Normal singular Cauchy integral operators with operator-valued symbols

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Abstract

In this paper we characterize a class of normal (isometric, coisometric, unitary, hyponormal) singular Cauchy integral operators with operator-valued (or matrix-valued) symbols on vector-valued (or $\mathbb{C}^n$-valued) L^2 spaces.

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1 Introduction

Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$ and let m denote the normalized Lebesgue measure on $\mathbb{T}$. Let $L^p = L^p(\mathbb{T})$ be the usual Lebesgue space on $\mathbb{T}$ and $H^p = H^p(\mathbb{T})$ denotes the usual Hardy space on $\mathbb{T}$, $1 \le p \le \infty$. Consider a singular integral operator S acting on L^2 with the Cauchy kernel, defined by

$$Sf(\zeta) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{f(z)}{z - \zeta} dz,$$

where the integral is understood in the sense of Cauchy's principal value of $f \in L^2$. If $f \in L^1$, then $Sf(\zeta)$ exists a.e. on $\mathbb{T}$ and $Sf(\zeta)$ is a measurable function. We define $P := (I + S)/2$ and $Q := (I - S)/2$, where I denotes the identity operator. Then $(P - Q)f(\zeta) = Sf(\zeta) = i\widetilde{f}(\zeta) + \int_{\mathbb{T}} f dm$, where

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$\widetilde{f}$ denotes the harmonic conjugate function of f with $\widetilde{f}(0)=0$. For functions $\varphi, \psi \in L^\infty$, the operator $S_{\varphi,\psi}$ on L^2 defined by

$$S_{\varphi,\psi} = \varphi P + \psi Q = \frac{\varphi - \psi}{2} S + \frac{\varphi + \psi}{2} I$$

is called a *singular Cauchy integral operator* on L^2. This operator has been studied by many authors. Very recently, Nakazi and Yamamoto [17] have considered the normality of $S_{\varphi,\psi}$, in the viewpoint of the Brown-Halmos theorem for normal Toeplitz operators on the Hardy space H^2 of the unit circle. In this paper we characterize classes of normal, isometric, unitary, hyponormal singular Cauchy integral operators with operator-valued symbols on vector-valued L^2 spaces. This work was inspired by the recent results of Nakazi and Yamamoto [17] on scalar-valued L^2 spaces. Even though our results and their proofs are much more complicated than the scalar case due to the noncommutativity of the symbols, our approach does give simple proofs of some results of Nakazi and Yamamoto. Our proofs use the recent work on normal Toeplitz operators with operator-valued symbols in [12] and also the work on commuting abstract Toeplitz and Hankel operators in [10].

Let E be a complex Hilbert space and $B(E)$ be the algebra of all bounded linear operators on E. Let $L^2_E = L^2_E(\mathbb{T})$ be the Hilbert space of E-valued norm square-integrable measurable functions on $\mathbb{T}$ and let $L^\infty_{B(E)}$ be the space of all bounded $B(E)$-valued functions with respect to operator supremum norm. For an operator-valued function $\Phi \in L^\infty_{B(E)}$, the operator M_Φ denotes the multiplication operator by Φ on L^2_E. For $\Phi, \Psi \in L^\infty_{B(E)}$, the singular integral operator $S_{\Phi,\Psi} : L^2_E \to L^2_E$ is defined by

$$S_{\Phi,\Psi}(f) = \Phi P f + \Psi Q f \quad (f \in L^2_E),$$

where P is the projection from L^2_E onto H^2_E and $Q = I - P$ is the projection onto $\left(H^2_E\right)^\perp = L^2_E \ominus H^2_E = \overline{zH^2_E}$. Throughout the paper, I will denote the identity operator on different Hilbert spaces. To emphasize, we sometimes use I_E to denote the identity operator on the space E. Noting that $S_{\Phi,\Psi} = M_\Psi + S_{\Phi-\Psi,0}$, we can think of $S_{\Phi,\Psi}$ is a perturbation of M_Ψ which is normal if and only if the symbol function Ψ is normal ($\Psi^*\Psi = \Psi\Psi^*$). Note that for $f, g \in L^2_E$,

$$\langle S_{\Phi,\Psi}(f), g\rangle = \langle \Phi P f + \Psi Q f, g\rangle = \langle f, P[\Phi^* g]\rangle + \langle f, Q[\Psi^* g]\rangle .$$

Thus the adjoint $S^*_{\Phi,\Psi}$ is given by the formula

$$S^*_{\Phi,\Psi} g = P[\Phi^* g] + Q[\Psi^* g], g \in L^2_E. \tag{1}$$

Let J denote the unitary operator on L^2_E given by

$$(Jf)(z) = \overline{z} f(\overline{z}) \text{ for } f \in L^2_E.$$

Note that J maps H^2_E onto $\overline{zH^2_E}$, $J^2 = I$ and $J^* = J$ on L^2_E. We can also define J on $L^\infty_{B(E)}$ similarly,

$$(J\Phi)(z) = \overline{z}\Phi(\overline{z}), \Phi \in L^\infty_{B(E)}.$$

A Toeplitz operator T_Φ and a Hankel operator H_Φ on H^2_E are defined by

$$T_\Phi f = P(\Phi f) \text{ and } H_\Phi f = JQ(\Phi f) = PJ(\Phi f),\ f \in H^2_E.$$

Toeplitz operators T_Φ and Hankel operators H_Φ are characterized by the operator equations $S^* T_\Phi S = T_\Phi$ and $H_\Phi S = S^* H_\Phi$, respectively, where $S = T_{zI_E}$ is the unilateral shift on H^2_E with multiplicity $\dim E$. See [10] for an abstract approach to Toeplitz operators and Hankel operators where S is replaced by an arbitrary isometry. For $\Phi \in L^\infty_{B(E)}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}) = (\Phi(\overline{z}))^* = \overline{z}(J\Phi)^*.$$

One can easily see that $\widetilde{\widetilde{\Phi}}(z) = \Phi(z)$.

The following lemma lists basic identities for Toeplitz and Hankel operators which will be used throughout the paper, often without referring back to this lemma.

Lemma 1.1 *The following hold.*

$$\begin{aligned}
PJ &= JQ;\\
J[\Phi f] &= z(J\Phi)(Jf) \textit{ for } \Phi \in L^\infty_{B(E)} \textit{ and } f \in L^2_E;\\
T^*_\Phi &= T_{\Phi^*}, H^*_\Phi = H_{\widetilde{\Phi}};\\
T_{\Phi\Psi} - T_\Phi T_\Psi &= H^*_{\Phi^*} H_\Psi;\\
T^*_\Phi T_\Phi - T_\Phi T^*_\Phi &= T_{\Phi^*\Phi - \Phi\Phi^*} + H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi;\\
H_\Phi T_\Psi &= H_{\Phi\Psi}, T^*_{\widetilde{\Psi}} H_\Phi = H_{\Psi\Phi} \textit{ for } \Psi \in H^\infty_{B(E)}.
\end{aligned}$$

The following lemma provides a direct connection of singular integral operator $S_{\Phi,\Psi}$ with Hankel and Toeplitz operators.

Lemma 1.2 *With respect to the decomposition $L^2_E = H^2_E \oplus \overline{zH^2_E}$, we have*

$$\begin{aligned}
S_{\Phi,\Psi} &= \begin{bmatrix} T_\Phi & H^*_{\Psi^*} J \\ JH_\Phi & JT_{z(J\Psi)} J \end{bmatrix} = \begin{bmatrix} T_\Phi & H_{\widetilde{\Psi}^*} J \\ JH_\Phi & JT^*_{\widetilde{\Psi}} J \end{bmatrix}\\
&= \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} T_\Phi & H^*_{\Psi^*} \\ H_\Phi & T_{z(J\Psi)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}.
\end{aligned}$$

Proof. We will justify the $(2,2)$-entry in the first block matrix form. Let $g_1, g_2 \in \overline{zH^2_E}$. Then

$$\begin{aligned}
\langle (QS_{\Phi,\Psi})\, g_1, g_2 \rangle &= \langle (S_{\Phi,\Psi})\, g_1, g_2 \rangle = \langle \Phi P g_1 + \Psi Q g_1, g_2 \rangle\\
&= \langle \Psi g_1, g_2 \rangle = \langle J(\Psi Q g_1), J g_2 \rangle\\
&= \langle z(J\Psi)(Jg_1), Jg_2 \rangle,
\end{aligned}$$

where the last equality follows from Lemma 1.1. On the other hand,

$$\begin{aligned}
\left\langle \left(JT_{\overline{z}(J\Psi)} J\right) g_1, g_2 \right\rangle &= \left\langle \left(T_{z(J\Psi)} J\right) g_1, Jg_2 \right\rangle = \langle P[z(J\Psi)\, Jg_1], Jg_2 \rangle\\
&= \langle z(J\Psi)\, Jg_1, Jg_2 \rangle,
\end{aligned}$$

where the last equality follows from the fact $P(Jg_2) = Jg_2$. The second block matrix form follows by noting that

$$z(J\Psi) = \left[\widetilde{\Phi}(z)\right]^*.$$

The proof is complete. ■

The paper is organized as follows. In Section 2 we give characterizations of the self-adjoint-ness and the positivity of singular integral operators $S_{\Phi,\Psi}$. Section 3 is devoted to a characterization of the normality of $S_{\Phi,\Psi}$ with operator-valued symbols and Section 4 is devoted to characterizations of the normality, the isometry and the coisometry of $S_{\Phi,\Psi}$ with matrix-valued symbols. In Section 5, we consider the hyponormality of $S_{\Phi,\Psi}$ with matrix-valued symbols.

2 Self-Adjoint $S_{\Phi,\Psi}$

The characterizations of self-adjoint $S_{\Phi,\Psi}$ and positive $S_{\Phi,\Psi}$ are generalization of the results in the scalar-valued case in Section 2 of [17]. The proofs appear similar.

Theorem 2.1 *Let $\Phi, \Psi \in L^\infty_{B(E)}$. Then the followings are equivalent.*

(a) *$S_{\Phi,\Psi}$ is self-adjoint on L^2_E.*

(b) *Both Φ and Ψ are self-adjoint operator-valued functions such that $\Phi - \Psi$ is a (constant) self-adjoint operator in $B(E)$.*

(c) *There exist $F \in zH^\infty_{B(E)}$ and self-adjoint operators Φ_0 and Ψ_0 in $B(E)$ such that $\Phi = F + F^* + \Phi_0, \Psi = F + F^* + \Psi_0$.*

(d) *$S_{\Phi,\Psi} = M_\Psi + S_{G_0,0}$, where M_Ψ is self-adjoint and G_0 is a (constant) self-adjoint operator in $B(E)$.*

Proof. We first prove the equivalence of (a) and (b). By Lemma 1.2,

$$S_{\Phi,\Psi} = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} T_\Phi & H^*_{\Psi^*} \\ H_\Phi & T_{z(J\Psi)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix};$$
$$S^*_{\Phi,\Psi} = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} T^*_\Phi & H^*_\Phi \\ H_{\Psi^*} & T^*_{z(J\Psi)} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}.$$

Therefore $S_{\Phi,\Psi}$ is self-adjoint if and only if

$$T_\Phi = T^*_\Phi, T_{z(J\Psi)} = T^*_{z(J\Psi)}; \tag{2}$$
$$H_\Phi = H_{\Psi^*}. \tag{3}$$

Equation (2) holds if and only if both Φ and Ψ are self-adjoint operator-valued functions. Equivalently, $\Phi = F + F^* + \Phi_0$ and $\Psi = G + G^* + \Psi_0$ for some $F, G \in zH^\infty_{B(E)}$ and two self-adjoint operators Φ_0 and Ψ_0 in $B(E)$. Equation (3) holds if and only $\Phi - \Psi^* \in H^\infty_{B(E)}$. But $\Phi - \Psi^* = \Phi - \Psi$ is self-adjoint, so $\Phi - \Psi = \Phi_0 - \Psi_0$ and $F = G$. The proofs of other equivalences are similar. ■

Theorem 2.2 *Let $\Phi, \Psi \in L^\infty_{B(E)}$. Then the followings hold:*

(a) *If $S_{\Phi,\Psi}$ is a positive operator on L^2_E, then for each $\theta \in [0, 2\pi)$, both $\Phi(e^{i\theta})$ and $\Psi(e^{i\theta})$ are positive operators in $B(E)$ and $\Phi - \Psi$ is a (constant) self-adjoint operator in $B(E)$.*

(b) *If for each $\theta \in [0, 2\pi)$, both $\Phi(e^{i\theta})$ and $\Psi(e^{i\theta})$ are positive operator in $B(E)$ and $\Phi - \Psi = G_0$, where G_0 is either a positive or negative operator in $B(E)$, then $S_{\Phi,\Psi}$ is a positive operator on L^2_E.*

Proof. We generalize the proof of Theorem 2.2 in [17] for the scalar-valued case to the operator-valued case. For part (b), the additional condition on G_0 is automatically satisfied in the scalar case. The proof of part (b) is made possibly by this assumption.

We first prove (a). Let $f \in H^2$ be a scalar function and $e \in E$. Then $f(z)e \in H^2_E$, and hence

$$\begin{aligned}\langle S_{\Phi,\Psi} f(z)e, f(z)e\rangle_{L^2_E} &= \langle \Phi P[f(z)e] + \Psi Q[f(z)e], f(z)e\rangle_{L^2_E} \\ &= \langle \Phi(z) f(z)e, f(z)e\rangle_{L^2_E} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle \Phi(e^{i\theta}) f(e^{i\theta})e, f(e^{i\theta})e\right\rangle_E d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle \Phi(e^{i\theta})e, e\right\rangle_E \left|f(e^{i\theta})\right|^2 d\theta \geq 0.\end{aligned}$$

Since $f \in H^2$ is arbitrary, it follows that $\left\langle \Phi(e^{i\theta})e, e\right\rangle_E \geq 0$ for any $e \in E$ and $\Phi(e^{i\theta})$ is a positive operator for each $\theta \in [0, 2\pi)$. Similarly, let $g \in \overline{zH^2}$ be a scalar-valued function and $e \in E$. Then $g(z)e \in \overline{zH^2_E}$ and

$$\begin{aligned}\langle S_{\Phi,\Psi} g(z)e, g(z)e\rangle_{L^2_E} &= \langle \Psi(z) g(z)e, g(z)e\rangle_{L^2_E} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle \Psi(e^{i\theta})e, e\right\rangle_E g(e^{i\theta})\overline{g(e^{i\theta})} d\theta \geq 0.\end{aligned}$$

Therefore $\left\langle \Psi(e^{i\theta})e, e\right\rangle_E \geq 0$ for any $e \in E$ and $\Psi(e^{i\theta})$ is a positive operator for each $\theta \in [0, 2\pi)$.

To prove (b), write $\Phi - \Psi = G_0$ for some self-adjoint operator $G_0 \in B(E)$. Let $h \in L^2_E$ be fixed. Write $h = f + g$ ($f = Pf,\ g = Qf$). Since G_0 is a constant operator, $G_0 g \in \overline{zH^2}$ and $\langle G_0 g, f\rangle_{L^2_E} = \langle G_0 f, g\rangle_{L^2_E} = 0$. If G_0 is a negative operator, then $\langle G_0 g, g\rangle_{L^2_E} \leq 0$ and

$$\begin{aligned}\langle S_{\Phi,\Psi} h, h\rangle_{L^2_E} &= \langle S_{\Phi,\Psi}(f+g), (f+g)\rangle_{L^2_E} = \langle \Phi f + (\Phi - G_0) g, f+g\rangle_{L^2_E} \\ &= \langle \Phi(f+g), f+g\rangle_{L^2_E} - \langle G_0 g, f+g\rangle_{L^2_E} \\ &= \langle \Phi(f+g), f+g\rangle_{L^2_E} - \langle G_0 g, g\rangle_{L^2_E} \geq \langle \Phi(f+g), f+g\rangle_{L^2_E} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\langle \Phi(e^{i\theta}) h(e^{i\theta}), h(e^{i\theta})\right\rangle_E d\theta \geq 0\end{aligned}$$

since for each $\theta, \Phi(e^{i\theta})$ is a positive operator. If instead G_0 is a positive operator, then similarly, $\langle S_{\Phi,\Psi}h, h\rangle_{L^2_E} \geq 0$. ■

3 Normal $S_{\Phi,\Psi}$ with operator-valued symbols

In this section we consider when $S_{\Phi,\Psi}$ with operator-valued symbols is normal. Here we also characterize when $S_{\Phi,\Psi}$ is an isometry, a coisometry or a unitary operator. Set

$$U = \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}.$$

Lemma 3.1 *Let $\Phi, \Psi \in L^\infty_{B(E)}$ and let $\Omega := \Phi - \Psi$. Then $S_{\Phi,\Psi}$ is normal if and only if the following three conditions are satisfied:*

$$T_{\Phi^*\Phi-\Phi\Phi^*} + H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Psi^*}H_{\Psi^*} = 0; \tag{4}$$

$$T_{\widetilde{\Psi}\widetilde{\Psi}^*-\widetilde{\Psi}^*\widetilde{\Psi}} + H^*_{\widetilde{\Psi}}H_{\widetilde{\Psi}} - H^*_{\widetilde{\Phi}}H_{\widetilde{\Phi}} = 0; \tag{5}$$

$$H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*}H_{\Psi^*} - H_\Phi T_{\Omega^*} = 0. \tag{6}$$

Proof. By Lemma 1.2, we have

$$\begin{aligned} US^*_{\Phi,\Psi}S_{\Phi,\Psi}U &= \begin{bmatrix} T^*_\Phi & H^*_\Phi \\ H_{\Psi^*} & T^*_{z(J\Psi)} \end{bmatrix} \begin{bmatrix} T_\Phi & H^*_{\Psi^*} \\ H_\Phi & T_{z(J\Psi)} \end{bmatrix} \\ &= \begin{bmatrix} T^*_\Phi T_\Phi + H^*_\Phi H_\Phi & T^*_\Phi H^*_{\Psi^*} + H^*_\Phi T_{z(J\Psi)} \\ H_{\Psi^*}T_\Phi + T^*_{z(J\Psi)}H_\Phi & H_{\Psi^*}H^*_{\Psi^*} + T^*_{z(J\Psi)}T_{z(J\Psi)} \end{bmatrix}; \end{aligned} \tag{7}$$

$$US_{\Phi,\Psi}S^*_{\Phi,\Psi}U = \begin{bmatrix} T_\Phi T^*_\Phi + H^*_{\Psi^*}H_{\Psi^*} & T_\Phi H^*_\Phi + H^*_{\Psi^*}T_{z(J\Psi)} \\ H_\Phi T^*_\Phi + T_{z(J\Psi)}H_{\Psi^*} & H_\Phi H^*_\Phi + T_{z(J\Psi)}T^*_{z(J\Psi)} \end{bmatrix}. \tag{8}$$

Then a direct calculation together with (7), (8), and Lemma 1.1 gives

$$\begin{aligned} &U\left[S^*_{\Phi,\Psi}S_{\Phi,\Psi} - S_{\Phi,\Psi}S^*_{\Phi,\Psi}\right]U \\ &= \begin{bmatrix} T_{\Phi^*\Phi-\Phi\Phi^*} + H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Psi^*}H_{\Psi^*} & H^*_{\Psi^*\Phi-\Phi\Psi^*} + H^*_{\Psi^*}T_{\widetilde{\Omega}} - T_\Omega H^*_\Phi \\ H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*}H_{\Psi^*} - H_\Phi T_{\Omega^*} & T_{\widetilde{\Psi}\widetilde{\Psi}^*-\widetilde{\Psi}^*\widetilde{\Psi}} + H^*_{\widetilde{\Psi}}H_{\widetilde{\Psi}} - H^*_{\widetilde{\Phi}}H_{\widetilde{\Phi}} \end{bmatrix}, \end{aligned}$$

giving the result. ■

The first two equations (4) and (5) of Lemma 3.1 are of the same nature and we can use the techniques in [12] to study them.

Lemma 3.2 *Let $\Phi, \Psi \in L^\infty_{B(E)}$. Then $T_{\Phi^*\Phi-\Phi\Phi^*} + H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Psi^*}H_{\Psi^*} = 0$ if and only if $T_{\Phi^*\Phi-\Phi\Phi^*} = 0$ and $H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Psi^*}H_{\Psi^*} = 0$.*

Proof. Note that for $m \geq 1$,

$$\begin{aligned} 0 &= S^{*m}(T_{\Phi^*\Phi-\Phi\Phi^*} + H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Psi^*}H_{\Psi^*})S^m \\ &= S^{*m}T_{\Phi^*\Phi-\Phi\Phi^*}S^m + S^{*m}H^*_{\Phi^*}H_{\Phi^*}S^{*m} - S^{*m}H^*_{\Psi^*}H_{\Psi^*}S^m \\ &= T_{\Phi^*\Phi-\Phi\Phi^*} + H^*_{\Phi^*}S^mS^{*m}H_{\Phi^*} - H^*_{\Psi^*}S^mS^{*m}H_{\Psi^*}. \end{aligned}$$

Since $S^m S^{*m}$ is the projection on the space $z^m H_E^2$, $S^m S^{*m} \to 0$ strongly as m tends to ∞. Thus both $S^{*m} H_{\Phi^*}^* H_{\Phi^*} S^m$ and $S^{*m} H_\Phi^* H_\Phi S^m$ strongly converge to 0. Therefore $T_{\Phi^*\Phi - \Phi\Phi^*} = 0$, which in turn implies $H_{\Phi^*}^* H_{\Phi^*} - H_{\Psi^*}^* H_{\Psi^*} = 0$ as well. ■

It is known ([12]) that an operator-valued L^∞-function $\Phi \in L^\infty_{B(E)}$ can be decomposed as

$$\Phi(z) = \Phi_+(z) + \Phi_0 + \Phi_-^*(z),$$

where $\Phi_+, \Phi_- \in zH^2_{B(E)}$ and $\Phi_0 \in B(E)$.

The following theorem is essentially one of the main theorems in [12], see Theorem 11 in [12]. Here we state it in terms of Hankel operators. The result stated here is in fact in a slightly more general form in the sense that the functions Φ_+ and Ψ_+ come from two different $L^\infty_{B(E)}$ functions Φ and Ψ while in [12] two functions Φ_+ and Φ_- come from one $L^\infty_{B(E)}$ function Φ.

Theorem 3.3 [12] *Let $\Phi, \Psi \in L^\infty_{B(E)}$. Then $H_{\Phi^*}^* H_{\Phi^*} = H_{\Psi^*}^* H_{\Psi^*}$ if and only if there exist a coisometry W such that $\Phi_+ = \Psi_+ W$ or $\Psi_+ = \Phi_+ W$.*

It turns out that operator equations similar to equation (6) were studied in [10]. Since the approach in [10] is quite abstract, we provide a self-contained treatment here. See Lemma 1, Theorem 1, Theorem 4 and Proposition 3 in [10] for related results. Let $P_0 = (I - SS^*)$ denote the projection of L_E^2 onto E which is comprised of all constant elements in L_E^2, that is,

$$P_0(f) = P_0(f_+ + f_0 + \overline{f_-}) = f_0 \text{ for } f \in L_E^2,$$

where $f_+, f_- \in zH_E^2$ and $f_0 \in E$. The following lemma is essentially Lemma 1 in [10] and we include the proof for completeness.

Lemma 3.4 [10] *Let $\Phi_i \in L^\infty_{B(E)}$ for $i = 1, 2, 3, 4, 5$. Then we have*

$$\begin{aligned} &S^*(T_{\Phi_1} H_{\Phi_2} - H_{\Phi_3} T_{\Phi_4} + H_{\Phi_5}) - (T_{\Phi_1} H_{\Phi_2} - H_{\Phi_3} T_{\Phi_4} + H_{\Phi_5}) S \\ &= T_{\overline{z}\Phi_1} P_0 H_{\Phi_2} + H_{\Phi_3} P_0 T_{z\Phi_4}. \end{aligned}$$

Proof. Immediate from a direct calculation. ■

The following lemma is essentially Theorem 1 in [10] and is in some sense in a slightly more general form.

Lemma 3.5 *Let $\Phi_i \in L^\infty_{B(E)}$ for $i = 1, 2, 3, 4, 5$. Then $T_{\Phi_1} H_{\Phi_2} - H_{\Phi_3} T_{\Phi_4} + H_{\Phi_5} = 0$ if and only if*

$$T_{\overline{z}\Phi_1} P_0 H_{\Phi_2} + H_{\Phi_3} P_0 T_{z\Phi_4} = 0; \tag{9}$$

$$(T_{\Phi_1} H_{\Phi_2} - H_{\Phi_3} T_{\Phi_4} + H_{\Phi_5}) P_0 = 0. \tag{10}$$

Proof. The necessity is clear from Lemma 3.4. Now we prove sufficiency. Let

$$D = T_{\Phi_1} H_{\Phi_2} - H_{\Phi_3} T_{\Phi_4} + H_{\Phi_5}.$$

By Lemma 3.4, the assumption (9) says $S^*D = DS$. Thus if $h \in \ker(D)$, then $DSh = S^*Dh = 0$ implies that $Sh \in \ker(D)$. Therefore $\ker(D)$ is an invariant subspace for S. But the assumption (10) says $\ker(D) \supseteq \operatorname{ran}(P_0) = E$. By iteration, $\ker(D) \supseteq S^nE$ for $n \geq 0$. That is, $\ker(D) = L^2_E$ and $D = 0$. ■

Remark 3.6 When E is finite dimensional, equations (9) and (10) can be studied more explicitly as we will see in next section.

Theorem 3.7 *Let $\Phi, \Psi \in L^\infty_{B(E)}$. Then $S_{\Phi,\Psi}$ is normal if and only if*

(a) $\Phi^*\Phi - \Phi\Phi^* = 0, \Psi^*\Psi - \Psi\Psi^* = 0$;

(b) *There exist coisometries W_1 and W_2 in $B(E)$ such that $\Phi_+ = \Psi_+W_1$ or $\Psi_+ = \Phi_+W_1$ and $\Phi_- = W_2\Psi_-$ or $\Psi_- = W_2\Phi_-$;*

(c) $T_{\bar{z}\widetilde{\Omega}^*}P_0H_{\Psi^*} + H_\Phi P_0 T_{z\Omega^*} = 0$ *and* $(H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*}H_{\Psi^*} - H_\Phi T_{\Omega^*})P_0 = 0.$

Proof. (a) follows from applying Lemma 3.2 to (4), (b) follows from applying Theorem 3.3 to (5) and (c) follows from applying Lemma 3.5 to (6). ■

The special case when $W_1 = W_2 = I$ gives the following result which generalizes Theorem 3.3 in [17].

Corollary 3.8 *Let $\Phi, \Psi \in L^\infty_{B(E)}$. Assume $\Phi - \Psi = G_0$ for some operator G_0 in $B(E)$. Then $S_{\Phi,\Psi}$ is normal if and only if $\Phi^*\Phi - \Phi\Phi^* = 0, \Psi^*\Psi - \Psi\Psi^* = 0$ and $\Psi^*G_0 - \Psi G_0^* = F_0$ for some self-adjoint operator F_0 in $B(E)$.*

Proof. Immediate from a direct calculation. ■

If G_0 is an invertible operator, then equation $\Psi^*G_0 - \Psi G_0^* = F_0$ leads to the following representation of Ψ. Write

$$\Psi = \Psi_+ + \Psi_0 + \Psi_-^*.$$

Then $\Psi^*G_0 - \Psi G_0^* = F_0$ implies that $\Psi_+^*G_0 = \Psi_-^*G_0^*, \Psi_-G_0 = \Psi_+^*G_0^*$. Therefore

$$\begin{aligned}\Psi &= \Psi_+ + \Psi_0 + \Psi_-^* = G_0^{*-1}G_0\Psi_- + \Psi_-^* + \Psi_0\\ &= G_0^{*-1}(G_0\Psi_- + G_0^*\Psi_-^*) + \Psi_0\\ &= G_0^{*-1}(F + F^*) + \Psi_0,\end{aligned}$$

where $F = G_0\Psi_-$ and $F+F^*$ is a self-adjoint operator-valued function in $L^\infty_{B(E)}$. If Ψ is a scalar function, then T_Ψ is a normal Toeplitz operator characterized in [1].

To see the power of the Theorem 3.7, we first give a proof of normal singular integral operators obtained in Theorem 3.1 of [17]. This proof will also guide us to the matrix-valued case in next section.

Theorem 3.9 [17] *Assume E is one dimensional. Then $S_{\Phi,\Psi}$ is normal if and only if there exist unimodular constant w and constant b such that $\Phi = w\Psi + b$ and $(w-1)[\Psi\Psi^*] + b\Psi^* - w\bar{b}\Psi \in H^\infty$.*

Proof. Condition (a) in Theorem 3.7 is automatically satisfied in the scalar-valued case. Condition (b) says that there exist two unimodular constants w_1 and w_2 such that

$$\Phi_+ = w_1\Psi_+ \text{ and } \Phi_- = \overline{w_2}\Psi_-. \tag{11}$$

Note that $P_0 = e_0 \otimes e_0$, where $e_0 = 1$ is a constant function. Equation $T_{\overline{z}\widetilde{\Omega}^*}P_0H_{\Psi^*} + H_\Phi P_0 T_{z\Omega^*} = 0$ becomes

$$T_{\overline{z}\widetilde{\Omega}^*}(e_0 \otimes e_0) H_{\Psi_+^*} + H_{\Phi_-^*}(e_0 \otimes e_0) T_{z\Omega^*} = 0;$$
$$T_{\overline{z}\widetilde{\Omega}^*}e_0 \otimes H_{\Psi_+^*}^* e_0 + H_{\Phi_-^*}e_0 \otimes T_{z\Omega^*}^* e_0 = 0.$$

By definitions,

$$T_{\overline{z}\widetilde{\Omega}^*}e_0 = \overline{z}\widetilde{\Omega_-}, H_{\Psi_+^*}^* e_0 = \overline{z}\Psi_+, H_{\Phi_-^*}e_0 = \overline{z}\widetilde{\Phi_-}, T_{z\Omega^*}^* e_0 = \overline{z}\Omega_+. \tag{12}$$

Therefore $\overline{z}\widetilde{\Omega_-} \otimes \overline{z}\Psi_+ + \overline{z}\widetilde{\Phi_-} \otimes \overline{z}\Omega_+ = 0$. If both Ψ_+ and Φ_- are not identically zero, then for some constant a, $\Omega_+ = a\Psi_+$ and $\widetilde{\Omega_-} = -\overline{a}\widetilde{\Phi_-}$. Hence

$$\Phi_+ = (a+1)\Psi_+, \Psi_- = (a+1)\Phi_-. \tag{13}$$

Comparing (13) with (11), we have $w_1 = w_2 = (a+1)$. If we set $w = w_1 = w_2$, then equation (11)

$$\Phi = w\Psi + b \tag{14}$$

for a unimodular constant w and a constant b. Note in the scalar case $\Psi^*\Phi - \Phi\Psi^* = 0$. Note also

$$H_{\Phi_-^*}T_{\Omega^*}e_0 = H_{\Phi_-^*}(\Omega_0^* + \Omega_-) = PJ\left[\Phi_-^*(\Omega_0^* + \Omega_-)\right];$$
$$T_{\widetilde{\Omega}^*}H_{\Psi^*} = T_{\widetilde{\Omega}^*}PJ\left[\Psi_+^*\right] = P\left[\widetilde{\Omega}^*J\left[\Psi_+^*\right]\right] = PJ\left[\Omega\Psi_+^*\right].$$

Now the equation $(H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*}H_{\Psi_+^*} - H_{\Phi_-^*}T_{\Omega^*})P_0 = 0$ is the same as

$$PJ\left[\Omega\Psi_+^* - \Phi_-^*(\Omega_0^* + \Omega_-)\right] = 0.$$

Equivalently,

$$\Omega\Psi_+^* - \Phi_-^*(\Omega_0^* + \Omega_-) \in H^2_{B(E)},$$

or equivalently, by (11) and the fact that $\left(\Psi_-^* - \Psi\right), (\Psi^* - \Psi_+^*) \in H^2$,

$$(w-1)[\Psi\Psi^*] + b\Psi^* - w\bar{b}\Psi \in H^\infty.$$

The proof is complete. ■

Remark 3.10 A reflection reveals that we do not need the scalar version of Theorem 3.3 (Lemma 3.2 in [17]) in the above proof since the existence of w_1, w_2 such $w_1 = w_2$ can be seen from equation (13) and then it follows from $H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Psi^*}H_{\Psi^*} = 0$ that $|w_1| = 1$.

We now characterize when $S_{\Phi,\Psi}$ is an isometry or a coisometry.

Proposition 3.11 *$S_{\Phi,\Psi}$ is an isometry if and only if $\Phi^*\Phi = I_E, \Psi^*\Psi = I_E$ and $\Psi^*\Phi \in H^\infty_{B(E)}$.*

Proof. By equation (7),

$$US^*_{\Phi,\Psi}S_{\Phi,\Psi}U = \begin{bmatrix} T^*_\Phi T_\Phi + H^*_\Phi H_\Phi & T^*_\Phi H^*_{\Psi^*} + H^*_\Phi T_{z(J\Psi)} \\ H_{\Psi^*}T_\Phi + T^*_{z(J\Psi)}H_\Phi & H_{\Psi^*}H^*_{\Psi^*} + T^*_{z(J\Psi)}T_{z(J\Psi)} \end{bmatrix}$$
$$= \begin{bmatrix} T^*_\Phi T_\Phi + H^*_\Phi H_\Phi & H^*_{\Psi^*\Phi} \\ H_{\Psi^*\Phi} & H_{\Psi^*}H^*_{\Psi^*} + T^*_{z(J\Psi)}T_{z(J\Psi)} \end{bmatrix}. \tag{15}$$

Therefore $S_{\Phi,\Psi}$ is an isometry if and only if

$$T^*_\Phi T_\Phi + H^*_\Phi H_\Phi = I = T_{I_E}; \tag{16}$$
$$H_{\Psi^*}H^*_{\Psi^*} + T^*_{z(J\Psi)}T_{z(J\Psi)} = I = T_{I_E}; \tag{17}$$
$$H_{\Psi^*\Phi} = 0. \tag{18}$$

From equation (16) we have

$$T^*_\Phi T_\Phi - T_{\Phi^*\Phi} + H^*_\Phi H_\Phi = T_{I_E - \Phi^*\Phi}.$$

Thus we have $-H^*_\Phi H_\Phi + H^*_\Phi H_\Phi = T_{I_E-\Phi^*\Phi}$, from which we conclude that $\Phi^*\Phi = I_E$. In a similar way, equation (17) gives

$$\overline{z}\,(J\Psi)^*\, z\,(J\Psi) = \widetilde{\Psi}\widetilde{\Psi}^* = I_E \text{ or } \Psi^*\Psi = I_E.$$

Equation (18) gives $\Psi^*\Phi \in H^\infty_{B(E)}$. The proof is complete. ■

Proposition 3.12 *$S_{\Phi,\Psi}$ is a coisometry if and only if the following conditions are satisfied.*

(a) $\Phi\Phi^* = I_E, \Psi\Psi^* = I_E$.

(b) *There exist coisometries $W_1, W_2 \in B(E)$ such that W_1 satisfies either $\Phi_+ = \Psi_+ W_1$ or $\Psi_+ = \Phi_+ W_1$ and W_2 satisfies either $\Psi_- = W_2\Phi_-$ or $\Phi_-\Psi_- = W_2\Psi_-$.*

(c) $H_\Phi P_0 T_{z\Phi^*} + T_{(J\Psi)}P_0 H_{\Psi^*} = 0$ *and* $\left(H_\Phi T^*_\Phi + T_{z(J\Psi)}H_{\Psi^*}\right)P_0 = 0$.

Proof. By equation (3.12),

$$US_{\Phi,\Psi}S^*_{\Phi,\Psi}U = \begin{bmatrix} T_\Phi T^*_\Phi + H^*_{\Psi^*}H_{\Psi^*} & T_\Phi H^*_\Phi + H^*_{\Psi^*}T_{z(J\Psi)} \\ H_\Phi T^*_\Phi + T_{z(J\Psi)}H_{\Psi^*} & H_\Phi H^*_\Phi + T_{z(J\Psi)}T^*_{z(J\Psi)} \end{bmatrix}.$$

Therefore $S_{\Phi,\Psi}$ is a coisometry if and only if

$$T_\Phi T_\Phi^* + H_{\Psi^*}^* H_{\Psi^*} = I = T_{I_E}; \quad (19)$$
$$H_\Phi H_\Phi^* + T_{z(J\Psi)} T_{z(J\Psi)}^* = I = T_{I_E}; \quad (20)$$
$$H_\Phi T_\Phi^* + T_{z(J\Psi)} H_{\Psi^*} = 0. \quad (21)$$

From (19) we have

$$\Phi\Phi^* = I_E \text{ and } H_{\Psi^*}^* H_{\Psi^*} = H_{\Phi^*}^* H_{\Phi^*}.$$

By Theorem 3.3, there exist a coisometry W_1 in $B(E)$ such that $\Phi_+ = \Psi_+ W_1$ or $\Psi_+ = \Phi_+ W_1$. (20) gives

$$z\,(J\Psi)\,\overline{z}\,(J\Psi)^*\,z\,(J\Psi) = \widetilde{\Psi}^*\widetilde{\Psi} = I_E \text{ and}$$
$$H_\Phi H_\Phi^* = H_{\widetilde{\Phi}}^* H_{\widetilde{\Phi}} = H_{\overline{z}(J\Psi)^*}^* H_{\overline{z}(J\Psi)^*} = H_{\widetilde{\Psi}}^* H_{\widetilde{\Psi}}.$$

Again by Theorem 3.3, there exists a coisometry W_2 in $B(E)$ such that $\widetilde{\Phi_-} = \widetilde{\Psi_-} W_2$ or $\widetilde{\Psi_-} = \widetilde{\Phi_-} W_2$. By Lemma 3.5, equation (21) holds if and only if

$$H_\Phi P_0 T_{z\Phi^*} + T_{(J\Psi)} P_0 H_{\Psi^*} = 0;$$
$$\left(H_\Phi T_{\Phi^*} + T_{z(J\Psi)} H_{\Psi^*}\right) P_0 = 0.$$

The proof is complete. ∎

Theorem 3.13 *$S_{\Phi,\Psi}$ is a unitary operator if and only if $\Phi^*\Phi = \Phi\Phi^* = I_E, \Psi^*\Psi = \Psi\Psi^* = I_E$ and $\Phi = \Psi W_0$ for some constant unitary operator W_0 in $B(E)$.*

Proof. Assume $S_{\Phi,\Psi}$ is a unitary operator. By Proposition 3.11, $\Phi^*\Phi = I_E, \Psi^*\Psi = I_E$ and $\Phi^*\Psi \in H_E^2$. By Proposition 3.12, $\Phi\Phi^* = I_E$ and $\Psi\Psi^* = I_E$.

Let $e \in E$. Then by (1),

$$\begin{aligned} S_{\Phi,\Psi} S_{\Phi,\Psi}^* (\Psi e) &= S_{\Phi,\Psi} \left(P\left[\Phi^*\Psi e\right] + Q\left[\Psi^*\Psi e\right]\right) \\ &= S_{\Phi,\Psi} \left(P\left[\Phi^*\Psi e\right] + Q\left[e\right]\right) \\ &= S_{\Phi,\Psi} P\left[\Phi^*\Psi e\right] = \Phi P\left[\Phi^*\Psi e\right] = \Psi e, \end{aligned}$$

where in second equality, we use $\Psi^*\Psi = I$. Hence for any $e \in E$, by $\Phi^*\Phi = I$,

$$\Phi P\left[\Phi^*\Psi e\right] = \Psi e \text{ or } P\left[\Phi^*\Psi e\right] = \Phi^*\Psi e.$$

Thus $\Phi^*\Psi e \in H_E^2$ and $\Phi^*\Psi \in H_{B(E)}^\infty$. Since both $\Phi^*\Psi$ and $\Phi^*\Psi$ are in $H_{B(E)}^\infty$, we conclude $\Psi^*\Phi = W_0$ or $\Phi = \Psi W_0$ for some constant unitary operator W_0 in $B(E)$.

We now prove the other direction. Suppose $\Phi^*\Phi = \Phi\Phi^* = I_E, \Psi^*\Psi = \Psi\Psi^* = I_E$ and $\Phi = \Psi W_0$. Then by Proposition 3.11, $S_{\Phi,\Psi}$ is an isometry. We now directly verify that $S_{\Phi,\Psi}$ is is a coisometry as well. It follows from the proof of Proposition 3.12 that we need only to verify the equation $H_\Phi T_\Phi^* + T_{z(J\Psi)} H_{\Psi^*} = 0$ as follows:

$$\begin{aligned} H_\Phi T_\Phi^* + T_{z(J\Psi)} H_{\Psi^*} &= H_{\Psi W_0} T_{\Psi W_0}^* + T_{z(J\Psi)} H_{\Psi^*} \\ &= H_\Psi T_{W_0} T_{W_0}^* T_\Psi^* + T_{z(J\Psi)} H_{\Psi^*} \\ &= H_\Psi T_\Psi^* + T_{z(J\Psi)} H_{\Psi^*} \\ &= H_{\Psi\Psi^*} = 0, \end{aligned}$$

where the 4th equality follows from (7). The proof is complete. ■

The scalar version of the above theorem is Theorem 3.4 in [17].

4 Normal $S_{\Phi,\Psi}$ with matrix-valued symbols

In this section we assume that E is a Hilbert space of finite dimension n. We first fix a basis for E. Then a function $\Phi \in L^\infty_{B(E)}$ is an $n \times n$ matrix whose entries are scalar-valued L^∞-functions. Thus we let $E = \mathbb{C}^n$ and we will write $L^\infty_{M_n}$ instead of $L^\infty_{B(E)}$, where M_n denotes the set of all $n \times n$ complex matrices. Let $det(\Phi_+)$ denote the determinant of Φ_+.

Let H be a Hilbert space. For $a, b \in H$, the rank one operator $a \otimes b$ on H is defined by

$$[a \otimes b]\, e = \langle b, e\rangle\, a, e \in H.$$

Note that $[a \otimes b]^* = b \otimes a$ and for a complex number λ,

$$\lambda\, [a \otimes b] = (\lambda a) \otimes b = a \otimes \left(\overline{\lambda} b\right).$$

We first need a lemma on operator equations involving finite rank operators.

Lemma 4.1 *Let a_i, b_i, c_i, d_i for $i = 1, 2, \cdots, n$ be vectors in a Hilbert space H. Assume $\{b_i\}_{i=1}^n$ are linearly independent and $\{c_i\}_{i=1}^n$ are linearly independent. Then*

$$\sum_{i=1}^n a_i \otimes b_i + \sum_{i=1}^n c_i \otimes d_i = 0$$

if and only if there exists a matrix A such that

$$d = Ab, a = -A^* c,$$

where

$$a = [a_1, \cdots, a_n]^T, b = [b_1, \cdots, b_n]^T, c = [c_1, \cdots, c_n]^T, d = [d_1, \cdots, d_n]^T.$$

Proof. We prove lemma for $n = 2$ since the proof for the general case is analogous. In fact, $n = 2$ case illustrates the idea most clearly. We also substitute c by $-c$ for convenience in the proof. Assume

$$a_1 \otimes b_1 + a_1 \otimes b_1 = c_1 \otimes d_1 + c_2 \otimes d_2 \tag{22}$$

Since c_1 and c_2 are linearly independent, by equating the range of the operators on two sides of the above equation, we see that there exists a constant 2×2 matrix $A = (a_{ij})$ such that

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Plugging the above equation into the right side of (22), we have

$$\begin{aligned} a_1 \otimes b_1 + a_2 \otimes b_2 &= c_1 \otimes (a_{11}b_1 + a_{12}b_2) + c_2 \otimes (a_{21}b_1 + a_{22}b_2) \\ &= (\overline{a_{11}}c_1 + \overline{a_{21}}c_2) \otimes b_1 + (\overline{a_{12}}c_1 + \overline{a_{22}}c_2) \otimes b_2. \end{aligned}$$

Since b_1 and b_2 are linearly independent,

$$a_1 = (\overline{a_{11}}c_1 + \overline{a_{21}}c_2)\,, a_2 = (\overline{a_{12}}c_1 + \overline{a_{22}}c_2)\,.$$

Equivalently,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^* \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The proof is complete. ■

We are ready to characterize normal operator $S_{\Phi,\Psi}$ under a nondegenerate condition on the matrix-valued symbol functions Φ and Ψ.

Theorem 4.2 *Let $\Phi, \Psi \in L^\infty_{M_n}$. Assume that neither of* $\det(\Phi_+)$ *and* $\det(\Phi_-)$ *is identically zero. Then $S_{\Phi,\Psi}$ is normal if and only if the following three conditions hold.*

(a) $\Phi^*\Phi - \Phi\Phi^* = 0, \Psi^*\Psi - \Psi\Psi^* = 0.$

(b) *There exists a unitary matrix W_0 and constant matrix G_0 such that $\Phi = \Psi W_0 + G_0$.*

(c) $\Psi^*\Psi(W_0 - I) + \Psi^* G_0 - \Psi W_0 G_0^* \in H^\infty_{M_n}.$

Proof. Condition (a) follows from Theorem 3.7. Since a coisometry in M_n is just an unitary matrix. By Theorem 3.7(b), there exist unitary matrices W_1 and W_2 in M_n such that

$$\Phi_+ = \Psi_+ W_1 \text{ and } \Psi_- = W_2 \Phi_-. \tag{23}$$

We will prove $W_1 = W_2$. Let $\{e_i, i = 1, 2, \cdots, n\}$ be the standard basis of $\mathbb{C}^n$. Equation $T_{\bar{z}\widetilde{\Omega}^*} P_0 H_{\Psi^*} + H_\Phi P_0 T_{z\Omega^*} = 0$ becomes

$$\begin{aligned} T_{\bar{z}\widetilde{\Omega}^*} \left(\sum_{i=1}^n e_i \otimes e_i \right) H_{\Psi_+^*} + H_{\Phi_-^*} \left(\sum_{i=1}^n e_i \otimes e_i \right) T_{z\Omega^*} &= 0; \\ \sum_{i=1}^n T_{\bar{z}\widetilde{\Omega}^*} e_i \otimes H^*_{\Psi_+^*} e_i + \sum_{i=1}^n H_{\Phi_-^*} e_i \otimes T^*_{z\Omega^*} e_i &= 0. \end{aligned}$$

Since we assumed neither of $\det(\Phi_+)$ and $\det(\Phi_-)$ is identically zero, equation (23) says $\det(\Psi_+)$ and $\det(\Psi_-)$ are both nonzero functions. Hence

$$\left\{H^*_{\Psi^*_+} e_i, i = 1, 2, \cdots, n\right\}$$

are linearly independent and $\left\{H_{\Phi^*_-} e_i, i = 1, 2, \cdots, n\right\}$ are linearly independent. By Lemma 4.1, there exists a constant $n \times n$ matrix A such that

$$\begin{aligned}
[T^*_{z\Omega^*} e_1, \cdots, T^*_{z\Omega^*} e_n]^T &= A\left[H^*_{\Psi^*_+} e_1, \cdots, H^*_{\Psi^*_+} e_n\right]^T; \\
\left[T_{\overline{z}\widetilde{\Omega}^*} e_1, \cdots, T_{\overline{z}\widetilde{\Omega}^*} e_n\right]^T &= -A^*\left[H_{\Phi^*_-} e_1, \cdots, H_{\Phi^*_-} e_n\right]^T.
\end{aligned}$$

Note that $T_{\overline{z}\widetilde{\Omega}^*} e_i$ is the i-th column of the matrix $\overline{z}\widetilde{\Omega_-}$ and so on. The above two equations are equivalent to

$$\overline{z}\Omega_+ = \overline{z}\Psi_+ A^T \text{ and } \overline{z}\widetilde{\Omega_-} = -\overline{z}\widetilde{\Phi_-} A^{*T}.$$

Therefore we have

$$\begin{aligned}
&\Omega_+ = \Psi_+ A^T, \Omega_- = -A^T\Phi_-; \\
&\Phi_+ = \Psi_+(A^T + 1), \Psi_- = (A^T + 1)\Phi_-.
\end{aligned}$$

Since neither of $\det(\Phi_+)$ and $\det(\Phi_-)$ is identically zero, comparing the above equations with (23) yields $W_1 = W_2 = (A + 1)$. Set $W = W_1 = W_2$, then equation (23) implies

$$\begin{aligned}
\Phi - \Psi W_0 &= \Phi_+ - \Psi_+ W_1 + \Phi^*_- - \Psi^*_+ W_2 + \Phi_0 - \Psi_0 W \\
&= \Phi_0 - \Psi_0 W.
\end{aligned}$$

Thus we conclude that

$$\Phi = \Psi W + G \tag{24}$$

for some unitary matrix W and constant matrix G. This proves Condition (b).

Next we prove Condition (c). The equation

$$(H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*} H_{\Psi^*_+} - H_{\Phi^*_-} T_{\Omega^*}) P_0 = 0$$

is the same as

$$(H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*} H_{\Psi^*} - H_{\Phi} T_{\Omega^*}) e_i = 0, i = 1, 2, \cdots, n. \tag{25}$$

Note that

$$\begin{aligned}
H_{\Phi^*_-} T_{\Omega^*} e_i &= H_{\Phi^*_-}(\Omega^*_0 + \Omega_-) e_i = PJ\left[\Phi^*_-(\Omega^*_0 + \Omega_-) e_i\right]; \\
T_{\widetilde{\Omega}^*} H_{\Psi^*} e_i &= T_{\widetilde{\Omega}^*} PJ\left[\Psi^*_+ e_i\right] = P\left[\widetilde{\Omega}^* J\left[\Psi^*_+ e_i\right]\right] = PJ\left[\Omega\Psi^*_+ e_i\right]; \\
H_{\Psi^*\Phi-\Phi\Psi^*} e_i &= PJ\left[\Psi^*\Phi e_i - \Phi\Psi^* e_i\right].
\end{aligned}$$

Thus equation (25) becomes

$$PJ\left[\Psi^*\Phi - \Phi\Psi^* + \Omega\Psi_+^* - \Phi_-^*(\Omega_0^* + \Omega_-)\right] = 0; \tag{26}$$
$$\Psi^*\Phi - \Phi\Psi^* + \Omega\Psi_+^* - \Phi_-^*(\Omega_0^* + \Omega_-) \in H^2_{M_n}.$$

But since $\Psi\Psi^* = \Psi^*\Psi$ and $\left(\Psi_-^* - \Psi\right), \left(\Psi^* - \Psi_+^*\right) \in H^2_{M_n}$, (26) becomes

$$\Psi^*\Psi(W - I) + \Psi^*G - \Psi W G^* \in H^\infty_{M_n}.$$

The proof is complete. ■

A reflection on the above proof leads us to the following result.

Proposition 4.3 *Let $\Phi \in L^\infty_{M_n}$ and $\Phi = \Psi W_0 + G_0$ for some unitary matrix W_0 and constant matrix G_0 in M_n. Then $S_{\Phi,\Psi}$ is normal if and only if the following two conditions hold.*

(a) $\Phi^*\Phi - \Phi\Phi^* = 0, \Psi^*\Psi - \Psi\Psi^* = 0.$

(b) $\Psi^*\Psi(W_0 - I) + \Psi^*G_0 - \Psi W_0 G_0^* \in H^\infty_{B(E)}.$

Proof. Note that $\det(\Phi_+)$ and $\det(\Phi_-)$ are not assumed to be nonzero functions. The necessity of (b) follows from the proof of Condition (c) in the above theorem. For sufficiency, from the proof of $W_1 = W_2$ in the above theorem, the assumption $\Phi = \Psi W_0 + G_0$ implies $T_{\overline{z}\widetilde{\Omega}^*}P_0 H_{\Psi^*} + H_\Phi P_0 T_{z\Omega^*} = 0$. Also by the proof of Condition (c) in the above theorem, Condition (b) implies

$$(H_{\Psi^*\Phi - \Phi\Psi^*} + T_{\widetilde{\Omega}^*}H_{\Psi_+^*} - H_{\Phi_-^*}T_{\Omega^*})P_0 = 0.$$

Now by Theorem 3.7, $S_{\Phi,\Psi}$ is normal. ■

If $W_0G_0 = G_0W_0$, $\Psi G_0 = G_0\Psi$, $\Psi W_0 = W_0\Psi$ and $(W_0 - I)$ is invertible (this is true in the scalar-valued case except when $\Phi - \Psi$ is a constant), Condition (ii) above reduces to the following more compact condition which is observed in Theorem 3.2 of [17] in the scalar-valued case.

Since W_0 is a unitary operator, $W_0G_0 = G_0W_0$, which implies that $W_0G_0^* = G_0^*W_0$. Note also $W_0(W_0 - I)^{-1} = -(W_0^* - I)^{-1}$. Thus

$$\begin{aligned}
&[\Psi^*\Psi(W_0 - I) + \Psi^*G_0 - \Psi W_0 G_0^*](W_0 - I)^{-1} \\
&\qquad = \left[\Psi^*\Psi + \Psi^*G_0(W_0 - I)^{-1} - \Psi W_0 G_0^*(W_0 - I)^{-1}\right] \\
&\qquad = \left[\Psi^*\Psi + \Psi^*G_0(W_0 - I)^{-1} - \Psi G_0^* W_0(W_0 - I)^{-1}\right] \\
&\qquad = \left[\Psi^*\Psi + \Psi^*G_0(W_0 - I)^{-1} - \Psi G_0^* W_0(W_0 - I)^{-1}\right] \\
&\qquad = \left[\Psi^*\Psi + \Psi^*G_0(W_0 - I)^{-1} + \Psi G_0^*(W_0^* - I)^{-1}\right] \\
&\qquad = \left(\Psi + G_0^*(W_0^* - I)^{-1}\right)^* \left(\Psi + G_0(W_0 - I)^{-1}\right) \\
&\qquad\qquad - G_0(W_0 - I)^{-1}G_0^*(W_0^* - I)^{-1}.
\end{aligned}$$

Therefore $\Psi^*\Psi(W_0 - I) + \Psi^* G_0 - \Psi W_0 G_0^* \in H^\infty_{B(E)}$ if and only if

$$\left(\Psi + G_0(W_0 - I)^{-1}\right)^* \left(\Psi + G_0(W_0 - I)^{-1}\right) \in H^\infty_{B(E)}.$$

Thus we have

$$\left(\Psi + G_0(W_0 - I)^{-1}\right)^* \left(\Psi + G_0(W_0 - I)^{-1}\right) = F_0^* F_0$$

for some constant operator $F_0 \in B(E)$.

Next we study when $S_{\Phi,\Psi}$ is an isometry.

Theorem 4.4 *Let $\Phi, \Psi \in L^\infty_{M_n}$. Then $S_{\Phi,\Psi}$ is an isometry if and only if $\Psi^*\Psi = I$ and $\Phi = \Psi\Theta$ for some inner matrix Θ. Furthermore $S_{\Phi,\Psi}$ is a unitary operator if and only if Θ is a constant unitary matrix.*

Proof. By Proposition 3.11, $\Phi^*\Phi = I, \Psi^*\Psi = I$ and $\Psi^*\Phi \in H^\infty_{M_n}$. Since Ψ is matrix-valued, $\Psi\Psi^* = I$. Set $\Psi^*\Phi = \Theta$. Then $\Theta^*\Theta = \Phi^*\Psi\Psi^*\Phi = \Phi^*\Phi = I$. Hence Θ is an inner matrix. It is also clear that $\Phi = \Psi\Theta$. The characterization of unitary $S_{\Phi,\Psi}$ follows from Theorem 3.13. ■

In the scalar case, the above theorem reduces to Lemma 3.10 and Theorem 3.4 in [17].

Surprisingly, in the matrix-valued case we prove that if $S_{\Phi,\Psi}$ is a coisometry, then $S_{\Phi,\Psi}$ is an unitary.

Theorem 4.5 *Let $\Phi, \Psi \in L^\infty_{M_n}$. Then $S_{\Phi,\Psi}$ is a coisometry if and only if $S_{\Phi,\Psi}$ is an unitary operator.*

Proof. If $S_{\Phi,\Psi}$ is a coisometry, by Proposition 3.12, $\Phi\Phi^* = I, \Psi\Psi^* = I$. Since Φ and Ψ are matrix-valued functions, $\Phi^*\Phi = I$ and $\Psi^*\Psi = I$ as well. The remaining proof is similar to that of Theorem 3.13. ■

In the scalar-valued case ($n = 1$), the above result was first noted in [11].

5 Hyponormal $S_{\Phi,\Psi}$ with matrix-valued symbols

In this section we consider hyponormality of $S_{\Phi,\Psi}$ with matrix-valued symbols.

For matrix-valued functions

$$A(z) := \sum_{j=-\infty}^{\infty} A_j z^j \in L^2_{M_{n\times m}} \text{ and } B(z) := \sum_{j=-\infty}^{\infty} B_j z^j \in L^2_{M_{n\times m}},$$

we define the inner product of A and B by

$$\langle A, B\rangle := \int_{\mathbb{T}} \mathrm{tr}\,(B^*A)\, d\mu = \sum_{j=-\infty}^{\infty} \mathrm{tr}\,(B_j^* A_j)\,,$$

where $\mathrm{tr}\,(\cdot)$ denotes the trace of a matrix and define $||A||_2 := \langle A, A\rangle^{\frac{1}{2}}$. We also define, for $A \in L^\infty_{M_{n\times m}}$,

$$||A||_\infty := \text{ess sup}_{z\in\mathbb{T}} ||A(z)|| \quad (||\cdot|| \text{ denotes the spectral norm of a matrix}).$$

In 2006, the hyponormality of Toeplitz operators with matrix-valued symbols was characterized by their symbols.

Lemma 5.1 [13] *For each $\Phi \in L^\infty_{M_n}$, let*

$$\mathcal{E}(\Phi) := \left\{K \in H^\infty_{M_n} : \ ||K||_\infty \le 1 \ \ \text{and} \ \ \Phi - K\Phi^* \in H^\infty_{M_n}\right\}.$$

Then T_Φ is hyponormal if and only if Φ is normal and $\mathcal{E}(\Phi)$ is nonempty.

As in the hyponormality of T_Φ, the normality of the symbols is necessary for the hyponormality of $S_{\Phi,\Psi}$.

Lemma 5.2 *Let $\Phi, \Psi \in L^\infty_{M_n}$. If $S_{\Phi,\Psi}$ is hyponormal then Φ and Ψ are normal on $\mathbb{T}$.*

Proof. Suppose $S_{\Phi,\Psi}$ is hyponormal. Then it follows from Lemma 3.1 that

$$\begin{bmatrix} T_{\Phi^*\Phi-\Phi\Phi^*} + H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Psi^*}H_{\Psi^*} & H^*_{\Psi^*\Phi-\Phi\Psi^*} + H^*_{\Psi^*}T_{\widetilde{\Omega}} - T_\Omega H^*_{\widetilde{\Phi}} \\ H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*}H_{\Psi^*} - H_{\Phi}T_{\Omega^*} & T_{\widetilde{\Psi}\widetilde{\Psi}^*-\widetilde{\Psi}^*\widetilde{\Psi}} + H^*_{\widetilde{\Psi}}H_{\widetilde{\Psi}} - H^*_{\widetilde{\Phi}}H_{\widetilde{\Phi}} \end{bmatrix} \ge 0,$$

where $\Omega = \Phi - \Psi$. Thus $T_{\Phi^*\Phi-\Phi\Phi^*} + H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Psi^*}H_{\Psi^*} \ge 0$. Therefore for each $m \in \mathbb{N}$,

$$\begin{aligned} 0 &\le S^{*m}(T_{\Phi^*\Phi-\Phi\Phi^*} + H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Psi^*}H_{\Psi^*})S^m \\ &= S^{*m}T_{\Phi^*\Phi-\Phi\Phi^*}S^m + S^{*m}H^*_{\Phi^*}H_{\Phi^*}S^m - S^{*m}H^*_{\Psi^*}H_{\Psi^*}S^m \\ &= T_{\Phi^*\Phi-\Phi\Phi^*} + H^*_{\Phi^*}S^mS^{*m}H_{\Phi^*} - H^*_{\Psi^*}S^mS^{*m}H_{\Psi^*}. \end{aligned}$$

Since S^mS^{*m} is the projection on the space $z^mH^2_{\mathbb{C}^n}$, $S^mS^{*m} \to 0$ strongly as m tends to ∞. Thus both $H^*_{\Phi^*}S^mS^{*m}H_{\Phi^*}$ and $H^*_{\Psi^*}S^mS^{*m}H_{\Psi^*}$ strongly converge to 0. Therefore $T_{\Phi^*\Phi-\Phi\Phi^*} \ge 0$, which in turn implies the Poisson integral of $\Phi^*\Phi - \Phi\Phi^*$ is positive semidefinite for $z \in \mathbb{D}$. Since Φ is finite matrix, we have $\Phi^*\Phi = \Phi\Phi^*$ a.e. on $\mathbb{T}$. Similarly, we also have that Ψ is normal on $\mathbb{T}$. ■

By Lemma 5.2, $S_{\Phi,\Psi}$ is hyponormal if and only if Φ and Ψ are normal and

$$\begin{bmatrix} H^*_{\Phi^*}H_{\Phi^*} - H^*_{\Psi^*}H_{\Psi^*} & H^*_{\Psi^*\Phi-\Phi\Psi^*} + H^*_{\Psi^*}T_{\widetilde{\Omega}} - T_\Omega H^*_{\widetilde{\Phi}} \\ H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*}H_{\Psi^*} - H_{\Phi}T_{\Omega^*} & H^*_{\widetilde{\Psi}}H_{\widetilde{\Psi}} - H^*_{\widetilde{\Phi}}H_{\widetilde{\Phi}} \end{bmatrix} \ge 0. \tag{27}$$

Definition 5.3 Let $\Phi \in L^\infty_{M_n}$. The *pseudo-selfcommutator* of T_Φ is defined by

$$[T^*_\Phi, T_\Phi]_p := H^*_{\Phi^*}H_{\Phi^*} - H^*_\Phi H_\Phi.$$

The Toeplitz operator T_Φ is said to be *pseudo-hyponormal* if $[T^*_\Phi, T_\Phi]_p$ is positive semi-definite.

By Definition 5.3, we can see that T_Φ is hyponormal if and only if T_Φ is pseudo-hyponormal and Φ is normal and that (via Theorem 3.3 of [13]) T_Φ is pseudo-hyponormal if and only if $\mathcal{E}(\Phi) \ne \emptyset$.

We now have:

Theorem 5.4 *Let $\Phi \in M_n$ and $\Psi \in L^\infty_{M_n}$ be normal. Then the followings are equivalent.*

(a) $S_{\Phi,\Psi}$ *is hyponormal.*

(b) $\Psi^* \in H^\infty_{M_n}$.

(c) $S_{\Phi,\Psi}$ *is hyponormal and* $S^*_{\Phi,\Psi}S_{\Phi,\Psi} - S_{\Phi,\Psi}S^*_{\Phi,\Psi}$ *is diagonal.*

Proof. Let $\Phi \in M_n$ and $\Psi \in L^\infty_{M_n}$ be normal. Then $H_\Phi = H_{\widetilde{\Phi}} = H_{\Phi^*} = 0$, and hence

$$U\left[S^*_{\Phi,\Psi}S_{\Phi,\Psi} - S_{\Phi,\Psi}S^*_{\Phi,\Psi}\right]U = \begin{bmatrix} -H^*_{\Psi^*}H_{\Psi^*} & H^*_{\Psi^*\Phi-\Phi\Psi^*} + H^*_{\Psi^*}T_{\widetilde{\Omega}} \\ H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*}H_{\Psi^*} & H^*_{\widetilde{\Psi}}H_{\widetilde{\Psi}} \end{bmatrix}. \tag{28}$$

Suppose $S_{\Phi,\Psi}$ is hyponormal. Then it follows from (28) that

$$\begin{bmatrix} -H^*_{\Psi^*}H_{\Psi^*} & H^*_{\Psi^*\Phi-\Phi\Psi^*} + H^*_{\Psi^*}T_{\widetilde{\Omega}} \\ H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*}H_{\Psi^*} & H^*_{\widetilde{\Psi}}H_{\widetilde{\Psi}} \end{bmatrix} \geq 0.$$

Thus $-H^*_{\Psi^*}H_{\Psi^*} \geq 0$, and hence $H_{\Psi^*} = 0$, so that $\Psi^* \in H^\infty_{M_n}$. This proves the implication $(a) \Rightarrow (b)$. For the implication $(b) \Rightarrow (c)$, suppose $\Psi^* \in H^\infty_{M_n}$. Then $H_{\Psi^*} = 0$, and hence it follows from Lemma 1.1 that $H_{\Psi^*\Phi-\Phi\Psi^*} = H_{\Psi^*}T_\Phi - T^*_{\widetilde{\Phi}}H_{\Psi^*} = 0$. It thus follows from (28) that

$$U\left[S^*_{\Phi,\Psi}S_{\Phi,\Psi} - S_{\Phi,\Psi}S^*_{\Phi,\Psi}\right]U = \begin{bmatrix} 0 & 0 \\ 0 & H^*_{\widetilde{\Psi}}H_{\widetilde{\Psi}} \end{bmatrix} \geq 0$$

and $\left[S^*_{\Phi,\Psi}S_{\Phi,\Psi} - S_{\Phi,\Psi}S^*_{\Phi,\Psi}\right]$ is diagonal. This proves the implication $(b) \Rightarrow (c)$. The implication $(c) \Rightarrow (a)$ is clear. ∎

Corollary 5.5 *Let* $\Phi \in L^\infty_{M_n}$ *and* $\Psi \in M_n$ *be normal. Then the followings are equivalent.*

(a) $S_{\Phi,\Psi}$ *is hyponormal.*

(b) $\Phi \in H^\infty_{M_n}$.

(c) $S_{\Phi,\Psi}$ *is hyponormal and* $S^*_{\Phi,\Psi}S_{\Phi,\Psi} - S_{\Phi,\Psi}S^*_{\Phi,\Psi}$ *is diagonal.*

Proof. Same as Theorem 5.4. ∎

To proceed, we recall:

Definition 5.6 ([4]) For $\Phi \in L^\infty_{M_n}$, we say that Φ has a *matrix pole* if there exists $\alpha \in \mathbb{D}$ such that $\{0\} \neq \ker H_\Phi \subseteq (z-\alpha)H^2_{\mathbb{C}^n}$.

If $\Theta \in H^2_{M_{n\times m}}$ is an inner matrix function, we write

$$\mathcal{H}(\Theta) := H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^m}.$$

Note that

$$\begin{aligned} f \in \mathcal{H}(\Theta) &\Longleftrightarrow \langle f,\ \Theta g\rangle = 0 \ \text{ for all } g \in H^2_{\mathbb{C}^m} \\ &\Longleftrightarrow \langle \Theta^* f,\ g\rangle = 0 \ \text{ for all } g \in H^2_{\mathbb{C}^m} \\ &\Longleftrightarrow \Theta^* f \in (H^2_{\mathbb{C}^m})^\perp \equiv L^2_{\mathbb{C}^m} \ominus H^2_{\mathbb{C}^m}. \end{aligned}$$

Lemma 5.7 *Let $\Theta \in H^2_{M_n}$ and $\Delta \in H^2_{M_{n\times m}}$ be inner matrix functions. Then*

$$\mathcal{H}(\Theta\Delta) = \mathcal{H}(\Theta) \oplus \Theta\mathcal{H}(\Delta).$$

Proof. Since $\Theta\Delta H^2_{\mathbb{C}^m} \subseteq \Theta H^2_{\mathbb{C}^n}$, it follows that $\mathcal{H}(\Theta) \subseteq \mathcal{H}(\Theta\Delta)$. Observe that for each $g \in \mathcal{H}(\Delta)$, $(\Theta\Delta)^*\Theta g = \Delta^* g \in (H^2_{\mathbb{C}^m})^\perp$, which implies $\Theta\mathcal{H}(\Delta) \subseteq \mathcal{H}(\Theta\Delta)$. Thus $\mathcal{H}(\Theta) \oplus \Theta\mathcal{H}(\Delta) \subseteq \mathcal{H}(\Theta\Delta)$. For the reverse inclusion, let $f \in \mathcal{H}(\Theta\Delta)$. Write $f_1 := P_{\mathcal{H}(\Theta)} f$. Then $f - f_1 \in P_{\Theta H^2_{\mathbb{C}^n}}$, so that $f - f_1 = \Theta f_2$ for some $f_2 \in H^2_{\mathbb{C}^n}$. Thus it suffices to show $f_2 \in \mathcal{H}(\Delta)$. Since $f_1 \in \mathcal{H}(\Theta\Delta)$, we have $\Theta f_2 = f - f_1 \in \mathcal{H}(\Theta\Delta)$. Thus $(\Theta\Delta)^*\Theta f_2 \in (H^2_{\mathbb{C}^m})^\perp$, and hence $\Delta^* f_2 \in (H^2_{\mathbb{C}^m})^\perp$, which implies $f_2 \in \mathcal{H}(\Delta)$. ■

We write $\mathcal{Z}(\theta)$ for the set of zeros of an inner function θ. We recall:

Lemma 5.8 [6] *Let $B \in H^2_{M_n}$ and $\Theta := \theta I_n$ with a finite Blaschke product θ. Then the followings are equivalent.*

(a) *$B(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$.*

(b) *B and Θ are right coprime.*

(c) *B and Θ are left coprime.*

We now have:

Theorem 5.9 *Let $\Phi, \Psi \in L^\infty_{M_n}$ be normal and $\Phi_+ = \Psi_+$. If Ψ^* (or $\widetilde{\Psi}^*$) has a matrix pole, then the followings are equivalent.*

(a) *$S^*_{\Phi,\Psi} S_{\Phi,\Psi} - S_{\Phi,\Psi} S^*_{\Phi,\Psi}$ is diagonal.*

(b) *$\Omega := \Phi - \Psi \in M_n$ and $\Psi^*\Omega - \Psi\Omega^* \in M_n$.*

(c) *$S_{\Phi,\Psi}$ is normal.*

Proof. For the implication $(a) \Rightarrow (b)$, suppose $\Phi_+ = \Psi_+$, and Ψ^* has a matrix pole. If $S^*_{\Phi,\Psi} S_{\Phi,\Psi} - S_{\Phi,\Psi} S^*_{\Phi,\Psi}$ is diagonal, then $\Omega^* = (\Phi - \Psi)^* \in H^2_{M_n}$ and $H_{\Psi^*\Phi - \Phi\Psi^*} + T_{\widetilde{\Omega}^*} H_{\Psi^*} - H_\Phi T_{\Omega^*} = 0$. Thus it follows from Lemma 3.5 that

$$T_{\overline{z}\widetilde{\Omega}^*} P_0 H_{\Psi^*} + H_\Phi P_0 T_{z\Omega^*} = 0 \tag{29}$$

and

$$(H_{\Psi^*\Phi - \Phi\Psi^*} + T_{\widetilde{\Omega}^*} H_{\Psi^*} - H_\Phi T_{\Omega^*}) P_0 = 0. \tag{30}$$

Since $\Omega^* \in H^2_{M_n}$, we have $P_0 T_{z\Omega^*} = 0$, and hence, by (29), $T_{\overline{z}\widetilde{\Omega}^*} P_0 H_{\Psi^*} = 0$. We now claim that

$$\Omega \in M_n. \tag{31}$$

To prove (31) there are two cases to consider.

(Case 1) Suppose $\ker H_{\widetilde{\Psi}^*} = \{0\}$. Then H_{Ψ^*} has dense range. Thus $\operatorname{ran} P_0 H_{\Psi^*}$ is dense in $\mathbb{C}^n$, so that the condition $T_{\overline{z}\widetilde{\Omega}^*} P_0 H_{\Psi^*} = 0$, which implies $\overline{z}\widetilde{\Omega}^* \in (H^2_{M_n})^\perp$, and hence $\Omega \in M_n$.

(Case 2) Suppose $\ker H_{\widetilde{\Psi}^*} \neq \{0\}$. By Lemma 1.1, we have $H_{\Psi^*}T_{zI_n} = T^*_{zI_n}H_{\Psi^*}$, so that $\ker H_{\Psi^*}$ is invariant under T_{zI_n}. Since Ψ^* has a matrix pole, $\ker H_{\Psi^*} \neq \{0\}$. Thus it follows from the Beurling-Lax-Halmos Theorem that $\ker H_{\Psi^*} = \Theta H^2_{\mathbb{C}^m}$ for some inner matrix function Θ. Since Ψ^* has a matrix pole, there exists $\alpha \in \mathbb{D}$ such that

$$\Theta H^2_{\mathbb{C}^m} = \ker H_{\Psi^*} \subseteq (z-\alpha)H^2_{\mathbb{C}^n} = b_\alpha H^2_{\mathbb{C}^n} \ \Bigl(b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}\Bigr),$$

which implies that $B_\alpha \equiv b_\alpha I_n$ is a left inner divisor of Θ (cf.[8, Corollary IX.2.2]). Thus $\Theta = B_\alpha \Theta_1$ for some inner matrix function Θ_1 and we can write $\Psi^* = A(B_\alpha\Theta_1)^*$, where $A \in H^\infty_{M_n}$ and A and $B_\alpha\Theta$ are right coprime. Thus A and B_α are right coprime, so that by Lemma 5.8, $A(\alpha)$ is invertible. Observe that $\widetilde{\Psi}^* = \widetilde{B_\alpha}^* \widetilde{\Theta_1}^* \widetilde{A} = \widetilde{\Theta_1}^* \widetilde{A} B^*_{\overline{\alpha}}$. We thus have

$$\begin{aligned} f \in \ker H_{\widetilde{\Psi}^*} &\Longrightarrow \widetilde{\Theta_1}^* \widetilde{A} B^*_{\overline{\alpha}} f \in H^2_{\mathbb{C}^n} \\ &\Longrightarrow \widetilde{A} B^*_{\overline{\alpha}} f \in \widetilde{\Theta_1} H^2_{\mathbb{C}^n} \subseteq H^2_{\mathbb{C}^n} \\ &\Longrightarrow f \in \ker H_{\widetilde{A}B^*_{\overline{\alpha}}} = b_{\overline{\alpha}} H^2_{\mathbb{C}^n} \ \text{(since } \widetilde{A}(\overline{\alpha}) = A^*(\alpha) \text{ is invertible).} \end{aligned}$$

Since $\ker H_{\widetilde{\Psi}^*} \neq \{0\}$, by the Beurling-Lax-Halmos Theorem, we have

$$\Delta H^2_{\mathbb{C}^q} = \ker H_{\widetilde{\Psi}^*} \subseteq b_{\overline{\alpha}} H^2_{\mathbb{C}^n} \ \text{for some inner matrix function } \Delta.$$

Thus $\Delta = B_{\overline{\alpha}}\Delta_1$ for some inner matrix function Δ_1. By Lemma 5.7, $\operatorname{ran} H_{\Psi^*} = \mathcal{H}(B_{\overline{\alpha}}\Delta_1) = \mathcal{H}(B_{\overline{\alpha}}) \oplus B_{\overline{\alpha}}\mathcal{H}(\Delta_1)$. Since $\mathcal{H}(B_{\overline{\alpha}}) = \oplus_{i=1}^n \mathcal{H}(b_{\overline{\alpha}})$ and $T_{\overline{z}\widetilde{\Omega}^*}P_0H_{\Psi^*} = 0$, it follows $\overline{z}\widetilde{\Omega}^* \in (H^2_{M_n})^\perp$, so that $\Omega \in M_n$. This proves the claim (31). Now since $\Omega \in M_n$, it follows from (30) that

$$0 = (H_{\Psi^*\Phi-\Phi\Psi^*} + T_{\widetilde{\Omega}^*}H_{\Psi^*} - H_\Phi T_{\Omega^*})P_0 = H_{(\Psi^*\Phi-\Phi\Psi^*+\Omega\Psi^*-\Phi\Omega^*)}P_0,$$

which implies $\Psi^*\Phi - \Phi\Psi^* + \Omega\Psi^*_+ - \Phi^*_-\Omega^* \in H^2_{M_n}$. Thus (by the proof of Corollary 3.8) we have that $\Psi^*\Omega - \Psi\Omega^* = F_0$ for some self-adjoint $F_0 \in M_n$, which gives the result. If $\widetilde{\Psi}^*$ has a matrix pole, the same argument as the above gives the result. This proves the implication $(a) \Rightarrow (b)$. The implication $(b) \Rightarrow (c)$ follows at once from Corollary 3.8. The implication $(c) \Rightarrow (a)$ is clear. ■

Corollary 5.10 *Let $\Phi, \Psi \in L^\infty_{M_n}$ with $\Phi_- = \Psi_-$. If Ψ has a matrix pole, then the followings are equivalent.*

(a) $S^*_{\Phi,\Psi}S_{\Phi,\Psi} - S_{\Phi,\Psi}S^*_{\Phi,\Psi}$ *is diagonal.*

(b) $\Omega := \Phi - \Psi \in M_n$ *and* $\widetilde{\Psi}\Omega - \widetilde{\Psi}^*\Omega^* \in M_n$.

(c) $S_{\Phi,\Psi}$ *is normal.*

Proof. Apply Theorem 5.9 to $\widetilde{\Psi}^*$ in place of Ψ. ■

Theorem 5.11 *Let $\Phi, \Psi \in L^\infty_{M_n}$ be normal and $\Phi - \Psi \in M_n$. Then $S_{\Phi,\Psi}$ is hyponormal if and only if $S_{\Phi,\Psi}$ is normal.*

Proof. Suppose $S_{\Phi,\Psi}$ is hyponormal and $\Phi - \Psi \in M_n$. Then $H_{\Phi^*} = H_{\Psi^*}$ and $H_{\widetilde{\Psi}} = H_{\widetilde{\Phi}}$. Thus by (27), we must have $H^*_{\Psi^*\Phi - \Phi\Psi^*} + H^*_{\Psi^*} T_{\widetilde{\Omega}} - T_\Omega H^*_\Phi = 0$, which implies $S_{\Phi,\Psi}$ is normal. ■

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