^{(i)} \in M_n$ for each $i = 0, \cdots, r-p$. Then we have

$$\Phi_+^{(\gamma)} = z^{r-p+1}\left(\sum_{i=0}^{r-p-1} z^i C_1^{(i)} + z^{r-p}\big(C_1^{(r-p)} + \overline{\gamma} P_{\mathcal{K}_z} A_1\big)\right)^* \equiv z^{r-p+1} C';$$

$$\Phi_-^{(\gamma)} = z^{r-p+1}\left(\sum_{i=0}^{r-p-1} z^i D_1^{(i)} + z^{r-p}\big(D_1^{(r-p)} + \gamma B_1\big)\right)^* \equiv z^{r-p+1} D'.$$

Since $T_{\Phi^{(\gamma)}}$ is pseudo-hyponormal, there exists a matrix function $K_\gamma \in \mathcal{E}(\Phi^{(\gamma)})$. Write

$$K_\gamma(z) = \sum_{i=0}^{\infty} z^i K_\gamma^{(i)}.$$

Since I_z and $C_1^{(0)}$, $D_1^{(0)}$ are coprime, I_z and C', D' are coprime. Thus, it follows from Theorem 6.5 (with $\theta = z$) that

$$\begin{pmatrix} D_1^{(0)} \\ \vdots \\ D_1^{(r-p-1)} \\ D_1^{(r-p)} + \gamma B_1 \end{pmatrix} = \begin{pmatrix} K_\gamma^{(0)} & 0 & \cdots & 0 \\ K_\gamma^{(1)} & K_\gamma^{(0)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_\gamma^{(r-p)} & K_\gamma^{(r-p-1)} & \cdots & K_\gamma^{(0)} \end{pmatrix} \begin{pmatrix} C_1^{(0)} \\ \vdots \\ C_1^{(r-p-1)} \\ C_1^{(r-p)} + \overline{\gamma} A_1(0) \end{pmatrix}.$$

Thus for each $\gamma \in \mathbb{C}$,

$$\gamma B_1 - \overline{\gamma} K_\gamma^{(0)} A_1(0) = \sum_{j=0}^{r-p-1} K_\gamma^{(r-p-j)} C_1^{(j)} + K_\gamma^{(0)} C_1^{(r-p)} - D_1^{(r-p)}. \tag{9.23}$$

If we put $\gamma := \zeta + i\,\zeta$ $(\zeta \in \mathbb{R})$, then (9.23) can be written as

$$\begin{aligned} &\zeta\,(B_1 - K_\gamma^{(0)} A_1(0)) + i\,\zeta\,(B_1 + K_\gamma^{(0)} A_1(0)) \\ &\qquad = \sum_{j=0}^{r-p-1} K_\gamma^{(r-p-j)} C_1^{(j)} + K_\gamma^{(0)} C_1^{(r-p)} - D_1^{(r-p)}. \end{aligned} \tag{9.24}$$

But since $||K_\gamma||_\infty \le 1$ (and consequently $\sup_\gamma ||\text{right-hand side of (9.24)}|| < \infty$), letting $\zeta \to \infty$ on both sides of (9.24) gives

$$B_1 = K_\gamma^{(0)} A_1(0) = -K_\gamma^{(0)} A_1(0)\,,$$

which implies that $B_1 = 0$. This contradicts the fact that $\det B_1 \neq 0$. This completes the proof. $\square$

THEOREM 9.17. *Let $\mathbf{T} := (T_\Phi, T_\Psi)$ be a pseudo-hyponormal Toeplitz pair with bounded type symbols $\Phi, \Psi \in L^\infty_{M_n}$. Suppose all inner parts of right coprime factorizations of $\Phi_\pm$ and $\Psi_\pm$ are diagonal-constant. If Φ and Ψ have a common tensored-scalar pole then the inner parts of Φ_- and Ψ_- coincide.*

PROOF. By assumption, write

$$\Phi_+ = \theta_0\theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta_2\theta_3 C^*, \quad \Psi_- = \theta_2 D^* \quad \text{(right coprime)},$$

where the θ_i are inner and $A, B, C, D \in H^2_{M_n}$. Suppose Φ and Ψ have a common tensored-scalar pole. We want to show that $\theta_0 = \theta_2$. By Theorem 4.40 and Lemmas 9.14 and 9.16, we may write, without loss of generality,

$$\theta_0 = z^p\delta_0, \quad \theta_1 = z^q\delta_1, \quad \theta_2 = z^p\delta_2, \quad \theta_3 = z^s\delta_3,$$

where $p \geq 1, q, s \geq 0$, $\delta_i(0) \neq 0$ for $i = 0, 1, 2, 3$. Assume to the contrary that $\theta_0 \neq \theta_2$. Then $\delta_0 \neq \delta_2$. Now if we combine Lemmas 4.18, 9.8, 9.12 9.15 and Proposition 4.19, then we can see that $\mathbf{T} := (T_{\Phi'}, T_{\Psi'})$ is pseudo-hyponormal, under the following coprime factorizations:

$$\Phi'_+ = z\omega A_1^*, \quad \Phi'_- = zB_1^*, \quad \Psi'_+ = z\omega^p C_1^*, \quad \Psi'_- = z\omega D_1^* \quad (p \geq 2),$$

where $\omega = b_\beta$ for some nonzero β. Let $\gamma \in \mathbb{C}$. By Lemma 9.9, $(T_{\Phi'-\gamma\Psi'}, T_{\Psi'})$ is pseudo-hyponormal. Observe that

$$(\Phi' - \gamma\Psi')_+ = z\omega^p\Big(\omega^{p-1}A_1 - \overline{\gamma}C_1\Big)^*.$$

On the other hand, we can choose $\gamma_0 \in \mathbb{C}$ such that $\big(\omega^{p-1}A_1 - \overline{\gamma_0}C_1\big)(0)$ is not invertible (in fact, $\overline{\gamma_0}$ is an eigenvalue of $\omega^{p-1}(0)C_1(0)^{-1}A_1(0)$). Then by Theorem 4.16, $\omega^{p-1}A_1 - \overline{\gamma_0}C_1$ and I_z are not coprime. Thus there exists a nonconstant inner divisor Ω_1 of I_z (say, $I_z = \Omega_1\Omega_2$) such that

$$(\Phi' - \gamma_0\Psi')_+ = \omega^p\Omega_2\Big(\Omega_1^*\big(\omega^{p-1}A_1 - \overline{\gamma_0}C_1\big)\Big)^*.$$

We thus have

$$\omega^p\Omega_2 H^2_{\mathbb{C}^n} \subseteq \ker H_{(\Phi'-\gamma_0\Psi')_+^*}.$$

Hence, by Corollary 9.11,

$$\omega^p\Omega_2 H^2_{\mathbb{C}^n} \subseteq \ker H_{(\Phi'-\gamma_0\Psi')_+^*} \subseteq \ker H_{(\Psi'_-)^*} = z\omega H^2_{\mathbb{C}^n},$$

which implies that I_z is an inner divisor of Ω_2, and hence Ω_1 is a constant unitary, a contradiction. This completes the proof. $\square$

COROLLARY 9.18. *Let $\mathbf{T} := (T_\Phi, T_\Psi)$ be a hyponormal Toeplitz pair with bounded type symbols $\Phi, \Psi \in L^\infty_{M_n}$. Suppose all inner parts of right coprime factorizations of $\Phi_\pm$ and $\Psi_\pm$ are diagonal-constant. If $\det \Phi_-^*$ and $\det \Psi_-^*$ have a common pole then the inner parts of Φ_- and Ψ_- coincide.*

PROOF. Write

$$\Phi_- = \theta_0 B^* \quad \text{and} \quad \Psi_- = \theta_2 D^* \quad \text{(right coprime)}.$$

Suppose $\det \Phi_-^*$ and $\det \Psi_-^*$ have a common pole. Since

$$\det \Phi_-^* = \frac{\det B}{\theta_0^n} \quad \text{and} \quad \det \Psi_-^* = \frac{\det D}{\theta_2^n},$$

there exists $\alpha \in \mathbb{D}$ such that $\theta_0(\alpha)^n = \theta_2(\alpha)^n = 0$, so that $\theta_0(\alpha) = \theta_2(\alpha) = 0$. Thus Φ_-^* and Ψ_-^* have a common tensored-scalar pole and hence it follows from Theorem 9.17. $\square$

Corollary 9.19. *Let $\mathbf{T} := (T_\Phi, T_\Psi)$ be a hyponormal Toeplitz pair with matrix-valued trigonometric polynomial symbols whose outer coefficients are invertible. Then*

$$\deg(\Phi_-) = \deg(\Psi_-).$$

Proof. Let $\Phi(z) = \sum_{j=-m}^{N} A_j z^j$ and $\Psi(z) = \sum_{j=-l}^{M} B_j z^j$. Then Φ and Ψ have a common tensored-scalar pole at $z = 0$. Thus by Theorem 9.16, we have $m = l$, i.e., $\deg(\Phi_-) = \deg(\Psi_-)$. $\square$

The following theorem gives a characterization of hyponormality for Toeplitz pairs with bounded type symbols.

Theorem 9.20. (Hyponormality of Toeplitz Pairs with Bounded Type Symbols) *Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ be a Toeplitz pair with bounded type symbols $\Phi, \Psi \in L^\infty_{M_n}$ of the form*

$$\Phi_+ = \theta_0\theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta_2\theta_3 C^*, \quad \Psi_- = \theta_2 D^* \quad \text{(coprime)}, \tag{9.25}$$

where the θ_i are inner functions. Assume that

(a) *θ_0 or θ_2 is a finite Blaschke product;*
(b) *θ_0 and θ_2 have a common Blaschke factor b_α such that $B(\alpha)$ and $D(\alpha)$ are diagonal-constant.*

If the pair $\mathbf{T}$ is hyponormal then

$$\Phi - \Lambda\Psi \in H^2_{M_n} \quad \textit{for some } \Lambda \in M_n. \tag{9.26}$$

Moreover, $\mathbf{T}$ is hyponormal if and only if

(i) *Φ and Ψ are normal and $\Phi\Psi = \Psi\Phi$;*
(ii) *$\Phi_- = \Lambda^*\Psi_-$;*
(iii) *$T_{\Psi^{1,\Omega}}$ is pseudo-hyponormal with $\Omega := \theta_0\theta_1\theta_3\overline{\theta}\Delta^*$,*

where $\theta :=$ g.c.d.(θ_1, θ_3) and $\Delta :=$ left-g.c.d. $\big(I_{\theta_0\theta},\ \overline{\theta}(\theta_3 A - \theta_1 C\Lambda^*)\big)$.

Proof. Suppose $\mathbf{T}$ is hyponormal. Then by Theorem 9.17, $\theta_0 = \theta_2$. In view of Lemma 9.13 we may assume that θ_1 and θ_3 are coprime. By our assumption, we note that $B(\alpha)$ and $D(\alpha)$ are invertible. Put $\Lambda := B(\alpha)D(\alpha)^{-1}$ (note that Λ is diagonal-constant by assumption). Then $(B - \Lambda D)(\alpha) = 0$, so that

$$B - \Lambda D = b_\alpha B_1 \quad \text{and} \quad \theta_0 = b_\alpha \delta_1 \quad (B_1 \in H^2_{M_n};\ \delta_1 \ \text{inner}). \tag{9.27}$$

We now claim that

$$\Phi_- = \Lambda^*\Psi_-. \tag{9.28}$$

Assume to the contrary that $\Phi_- \neq \Lambda^*\Psi_-$. Then it follows from (9.27) that

$$\begin{aligned}\Phi - \Lambda\Psi &= \theta_0\theta_1\theta_3(\overline{\theta_3}A^* - \overline{\theta_1}\Lambda C^*) + \overline{\theta_0}(B - \Lambda D)\\ &= \theta_0\theta_1\theta_3(\overline{\theta_3}A^* - \overline{\theta_1}\Lambda C^*) + \overline{\delta}_1 B_1.\end{aligned}$$

Suppose that $\Phi - \Lambda\Psi \notin H^2_{M_n}$. Then δ_1 is nonconstant. Since, by Lemma 9.9, $(T_{\Phi-\Lambda\Psi}, T_\Psi)$ is pseudo-hyponormal, applying Theorem 9.17 with $(T_{\Phi-\Lambda\Psi}, T_\Psi)$ in place of (T_Φ, T_Ψ) gives $\theta_0 = \delta_1$, which is a contradiction. This proves (9.28) and hence (9.26).

Towards the second assertion, let $\theta := \text{g.c.d.}(\theta_1, \theta_3)$. Then we can write

$$\theta_1 = \theta\omega_1 \text{ and } \theta_3 = \theta\omega_3 \quad \text{(for some coprime inner functions } \omega_1 \text{ and } \omega_3).$$

We thus have

$$\Phi_\Lambda := \Phi - \Lambda\Psi = \theta_0\theta\omega_1\omega_3(\overline{\omega_3}A^* - \overline{\omega_1}\Lambda C^*) \in H^2_{M_n}.$$

We claim that

$$\omega_3 A - \omega_1 C\Lambda^* \text{ and } \omega_1\omega_3 \text{ are coprime.} \tag{9.29}$$

Assume, to the contrary, that $\omega_3 A - \omega_1 C\Lambda^*$ and ω_1 are not coprime. Then there exists a nonconstant inner matrix function $\Upsilon \in H^\infty_{M_n}$ such that

$$\omega_3 A - \omega_1 C\Lambda^* = \Upsilon A' \quad \text{and} \quad I_{\omega_1} = \Upsilon\Omega'.$$

Thus we have

$$\omega_3 A = \Upsilon\Omega' C\Lambda^* + \Upsilon A' = \Upsilon(\Omega' C\Lambda^* + A'). \tag{9.30}$$

Since A and I_{θ_1} are coprime, it follows that A and Υ are coprime and hence $\widetilde{A}$ and $\widetilde{\Upsilon}$ are coprime. It thus follows from (9.30) that

$$\begin{aligned}\Upsilon^*\omega_3 A \in H^2_{M_n} &\Longrightarrow H_{\widetilde{A}\widetilde{\omega_3}\widetilde{\Upsilon}^*} = 0\\ &\Longrightarrow \widetilde{I_{\omega_3}} H^2_{\mathbb{C}^n} \subseteq \ker H_{\widetilde{A}\widetilde{\Upsilon}^*} = \widetilde{\Upsilon} H^2_{\mathbb{C}^n},\end{aligned}$$

which implies that $\widetilde{\Upsilon}$ is a inner divisor of $\widetilde{I_{\omega_3}}$, and hence Υ is an inner divisor of I_{ω_3}, so that Υ is a nonconstant common inner divisor of I_{ω_1} and I_{ω_3}, a contradiction. This proves (9.29). Let

$$\Delta := \text{left-g.c.d.}\left(I_{\theta_0\theta},\ \omega_3 A - \omega_1 C\Lambda^*\right).$$

Then we may write

$$\Phi_\Lambda = (\Delta^*\theta_0\theta\omega_1\omega_3)\Big[\Delta^*(\omega_3 A - \omega_1 C\Lambda^*)\Big]^* \quad \text{(left coprime).}$$

It follows from Corollary 4.8 (with $A := \Delta^*(\omega_3 A - \omega_1 C\Lambda^*)$, $\theta := \omega_1\omega_3$ and $B := \theta_0\theta\Delta^*$) that $\Delta^*(\omega_3 A - \omega_1 C\Lambda^*)$ and $\theta_0\theta\omega_1\omega_3\Delta^*$ are left coprime, and by Lemma 9.6, $T_{\Psi^1,\Omega}$ is pseudo-hyponormal, where $\Omega := \theta_0\theta\omega_1\omega_3\Delta^* = \theta_0\theta_1\theta_3\overline{\theta}\Delta^*$. The converse is obtained by reversing the above argument. This completes the proof. $\square$

Corollary 9.21. (Hyponormality of Rational Toeplitz Pairs) *Let* $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ *be a Toeplitz pair with rational symbols* $\Phi, \Psi \in L^\infty_{M_n}$ *of the form*

$$\Phi_+ = \theta_0\theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta_2\theta_3 C^*, \quad \Psi_- = \theta_2 D^* \quad \text{(coprime).} \tag{9.31}$$

Assume that θ_0 *and* θ_2 *are not coprime. Assume also that* $B(\gamma_0)$ *and* $D(\gamma_0)$ *are diagonal-constant for some* $\gamma_0 \in \mathcal{Z}(\theta_0)$. *If the pair* $\mathbf{T}$ *is hyponormal then*

$$\Phi - \Lambda\Psi \in H^2_{M_n} \quad \textit{for some } \Lambda \in M_n. \tag{9.32}$$

Moreover, the pair $\mathbf{T}$ *is hyponormal if and only if*

(i) Φ *and* Ψ *are normal and* $\Phi\Psi = \Psi\Phi$*;*
(ii) $\Phi_- = \Lambda^*\Psi_-$ *(with* $\Lambda := B(\gamma_0)D(\gamma_0)^{-1}$*);*
(iii) $T_{\Psi^1,\Omega}$ *is pseudo-hyponormal with* $\Omega := \theta_0\theta_1\theta_3\overline{\theta}\Delta^*$*,*

where $\theta := \text{g.c.d.}(\theta_1, \theta_3)$ *and* $\Delta := \text{left-g.c.d.}\left(I_{\theta_0\theta},\ \overline{\theta}(\theta_3 A - \theta_1 C\Lambda^*)\right)$.

Proof. This follows from Theorem 9.20. $\square$

For the following result, we recall the notion of outer coefficient, defined on page 9.

COROLLARY 9.22. (Hyponormality of Polynomial Toeplitz Pairs) *Let $\Phi, \Psi \in L^\infty_{M_n}$ be matrix-valued trigonometric polynomials of the form*

$$\Phi(z) := \sum_{j=-m}^{N} A_j z^j \quad \textit{and} \quad \Psi(z) := \sum_{j=-\ell}^{M} B_j z^j \tag{9.33}$$

satisfying

(i) *the outer coefficients $A_{-m}, A_N, B_{-\ell}$ and B_M are invertible;*
(ii) *the "co-analytic" outer coefficients A_{-m} and $B_{-\ell}$ are diagonal-constant.*

If $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ is hyponormal then

$$\Phi - \Lambda\Psi \in H^2_{M_n} \quad \textit{for some constant matrix } \Lambda \in M_n. \tag{9.34}$$

Remark. If Φ and Ψ are scalar-valued trigonometric polynomials then all assumptions of the above result are trivially satisfied, and hence the above result reduces to (9.1).

PROOF OF COROLLARY 9.22. This follows from Corollary 9.21 together with the following observations: If Φ and Ψ are matrix-valued trigonometric polynomials of the form (9.33), then under the notation of Corollary 9.21 we have

(i) $\theta_i = z^{m_i}$ for some $m_i \in \mathbb{N}$, and hence $\mathcal{Z}(\theta_i) = \{0\}$;
(ii) $A(0) = A_N$, $B(0) = A_{-m}$, $C(0) = B_M$ and $D(0) = B_{-\ell}$;
(iii) if A_N, A_{-m}, B_M and $B_{-\ell}$ are invertible then the representation (9.33) are coprime factorizations;
(iv) $B(0)$ and $D(0)$ are diagonal-constant.

□

We next consider the self-commutators of the Toeplitz pairs with matrix-valued rational symbols and derive rank formulae for their self-commutators.

Let $\Phi, \Psi \in L^\infty_{M_n}$ be matrix-valued rational functions of the form

$$\Phi_+ = \theta_0\theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta_2\theta_3 C^*, \quad \Psi_- = \theta_2 D^* \quad \text{(coprime)}. \tag{9.35}$$

By Corollary 9.18, if the pair (T_Φ, T_Ψ) is hyponormal and if θ_0 and θ_2 are not coprime then $\theta_0 = \theta_2$. The following question arises at once.

QUESTION 9.23. *Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ be hyponormal, where Φ and Ψ are given in (9.35). Does it follow that $\theta_1 = \theta_3$?*

The answer to Question 9.23 is negative even for scalar-valued symbols, as shown by the following example.

EXAMPLE 9.24. Let $\varphi = \overline{\varphi_-} + \varphi_+ \in L^\infty$ and $\psi = \overline{\psi_-} + \psi_+ \in L^\infty$ be of the form

$$\varphi_+ = 4z, \quad \varphi_- = z, \quad \psi_+ = 2zb, \quad \psi_- = z \quad \left(\text{where } b(z) := \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z}\right).$$

Under the notation of Corollary 9.21, $\varphi_- = \psi_-$ and we have

$$a\theta_3 - \overline{\lambda} c\theta_1 = 4b - 2.$$

Since $(4b-2)(0)=0$, we have $\delta = z$ and

$$P_{H_0^2}(\theta\delta\overline{\theta}_1 c) = P_{H_0^2}(z\cdot 2) = 2z.$$

Thus we have $\upsilon = \overline{z} + 2z$, and T_υ is hyponormal. Therefore by Corollary 9.21, $\mathbf{T} := (T_\phi, T_\psi)$ is hyponormal. Note that $\theta_1 = 1 \neq b = \theta_3$. □

However, if the symbols are matrix-valued rational function then the answer to Question 9.23 is indeed affirmative under some assumptions on the symbols.

Theorem 9.25. *Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ be a hyponormal Toeplitz pair with rational symbols $\Phi, \Psi \in L^\infty_{M_n}$ of the form*

$$\Phi_+ = \theta_0\theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta_2\theta_3 C^*, \quad \Psi_- = \theta_2 D^* \quad \text{(coprime)}. \tag{9.36}$$

Assume that

(i) $\mathcal{Z}(\theta_0) = \mathcal{Z}(\theta_1)$;
(ii) $\mathcal{Z}(\theta_0) \cap \mathcal{Z}(\theta_2) \neq \emptyset$;
(iii) $B(\gamma_0)$ *and* $D(\gamma_0)$ *are diagonal-constant for some* $\gamma_0 \in \mathcal{Z}(\theta_0)$.

Then $\theta_0 = \theta_2$ and $\theta_1 = \theta_3$.

Remark 9.26. Note that $\mathcal{Z}(\theta_0) \neq \mathcal{Z}(\theta_1)$ and $\mathcal{Z}(\theta_2) \neq \mathcal{Z}(\theta_3)$ in Example 9.24.

Proof of Theorem 9.25. By Corollary 9.18, we have $\theta_0 = \theta_2$. In view of Corollary 9.13, we may write

$$\Phi_+ = \theta_0\theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta_0\theta_3 C^*, \quad \Psi_- = \theta_0 D^* \quad \text{(coprime)},$$

where the θ_1 and θ_3 are coprime. By Corollary 9.21, it follows that $T_{\Psi^{1,\Omega}}$ is pseudo-hyponormal with $\Omega := \theta_0\theta_1\theta_3\Delta^*$, where $\Delta :=$ left-g.c.d. $\big\{I_{\theta_0},\ (\theta_3 A - \theta_1 C\Lambda^*)\big\}$ (with $\Lambda := B(\gamma_0)D(\gamma_0)^{-1}$). Thus we can write

$$I_{\theta_0} = \Delta\Theta_0' \quad \text{and} \quad \theta_3 A - \theta_1 C\Lambda^* = \Delta A_1 \quad (\Theta_0', A_1 \in H^\infty_{M_n}). \tag{9.37}$$

Suppose $\mathcal{Z}(\theta_0) = \mathcal{Z}(\theta_1)$. Then we can write

$$\theta_0 = \prod_{k=1}^N b_{\alpha_k}^{p_k} \quad \text{and} \quad \theta_1 = \prod_{k=1}^N b_{\alpha_k}^{q_k} \quad (\alpha_k \in \mathbb{D},\ p_k, q_k \in \mathbb{Z}_+)\,.$$

Clearly,

$$\delta \equiv \text{g.c.d.}(\theta_0, \theta_1) = \prod_{k=1}^N b_{\alpha_k}^{r_k} \quad \big(r_k := \min{(p_k, q_k)}\big).$$

Thus, we may write

$$\theta_0 = \delta\delta_0 \quad \text{and} \quad \theta_1 = \delta\delta_1 \quad (\delta_i \text{ is a finite Blaschke product for } i = 0, 1). \tag{9.38}$$

Since θ_1 and θ_3 are coprime, it follows from (3.1) that $\mathcal{Z}(\theta_1) \cap \mathcal{Z}(\theta_3) = \emptyset$, and hence $\mathcal{Z}(\theta_0) \cap \mathcal{Z}(\theta_3) = \emptyset$, so that θ_0 and θ_3 are coprime. It follows from (9.37) and (9.38) that

$$I_{\theta_0} = I_{\delta\delta_0} = \Delta\Theta_0' \quad \text{and} \quad \theta_3 A - \delta\delta_1 C\Lambda^* = \Delta A_1.$$

By Corollary 4.8 and Theorem 4.16, we can easily see that

$$I_\delta \text{ and } \Delta \text{ are coprime.} \tag{9.39}$$

By Theorem 4.16, $I_{\delta'} \equiv D(\Delta)$ and I_δ are coprime. Thus, δ' and δ are coprime, so that $\mathcal{Z}(\delta') \cap \mathcal{Z}(\delta) = \emptyset$, and hence $\mathcal{Z}(\delta') \cap \mathcal{Z}(\theta_0) = \emptyset$. Since δ' is a inner divisor of θ_0, $\mathcal{Z}(\delta') \subseteq \mathcal{Z}(\theta_0)$ and hence $\mathcal{Z}(\delta') = \emptyset$. Thus, δ' is a constant, so that Δ is a constant unitary. We now want to show that $\theta_3 = 1$. Assume to the contrary that $\theta_3 \neq 1$. Observe that

$$H_{(\Psi^{1,\Omega})_+^*} = H_{[P_{H_0^2(\Psi_+\Omega^*)}]^*} = H_{\Omega\Psi_+^*} = H_{\theta_1 C} = 0 \quad (\Omega := \theta_0\theta_1\theta_3)\,. \tag{9.40}$$

It thus follows that

$$\theta_0 H^2_{\mathbb{C}_n} = \ker H_{(\Psi^{1,\Omega})_-^*} \supseteq \ker H_{(\Psi^{1,\Omega})_+^*} = H^2_{\mathbb{C}^n},$$

a contradiction. Therefore we must have $\theta_3 = 1$. This completes the proof. □

Corollary 9.27. *Let $\Phi, \Psi \in L^\infty_{M_n}$ be matrix-valued trigonometric polynomials such that the outer coefficients of Φ and Ψ are invertible and the co-analytic outer coefficients of Φ and Ψ are diagonal-constant. If $\mathbf{T} := (T_\Phi, T_\Psi)$ is hyponormal then*

$$\deg(\Phi_+) = \deg(\Psi_+).$$

Proof. Immediate from Theorem 9.25. □

If the matrix-valued rational symbols Φ and Ψ have the same co-analytic and analytic degrees we get a general necessary condition for the hyponormality of the pair $\mathbf{T} := (T_\Phi, T_\Psi)$. This plays an important role in getting a rank formula for the self-commutator $[\mathbf{T}^*, \mathbf{T}]$.

Theorem 9.28. *Let $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ be a hyponormal Toeplitz pair with rational symbols $\Phi, \Psi \in L^\infty_{M_n}$ of the form*

$$\Phi_+ = \theta_0\theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta_2\theta_1 C^*, \quad \Psi_- = \theta_2 D^* \quad \text{(coprime)}. \tag{9.41}$$

Suppose

(i) $\mathcal{Z}(\theta_0) \cap \mathcal{Z}(\theta_2) \neq \emptyset$;

(ii) $B(\gamma_0)$ *and* $D(\gamma_0)$ *are diagonal-constant.*

Then $\Phi - \Lambda\Psi \in \mathcal{K}_{z\theta_1}$ for some $\Lambda \in M_n$.

Proof. By Corollary 9.18, we have $\theta_0 = \theta_2$. It follows from Corollary 9.21 that $T_{\Psi^{1,\Omega}}$ is pseudo-hyponormal with

$$\Omega = \theta_0\theta_1\Delta^* \quad (\text{with } \Delta := \text{left-g.c.d.}\,(I_{\theta_0},\, A_0 - C_0\Lambda^*))\,,$$

where $A_0 := P_{\mathcal{K}_{\theta_0}} A$ and $C_0 := P_{\mathcal{K}_{\theta_0}} C$. Then we can easily see that

$$\Delta = I_{\theta_0}. \tag{9.42}$$

Also since Δ is a left inner divisor of $A_0 - C_0\Lambda^*$, it follows that $A_0 - C_0\Lambda^* \in \theta_0 H^2_{M_n}$. By Lemma 4.4, $C_0\Lambda^* \in \mathcal{K}_{\theta_0}$ and hence $A_0 - C_0\Lambda^* \in \mathcal{K}_{\theta_0}$. Therefore,

$$A_0 - C_0\Lambda^* \in \theta_0 H^2_{M_n} \bigcap \mathcal{K}_{\theta_0} = \{0\},$$

which implies $A_0 = C_0\Lambda^*$. Put $A_1 := A - A_0$, and $C_1 := C - C_0$. Then we may write

$$\Phi_+ - \Lambda\Psi_+ = \theta_0\theta_1(A_0^* + A_1^*) - \theta_0\theta_1\Lambda(C_0^* + C_1^*) = \theta_1(A_2 - C_2\Lambda^*)^*,$$

(where $A_2 := \overline{\theta_0} A_1, C_2 := \overline{\theta_1} C_1 \in H^2_{M_n}$) which implies $\Phi_+ - \Lambda\Psi_+ \in \mathcal{K}_{z\theta_1}$ by Lemma 4.4. It thus follows from Corollary 9.21 that

$$\Phi - \Lambda\Psi = \Phi_-^* - \Lambda\Psi_-^* + \Phi_+ - \Lambda\Psi_+ = \Phi_+ - \Lambda\Psi_+ \in \mathcal{K}_{z\theta_1}\,,$$

which proves the theorem. □

We now obtain the rank formula for self-commutators.

THEOREM 9.29. *Let* $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ *be a hyponormal Toeplitz pair with rational symbols* $\Phi, \Psi \in L^\infty_{M_n}$ *of the form*

$$\Phi_+ = \theta_0\theta_1 A^*, \quad \Phi_- = \theta_0 B^*, \quad \Psi_+ = \theta_2\theta_1 C^*, \quad \Psi_- = \theta_2 D^* \quad \text{(coprime)}.$$

If θ_0 *and* θ_2 *are not coprime and* $B(\gamma_0)$ *and* $D(\gamma_0)$ *are diagonal-constant for some* $\gamma_0 \in \mathcal{Z}(\theta_0)$, *then*

$$\begin{aligned}[\mathbf{T}^*, \mathbf{T}] &= \mathcal{W}\left(\big(I - K(M)^*K(M)\big) \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)\mathcal{W}^* \bigoplus 0 \\ &= \left([T^*_\Phi, T_\Phi] \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) \bigoplus 0\end{aligned} \tag{9.43}$$

with

$$\mathcal{W} := \begin{pmatrix} (T_A)^*_{\theta_0\theta_1} W & 0 \\ 0 & (T_C)^*_{\theta_0\theta_1} W \end{pmatrix},$$

where $K \in \mathcal{C}(\Phi)$ *is a polynomial (cf. (6.3)), and* W *and* M *are given by* (6.4) *and* (6.5) *with* $\theta = \theta_0\theta_1$. *In particular,*

$$\operatorname{rank}\,[\mathbf{T}^*, \mathbf{T}] = \operatorname{rank}\,[T^*_\Phi, T_\Phi] = \operatorname{rank}\Big(I - K^*(M)K(M)\Big). \tag{9.44}$$

REMARK 9.30. We note that the formula (9.43) does not say that the hyponormality of T_Φ implies the hyponormality of the pair $\mathbf{T}$. Indeed, (9.43) is true for the selfcommutators only when the pair $\mathbf{T}$ is already hyponormal (via Corollary 9.21).

PROOF OF THEOREM 9.29. By Corollary 9.18, we have $\theta_0 = \theta_2$. Let $\Lambda \equiv B(\gamma_0)D(\gamma_0)^{-1}$. Since $\mathbf{T} \equiv (T_\Phi, T_\Psi)$ is hyponormal, it follows from Corollary 9.21 that $\Phi_- = \Lambda^*\Psi_-$. Since T_Φ and T_Ψ are hyponormal we can find functions $K \in \mathcal{C}(\Phi)$ and $K_1 \in \mathcal{C}(\Psi)$. We thus have $H_{\Phi^*_-} = H_{K\Phi^*_+}$ and $H_{\Psi^*_-} = H_{K_1\Psi^*_+}$. A straightforward calculation with Lemma 6.8 shows that

$$\begin{aligned}
&\Big(H^*_{\Phi^*_+}H_{\Psi^*_+} - H^*_{\Psi^*_-}H_{\Phi^*_-}\Big)|_{\mathcal{H}(I_\theta)} = (T_A)^*_\theta\, W\Big(I - (\Lambda^{-1}K)^*(M)(\Lambda K_1)(M)\Big)W^*(T_C)_\theta;\\
&\Big(H^*_{\Phi^*_+}H_{\Phi^*_+} - H^*_{\Phi^*_-}H_{\Phi^*_-}\Big)|_{\mathcal{H}(I_\theta)} = (T_A)^*_\theta\, W\Big(I - K^*(M)K(M)\Big)W^*(T_A)_\theta;\\
&\Big(H^*_{\Psi^*_+}H_{\Psi^*_+} - H^*_{\Psi^*_-}H_{\Psi^*_-}\Big)|_{\mathcal{H}(I_\theta)} = (T_C)^*_\theta\, W\Big(I - K_1^*(M)K_1(M)\Big)W^*(T_C)_\theta,
\end{aligned}$$

where W and M are given by (6.4) and (6.5) with $\theta = \theta_0\theta_1$ and I is the identity on $\mathbb{C}^{n\times d}$. But since $\operatorname{ran}\,[\mathbf{T}^*, \mathbf{T}] \subseteq \mathcal{H}(I_\theta) \oplus \mathcal{H}(I_\theta)$, we have

$$\begin{aligned}
[\mathbf{T}^*, \mathbf{T}] &= [\mathbf{T}^*, \mathbf{T}]\,|_{\mathcal{H}(I_\theta)\oplus\mathcal{H}(I_\theta)} \bigoplus 0\\
&= \begin{pmatrix} \Big(H^*_{\Phi^*_+}H_{\Phi^*_+} - H^*_{\Phi^*_-}H_{\Phi^*_-}\Big)|_{\mathcal{H}(I_\theta)} & \Big(H^*_{\Phi^*_+}H_{\Psi^*_+} - H^*_{\Psi^*_-}H_{\Phi^*_-}\Big)|_{\mathcal{H}(I_\theta)} \\ \Big(H^*_{\Psi^*_+}H_{\Phi^*_+} - H^*_{\Phi^*_-}H_{\Psi^*_-}\Big)|_{\mathcal{H}(I_\theta)} & \Big(H^*_{\Psi^*_+}H_{\Psi^*_+} - H^*_{\Psi^*_-}H_{\Psi^*_-}\Big)|_{\mathcal{H}(I_\theta)} \end{pmatrix} \bigoplus 0\\
&= \mathcal{W}\begin{pmatrix} I - K(M)^*K(M) & I - (\Lambda^{-1}K)^*(M)(\Lambda K_1)(M) \\ I - (\Lambda K_1)^*(M)(\Lambda^{-1}K)(M) & I - K_1^*(M)K_1(M) \end{pmatrix}\mathcal{W}^* \bigoplus 0,
\end{aligned} \tag{9.45}$$

where

$$\mathcal{W} := \begin{pmatrix} (T_A)^*_\theta W & 0 \\ 0 & (T_C)^*_\theta W \end{pmatrix}.$$

On the other hand, since $\mathbf{T}$ is hyponormal it follows from Theorem 9.28 that $\Phi - \Lambda\Psi \in \mathcal{K}_{z\theta_1}$. Thus we can see that

$$\mathcal{C}(\Phi^{1,\theta_1}) = \mathcal{C}(\Lambda\Psi^{1,\theta_1}) = \mathcal{C}((\Lambda\Psi)^{1,\theta_1}).$$

It thus follows from Proposition 6.1 that

$$\mathcal{C}(\Phi) = \mathcal{C}(\Lambda\Psi), \tag{9.46}$$

which implies that

$$\mathcal{C}(\Psi) = \{\Lambda^*\Lambda^{-1}K : K \in \mathcal{C}(\Phi)\}. \tag{9.47}$$

Thus we can choose $K_1 \in \mathcal{C}(\Psi)$ such that

$$K_1 = \Lambda^*\Lambda^{-1}K.$$

In what follows, we recall that K is a polynomial and $K(M)$ is defined by (6.6). Therefore, it follows from Lemma 6.9 that

$$\begin{aligned}(\Lambda^{-1}K)^*(M)(\Lambda K_1)(M) &= (\Lambda^{-1}K)^*(M)(\Lambda^*K)(M)\\ &= K^*(M)(\Lambda^{*-1}\otimes I)(\Lambda^*\otimes I)K(M)\\ &= K^*(M)K(M)\end{aligned} \tag{9.48}$$

and similarly,

$$(\Lambda K_1)^*(M)(\Lambda^{-1}K)(M) = K^*(M)K(M). \tag{9.49}$$

Also, by (9.47) and Lemma 6.9, we have

$$\begin{aligned}K_1^*(M)K_1(M) &= (\Lambda^*\Lambda^{-1}K)^*(M)(\Lambda^*\Lambda^{-1}K)(M)\\ &= K^*(M)((\Lambda^{*-1}\Lambda)\otimes I)(\Lambda^*\Lambda^{-1}\otimes I)K(M)\\ &= K^*(M)K(M).\end{aligned} \tag{9.50}$$

Thus, by (9.45), (9.48), (9.49), and (9.50), we have

$$\begin{aligned}[\mathbf{T}^*, \mathbf{T}] &= \mathcal{W}\begin{pmatrix} I-K(M)^*K(M) & I-K(M)^*K(M)\\ I-K(M)^*K(M) & I-K(M)^*K(M)\end{pmatrix}\mathcal{W}^*\bigoplus 0\\ &= \mathcal{W}\left(\big(I-K(M)^*K(M)\big)\otimes\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right)\mathcal{W}\bigoplus 0\\ &= \left([T_\Phi^*, T_\Phi]\otimes\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right)\bigoplus 0\end{aligned}$$

which proves (9.43).

On the other hand, since $A(\alpha)$ is invertible for each zero α of $\theta_0\theta_1$, it follows from Corollary 5.8 that $(T_A)_{\theta_0\theta_1}$ and $(T_C)_{\theta_0\theta_1}$ are invertible. Therefore $\mathcal{W}$ is invertible, so that the rank formula (9.44) is obtained from (9.43). ☐

On the other hand, Theorem 9.29 can be extended to Toeplitz m-tuples. To see this we observe:

Lemma 9.31. *Let $\Phi_i \in L^\infty_{M_n}$ $(i = 1, 2, \ldots, m)$ and let σ be a permutation on $\{1, 2, \ldots, m\}$. Then $\mathbf{T} := (T_{\Phi_1}, \ldots, T_{\Phi_m})$ is hyponormal if and only if $\mathbf{T}_\sigma := (T_{\Phi_{\sigma(1)}}, \ldots, T_{\Phi_{\sigma(m)}})$ is hyponormal. Furthermore, we have*

$$\operatorname{rank}\,[\mathbf{T}^*, \mathbf{T}] = \operatorname{rank}\,[\mathbf{T}_\sigma^*, \mathbf{T}_\sigma].$$

Proof. Obvious. ☐

Lemma 9.32. *Let* $\Phi_i \in L^\infty_{M_n}$ $(i = 1, 2, \dots, m)$. *Then the* m*-tuple* $\mathbf{T} := (T_{\Phi_1}, \dots, T_{\Phi_m})$ *is hyponormal if and only if every sub-tuple of* $\mathbf{T}$ *is hyponormal.*

Remark. A tuple $\mathbf{T}$ is considered to be its own sub-tuple.

Proof of Lemma 9.32. In view of Lemma 9.31, it suffices to show that if $\mathbf{T}$ is hyponormal then for every $m_0 \le m$, the sub-tuple $\mathbf{T}_{m_0} := (T_{\Phi_1}, \dots, T_{\Phi_{m_0}})$ is hyponormal. But this is obvious since

$$[\mathbf{T}^*, \mathbf{T}] = \begin{pmatrix} [\mathbf{T}^*_{\Phi_{m_0}}, \mathbf{T}_{\Phi_{m_0}}] & * \\ * & * \end{pmatrix}.$$

$\square$

Corollary 9.33. *For each* $i = 1, 2, \dots, m$, *suppose* $\Phi_i = (\Phi_i)^*_- + (\Phi_i)_+ \in L^\infty_{M_n}$ *is a matrix-valued rational function of the form*

$$(\Phi_i)_+ = \theta_i \delta A_i^* \quad \textit{and} \quad (\Phi_i)_- = \theta_i B_i^* \quad \text{(coprime)}.$$

Assume that there exists j_0 $(1 \le j_0 \le m)$ *such that* θ_{j_0} *and* θ_i *are not coprime for each* $i = 1, 2, \dots, m$. *If* $B_i(\gamma_0)$ *is diagonal-constant for some* $\gamma_0 \in \mathcal{Z}(\theta_i)$ *and for each* $i = 1, 2, \dots, m$, *then the following statements are equivalent:*

(a) *The* m*-tuple* $\mathbf{T} := (T_{\Phi_1}, T_{\Phi_2}, \cdots T_{\Phi_m})$ *is hyponormal;*
(b) *Every subpair of* $\mathbf{T}$ *is hyponormal.*

Moreover, if $\mathbf{T}$ *is hyponormal, then*

$$\begin{aligned}
[\mathbf{T}^*, \mathbf{T}]_p &= \left([T^*_{\Phi_1}, T_{\Phi_1}]_p \otimes \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \right) \bigoplus 0 \\
&= \mathcal{W}^* \left(\big(I - K(M)^* K(M)\big) \otimes \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \right) \mathcal{W} \bigoplus 0,
\end{aligned}$$

where $K \in \mathcal{C}(\Phi_1)$ *is a polynomial* (cf. (6.3)), W *and* M *are given by* (6.4) *and* (6.5) *with* $\theta = \theta_1 \delta$, *and* $\mathcal{W} := \operatorname{diag}\big((T_{A_i})^*_{\theta_i \delta} W\big)$. *In particular,*

$$\operatorname{rank} [\mathbf{T}^*, \mathbf{T}]_p = \operatorname{rank} [T^*_{\Phi_1}, T_{\Phi_1}]_p = \operatorname{rank} (I - K^*(M) K(M)).$$

Proof. (a) $\Rightarrow$ (b): Immediate from Lemma 9.32.

(b) $\Rightarrow$ (a): We follow the idea in [**CuL1**, Corollary 2.11]. Since every subpair $\mathbf{T}_{ij} = (T_{\Phi_i}, T_{\Phi_j})$ of $\mathbf{T}$ is hyponormal for each i, j, it follows from Theorem 9.29 that

$$[\mathbf{T}^*_{ij}, \mathbf{T}_{ij}]_p = \left([T^*_{\Phi_i}, T_{\Phi_i}] \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \bigoplus 0 \quad (1 \le i, j \le m).$$

Therefore

$$
\begin{aligned}
[\mathbf{T}^*, \mathbf{T}]_p &= \begin{pmatrix} [T^*_{\Phi_1}, T_{\Phi_1}]_p & [T^*_{\Phi_2}, T_{\Phi_1}]_p & \dots & [T^*_{\Phi_m}, T_{\Phi_1}]_p \\ [T^*_{\Phi_1}, T_{\Phi_2}]_p & [T^*_{\Phi_2}, T_{\Phi_2}]_p & \dots & [T^*_{\Phi_m}, T_{\Phi_2}]_p \\ \vdots & \vdots & \ddots & \vdots \\ [T^*_{\Phi_1}, T_{\Phi_m}]_p & [T^*_{\Phi_2}, T_{\Phi_m}]_p & \dots & [T^*_{\Phi_m}, T_{\Phi_m}]_p \end{pmatrix} \\
&= \left([T^*_{\Phi_1}, T_{\Phi_1}]_p \otimes \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \right) \bigoplus 0 \\
&= \mathcal{W}^* \left(\big(I - K(M)^* K(M)\big) \otimes \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \right) \mathcal{W} \bigoplus 0,
\end{aligned}
$$

where $\mathcal{W} := \mathrm{diag}\big((T_{A_i})^*_{\theta_i \delta} W\big)$. The rank formula is obvious. This completes the proof. $\square$

CHAPTER 10

Concluding remarks

In this paper we have tried to answer a number of questions involving matrix functions of bounded type; there are, however, still many questions that we were unable to answer. These questions have to do with whether properties involving matrix rational functions can be transmitted to the case of matrix functions of bounded type. Concretely, this means that if we know property X in the case when θ is a finite Blaschke product in the decomposition (2.6) of a matrix function Φ whose adjoint is of bounded type, then property X is still true for the cases of any inner function θ. Consequently, the main problem lies on the cases where θ is a singular inner function. Below we pose some questions involving matrix functions of bounded type.

1. Mutual singularity of two finite positive Borel measures. In Chapter 3, we have considered coprime singular inner functions, to understand well functions of bounded type; this is helpful when considering coprime factorizations for functions of bounded type. Here is a relevant question: Let μ and λ be finite positive Borel measures on $\mathbb{T}$. For $x \in \mathbb{T}$ and $r > 0$, write

$$B(x,r) := \Big\{xe^{i\theta} : 0 < |\theta| < r\Big\}.$$

For $x \in \operatorname{supp}(\lambda)$, let

$$Q_r(\mu/\lambda)(x) := \frac{\mu(B(x,r))}{\lambda(B(x,r))}.$$

Now we may define *a derivative* of μ with respect to λ at x to be

$$D(\mu)_\lambda(x) := \lim_{r\to 0} Q_r(\lambda/\mu)(x)$$

at those points $x \in \operatorname{supp}(\lambda)$ at which this limit exists. In this case, we may ask:

Problem 10.1. *Let μ and λ be finite positive Borel measures on $\mathbb{T}$. Does it follow that $\mu \perp \lambda$ if and only if $D(\mu)_\lambda(x) = 0$ a.e.$[\lambda]$?*

If λ is the Lebesgue measure on $\mathbb{T}$, then the answer to Problem 10.1 is affirmative (cf. [**Ru**, Theorem 7.14]).

We also have:

Problem 10.2. *Let $\theta_1, \theta_2 \in H^\infty$ be singular inner functions with singular measures μ_1 and μ_2, respectively. Are the following statements equivalent?*

(a) *θ_1 and θ_2 are not coprime;*

(b) *There exists* $x \in S \equiv \text{supp}\,(\mu_1) \cap \text{supp}\,(\mu_2)$ *and* $0 < m < 1$ *such that*

$$m \le \left\{ \frac{\mu_1(B(x,r))}{\mu_2(B(x,r))} : r \ne 0 \right\},$$

where $B(x,r) = \{xe^{i\theta} : 0 < |\theta| < r\}$*;*

(c) $D(\mu_1)_{\mu_2} \ne 0\ [\mu_2]$.

2. Coprime factorizations for compositions. In Theorem 4.40, we have shown that if $\Phi \in H^\infty_{M_n}$ is such that Φ^* is of bounded type of the form $\Phi = \Theta A^*$ (right coprime), then for a Blaschke factor θ, we have

$$\Phi \circ \theta = \big(\Theta \circ \theta\big)\big(A \circ \theta\big)^* \quad \text{(right coprime)}. \tag{10.1}$$

However we were unable to decide whether (10.1) holds for any inner function θ.

Problem 10.3. *Let* $\Theta \in H^\infty_{M_n}$ *be inner and* $A \in H^\infty_{M_n}$*. If* Θ *and* A *are right coprime, does it follow that* $\Theta \circ \theta$ *and* $A \circ \theta$ *are right coprime for any inner function* $\theta \in H^\infty$*?*

3. Invertibility of the compressions. Let $A \in H^2_{M_n}$ and $\Theta \equiv I_\theta \in H^\infty_{M_n}$ be a rational function, i.e., θ is a finite Blaschke product. In Corollary 5.8, we have shown that if A and Θ are coprime then $(T_A)_\Theta$, the compression of T_A to $\mathcal{H}(\Theta)$, is invertible. However we were unable to decide when $(T_A)_\Theta$ is invertible.

Problem 10.4. *Give a necessary and sufficient condition for* $(T_A)_\Theta$ *to be invertible if* $\Theta \equiv I_\theta$ *for an arbitrary inner function* θ*.*

4. An interpolation problem for matrix functions of bounded type. In Chapter 6, we have considered an interpolation problem: For a matrix function $\Phi \in L^\infty_{M_n}$, when does there exist a bounded analytic matrix function $K \in H^\infty_{M_n}$ satisfying $\Phi - K\Phi^* \in H^\infty_{M_n}$? In particular, we are interested in the cases when Φ and Φ^* are of bounded type. As we have discussed in Chapter 6, $\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}$ is a necessary condition for the existence of a solution. If Φ is a matrix-valued rational function then using the Kronecker Lemma we can show that this condition is also sufficient for the existence of a solution (cf. [**CHL2**, Proposition 3.9]). Moreover, in this case, the solution K is given by a polynomial via the classical Hermite-Fejér Interpolation Problem. However we were unable to determine whether the condition $\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}$ is sufficient for the existence of a solution when $\Phi \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type.

Problem 10.5. *Let* $\Phi \in L^\infty_{M_n}$ *be such that* Φ *and* Φ^* *are of bounded type. If* $\ker H_{\Phi_+^*} \subseteq \ker H_{\Phi_-^*}$*, does there exist a solution* $K \in H^\infty_{M_n}$ *satisfying* $\Phi - K\Phi^* \in H^\infty_{M_n}$*?*

5. Square-hyponormal Toeplitz operators. In Theorem 7.3, we have shown that if $\Phi \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type, and if Φ has a tensored-scalar singularity, then the subnormality and the normality of T_Φ coincide. Also in [**CHL2**, Theorem 4.5], it was shown that if $\Phi \in L^\infty_{M_n}$ is a matrix-valued rational function whose inner part of the coprime factorization of its co-analytic part is diagonal-constant, and if T_Φ and T_Φ^2 are hyponormal, then T_Φ is either normal or analytic. However we were unable to decide whether this result still holds for matrix-valued bounded type symbols.

PROBLEM 10.6. *Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type of the form*

$$\Phi_- = \theta B^* \quad \text{(coprime)},$$

where θ is an inner function in H^∞. If T_Φ and T_Φ^2 are hyponormal, does it follow that T_Φ is either normal or analytic?

Bibliography

[Ab] M.B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43** (1976), 597–604.

[AC] P.R. Ahern and D.N. Clark, *On functions orthogonal to invariant subspaces*, Acta Math. **124** (1970), 191–204.

[At] A. Athavale, *On joint hyponormality of operators*, Proc. Amer. Math. Soc. **103** (1988), 417–423.

[BGR] J.A. Ball, I. Gohberg, and L. Rodman, *Interpolation of Rational Matrix Functions*, Oper. Th. Adv. Appl. vol. 45, Birkhäuser, Basel, 1990.

[BR] J.A. Ball and M.W. Raney, *Discrete-time dichotomous well-posed linear systems and generalized Schur-Nevanlinna-Pick interpolation*, Complex Anal. Oper. Theory **1** (2007), no. 1, 1–54.

[Be] H. Bercovici, *Operator Theory and Arithmetic in H^∞*, Mathematical Surveys and Monographs, vol. 26, Amer. Math. Soc., Providence, 1988.

[BS] A. Böttcher and B. Silbermann, *Analysis of Toeplitz Operators*, Springer, Berlin-Heidelberg, 2006.

[Br] J. Bram, *Subnormal operators*, Duke Math. J. **22** (1955), 75–94.

[BH] A. Brown and P.R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102.

[Con] J.B. Conway, *The Theory of Subnormal Operators*, Mathematical Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.

[CS] J.B. Conway and W. Szymanski, *Linear combination of hyponormal operators*, Rocky Mountain J. Math. **18** (1988), 695–705.

[Co1] C. Cowen, *On equivalence of Toeplitz operators*, J. Operator Theory **7** (1982), 167–172.

[Co2] C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103** (1988), 809–812.

[CoL] C. Cowen and J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351** (1984), 216–220.

[Cu] R.E. Curto, *Joint hyponormality: A bridge between hyponormality and subnormality*, Operator Theory: Operator Algebras and Applications (Durham, NH, 1988) (W.B. Arveson and R.G. Douglas, eds.), Proc. Sympos. Pure Math., vol. 51, Part II, Amer. Math. Soc., Providence, 1990, 69–91.

[CHKL] R.E. Curto, I.S. Hwang, D. Kang and W.Y. Lee, *Subnormal and quasinormal Toeplitz operators with matrix-valued rational symbols*, Adv. Math. **255** (2014), 561–585.

[CHL1] R.E. Curto, I.S. Hwang and W.Y. Lee, *Which subnormal Toeplitz operators are either normal or analytic ?*, J. Funct. Anal. **263(8)** (2012), 2333-2354.

[CHL2] R.E. Curto, I.S. Hwang and W.Y. Lee, *Hyponormality and subnormality of block Toeplitz operators*, Adv. Math. **230** (2012), 2094-2151.

[CHL3] R.E. Curto, I.S. Hwang and W.Y. Lee, *Hyponormality of bounded-type Toeplitz operators*, Math. Nachr. **287** (2014), 1207–1222.

[CuL1] R.E. Curto and W.Y. Lee, *Joint hyponormality of Toeplitz pairs*, Memoirs Amer. Math. Soc. **712**, Amer. Math. Soc., Providence, 2001.

[CuL2] R.E. Curto and W.Y. Lee, *Towards a model theory for 2–hyponormal operators*, Integral Equations Operator Theory **44** (2002), 290–315.

[CMX] R.E. Curto, P.S. Muhly and J. Xia, *Hyponormal pairs of commuting operators*, Contributions to Operator Theory and Its Applications (Mesa, AZ, 1987) (I. Gohberg, J.W. Helton and L. Rodman, eds.), Oper. Th. Adv. Appl., vol. 35, Birkhäuser, Basel–Boston, 1988, 1–22.

[Do1] R.G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[Do2] R.G. Douglas, *Banach Algebra Techniques in the Theory of Toeplitz Operators*, CBMS vol. 15, Amer. Math. Soc., Providence, 1973.

[DPY] R.G. Douglas, V.I. Paulsen, and K. Yan, *Operator theory and algebraic geometry*, Bull. Amer. Math. Soc. (N.S.) **20** (1989), 67–71.

[DSS] R.G. Douglas, H. Shapiro, and A. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator*, Ann. Inst. Fourier(Grenoble) **20** (1970), 37–76.

[DY] R.G. Douglas and K. Yan, *A multi-variable Berger-Shaw theorem*, J. Operator Theory **27** (1992), 205–217.

[FL] D.R. Farenick and W.Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348** (1996), 4153–4174.

[FM] D.R. Farenick and R. McEachin, *Toeplitz operators hyponormal with the unilateral shift*, Integral Equations Operator Theory **22** (1995), 273–280.

[FF] C. Foiaş and A. Frazho, *The commutant lifting approach to interpolation problems*, Oper. Th. Adv. Appl. vol. 44, Birkhäuser, Boston, 1993.

[Fu] P. Fuhrmann, *Linear Systems and Operators in Hilbert Spaces*, McGraw-Hill, New York, 1981.

[Ga] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[GGK] I. Gohberg, S. Goldberg, and M.A. Kaashoek, *Classes of Linear Operators, Vol II*, Birkhäuser, Basel, 1993.

[Gu1] C. Gu, *A generalization of Cowen's characterization of hyponormal Toeplitz operators*, J. Funct. Anal. **124** (1994), 135–148.

[Gu2] C. Gu, *On a class of jointly hyponormal Toeplitz operators*, Trans. Amer. Math. Soc. **354** (2002), 3275–3298.

[GHR] C. Gu, J. Hendricks and D. Rutherford, *Hyponormality of block Toeplitz operators*, Pacific J. Math. **223** (2006), 95–111.

[GS] C. Gu and J.E. Shapiro, *Kernels of Hankel operators and hyponormality of Toeplitz operators*, Math. Ann. **319** (2001), 553–572.

[Hal1] P.R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.

[Hal2] P.R. Halmos, *Ten years in Hilbert space*, Integral Equations Operator Theory **2** (1979), 529–564.

[Har] R.E. Harte, *Invertibility and Singularity for Bounded Linear Operators*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 109, Marcel Dekker, New York, 1988.

[He] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[HKL1] I.S. Hwang, I.H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols*, Math. Ann. **313(2)** (1999), 247-261.

[HKL2] I.S. Hwang, I.H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols: An extremal case*, Math. Nachr. **231** (2001), 25-38.

[HL1] I.S. Hwang and W.Y. Lee, *Hyponormality of trigonometric Toeplitz operators*, Trans. Amer. Math. Soc. **354** (2002), 2461-2474.

[HL2] I.S. Hwang and W.Y. Lee, *Hyponormality of Toeplitz operators with rational symbols*, Math. Ann. **335** (2006), 405–414.

[HL3] I.S. Hwang and W.Y. Lee, *Hyponormal Toeplitz operators with rational symbols*, J. Operator Theory **56** (2006), 47–58.

[HL4] I.S. Hwang and W.Y. Lee, *Joint hyponormality of rational Toeplitz pairs*, Integral Equations Operator Theory **65** (2009), 387–403; Erratum **69**(2011), 445–446.

[Le] W.Y. Lee, *Cowen sets for Toeplitz operators with finite rank selfcommutators*, J. Operator Theory **54(2)** (2005), 301-307.

[MAR] R.A. Martínez-Avendaño and P. Rosenthal, *An Introduction to Operators on the Hardy-Hilbert Space*, Springer, New York, 2007.

[McCP1] S. McCullough and V. Paulsen, *A note on joint hyponormality*, Proc. Amer. Math. Soc. **107** (1989), 187–195.

[McCP2] S. McCullough and V. Paulsen, *k-hyponormality of weighted shifts*, Proc. Amer. Math. Soc. **116** (1992), 165–169.

[Mo] B.B. Morrel, *A decomposition for some operators*, Indiana Univ. Math. J. **23** (1973), 497-511.

[NT] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338** (1993), 753–769.

[Ni1] N.K. Nikolski, *Treatise on the Shift Operator*, Springer, New York, 1986.

[Ni2] N.K. Nikolski, *Operators, Functions, and Systems: An Easy Reading Volume I: Hardy, Hankel, and Toeplitz*, Mathematical Surveys and Monographs, vol. 92, Amer. Math. Soc., Providence, 2002.

[Pe] V.V. Peller, *Hankel Operators and Their Applications*, Springer, New York, 2003.

[Po] V.P. Potapov, *On the multiplicative structure of J-nonexpansive matrix functions*, Tr. Mosk. Mat. Obs. (1955), 125-236 (in Russian); English trasl. in: Amer. Math. Soc. Transl. (2) 15 (1966), 131-243.

[Ru] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1987.

[SFBK] B. Sz.-Nagy, C. Foiaş, H. Berkovici, and L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, Springer, New York, 2010.

[Ta] S. Takenaka, *On the orthogonal functions and a new formula of interpolation*, Japan J. Math. **2** (1925), 129–145.

[Xi] D. Xia, *On the semi-hyponormal n-tuple of operators*, Integral Equations Operator Theory **6** (1983), 879–898.

[Zhu] K. Zhu, *Hyponormal Toeplitz operators with polynomial symbols*, Integral Equations Operator Theory **21** (1996), 376–381.

List of Symbols