Hyponormality of Toeplitz operators in several variables by the weighted shifts approach

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Abstract

In this paper, we characterize hyponormality of a class of operators related to a tuple of weighted shifts. As applications we construct examples of hyponormal Toeplitz operators on Hilbert spaces of holomorphic functions on the polydisk or the unit ball which include the Bergman space of the polydisk, the Hardy space of the unit ball and the Drury-Arveson space.

1 Introduction

Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. For $A, B \in B(H)$, the commutator of A and B is defined by

$$[A, B] = AB - BA.$$

Then the self-commutator of A is defined by

$$[A^*, A] = A^*A - AA^*.$$

If $[A^*, A] \geq 0$, then A is said to be hyponormal. For the systematic study of hyponormal operators, we refer to [23] [33]. A recent research monograph [15] finds new and important connections between hyponormal operators and the analysis of planar shapes.

Let L^2 be the space of square integrable functions with respect to the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Let H^2 be the Hardy space of the open unit disk $\mathbb{D}$. Let P be the projection from L^2 onto H^2. For $\varphi \in L^\infty$, the Toeplitz operator T_φ on H^2 is defined by $T_\varphi h = P[\varphi h]$, $h \in H^2$. Toeplitz operators are well-studied and has many applications [4].

In 1970, the fifth problem of ten problems by Halmos [16] asked: Is every subnormal Toeplitz operator T_φ either analytic or normal ? It was shown in 1975 by Amemiya, Ito, and Wong [3] that the answer to Halmos' question is positive for quasinormal Toeplitz operators. Abrahamse in 1976 [1] proved that the answer to Halmos' question is also positive if φ is a function of bounded type. Inspired by the work of Sun [31], a negative answer was given in 1984 by Cowen and Long [6] [7]. Along this line of research, hyponormal Toeplitz operators were characterized by Cowen [5] in 1988

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by using Sarason's commutant lifting theorem [30]. Since then intensive research has been done on this topic along with its connection with interpolation problems and function theory, many results are obtained, see for example [14] [20] [27] [34]. We also mentioned the extension of Cowen's result to hyponormal Toeplitz operators with matrix-valued or operator-valued symbols [11] [13] by using the general commutant lifting theorem of Nagy-Foias [10] [32] and the recent study of analogous question for subnormal Toeplitz operators with matrix-valued symbols [8] [9] [12] [19].

Let $L^2(\mathbb{D})$ be the L^2 space of of square integrable functions with respect to the normalized area measure on $\mathbb{D}$ and $A^2(\mathbb{D})$ be the closed subspace of $L^2(\mathbb{D})$ consisting of all analytic functions on $\mathbb{D}$. We still use P to denote the projection from $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$. For $\varphi \in L^\infty$, the Toeplitz operator T_φ on $A^2(\mathbb{D})$ is defined analogously by $T_\varphi h = P[\varphi h]$, $h \in A^2(\mathbb{D})$. The Hankel operator H_φ from $A^2(\mathbb{D})$ to $L^2(\mathbb{D}) \ominus A^2(\mathbb{D})$ is defined by $H_\varphi h = \varphi h - P[\varphi h]$, $h \in A^2(\mathbb{D})$. Since there is no commutant lifting theorem on the Bergman space $zA^2(\mathbb{D})$, Cowen's characterization of hyponormal Toeplitz operators does not carry to the Bergman space. Another fundamental difference between H^2 and $A^2(\mathbb{D})$ is that $L^2 \ominus H^2$ is just the conjugation of zH^2, and $L^2(\mathbb{D}) \ominus A^2(\mathbb{D})$ is much larger than the conjugation of $zA^2(\mathbb{D})$. This makes computations related to Hankel operators on $A^2(\mathbb{D})$ much more complicated. Thus the problem of characterizing hyponormal Toeplitz operators on the Bergman space seems very difficult. There is a necessary condition for the hyponormality obtained by [29], which states that for a class of analytic functions f and g, if $T_{f+\overline{g}}$ is hyponormal, then $|f'(z)| \geq |g'(z)|$ on the unit circle $\mathbb{T}$. The restriction on f and g is relaxed in [2]. More importantly, this necessary condition can be nicely viewed as a type of extension of mean-valued theorem for harmonic functions [2].

Apart from this necessary condition, a necessary and sufficient condition for the hyponormality of Toeplitz operator $T_{f+\overline{g}}$ on the Bergman space and weighted Bergman space is known only when the symbols f and g are of special forms such as $f = az^n + bz^m$ and $g = cz^n + dz^m$ [17] [18] [21] [25] [26] [28]. Even for these special symbols the results are not complete and their proofs are often algebraically complicated. As in the Hardy space case,

$$\left[T^*_{f+\overline{g}}, T_{f+\overline{g}}\right] = H^*_{\overline{f}} H_{\overline{f}} - H^*_g H_g.$$

On the Hardy space, this connection with Hankel operators was crucial for the study of hyponormal and subnormal Toeplitz operators [5] [11]. It is natural on the Bergman space to use Hankel operators and their properties as well, which was the approach in previous work.

However, we observe in this paper that for symbols such as $f = az^n + bz^m$, $g = cz^n + dz^m$, it is possible to direct work with $T_{f+\overline{g}}$ whose matrix representation is a banded matrix with four bands. In particular we will find a condition on a, b, c and d that the self-commutator of $T_{f+\overline{g}}$ is a diagonal operator. Such an observation also arises from the fact that T_{z^n} can be viewed as a weighted shift. The approach of using general weighted shifts also allows us to consider the hyponormality of slightly more general operators than the Toeplitz operators on weighted Bergman space. Furthermore, this new approach considerably simplifies the proofs of several previous results.

Surprisingly, or not so surprisingly because of the simple and general nature of our approach, we are able to extend the results to Toeplitz operators on weighted Bergman space in several variables for which there is almost no previous research. Let $\mathbb{D}^d$ be the polydisk. We use $\mathcal{K}_\rho(\mathbb{D}^d)$ (for $\rho > 1$) to denote the Hilbert space of analytic functions on the polydisk $\mathbb{D}^d$ with reproducing kernel

$$K(z, \lambda) = \frac{1}{\prod_{i=1}^d \left(1 - z_i \overline{\lambda_i}\right)^\rho}.$$

This scale of spaces contains the Bergman space ($\rho = 2$) and Hardy space ($\rho = 1$). When $\rho > 2$, $\mathcal{K}_\rho(\mathbb{D}^d)$ is often called the weighted Bergman space on $\mathbb{D}^d$ since it is a Hilbert space of holomorphic functions with respect to certain measure on $\mathbb{D}^d$. Let $\mathbb{B}^d$ be the unit ball in $\mathbb{C}^d$. Let $\mathcal{K}_\rho(\mathbb{B}^d)$ ($\rho \geq 1$) be the Hilbert space of analytic functions on $\mathbb{B}^d$ with reproducing kernel

$$K(z,w) = \frac{1}{(1 - \langle z, w\rangle)^\rho}.$$

This scale of spaces contains the Bergman space ($\rho = d+1$), the Hardy space ($\rho = d$), and the Drury-Arveson space ($\rho = 1$). We will discuss hyponormal Toeplitz operators on weighted Hardy and Bergman space $\mathcal{K}_\rho(\mathbb{D}^d)$ and $\mathcal{K}_\rho(\mathbb{B}^d)$. The results on $\mathcal{K}_\rho(\mathbb{D}^d)$ and $\mathcal{K}_\rho(\mathbb{B}^d)$ are different in that it is impossible for a class of Toeplitz operators on $\mathcal{K}_\rho(\mathbb{B}^d)$ to be hyponormal even though they are hyponormal on $\mathcal{K}_\rho(\mathbb{D}^d)$. Furthermore, the result on $\mathcal{K}_\rho(\mathbb{B}^d)$ is not as complete as on $\mathcal{K}_\rho(\mathbb{D}^d)$, and the proof on $\mathcal{K}_\rho(\mathbb{B}^d)$ is more difficult due to the fact the certain analytic Toeplitz operators on $\mathcal{K}_\rho(\mathbb{D}^d)$ is double commuting, but they are only commuting on $\mathcal{K}_\rho(\mathbb{B}^d)$. For example, by Theorem 5.1 and Proposition 5.3 below, we have the following result.

Theorem. Assume $a\bar{b} = c\bar{d}$. Let $M = (M_1, \cdots, M_d)$ and $N = (N_1, \cdots, N_d)$ be two multi-indexes with $M < N$ and $|a| < |c|$. For $\varphi = az^M + bz^N + \overline{cz}^M + \overline{d}\overline{z}^N$, T_φ on $\mathcal{K}_\rho(\mathbb{D}^d)$ ($\rho > 1$) is hyponormal if and only if

$$\frac{|c|^2 - |a|^2}{|b|^2 - |d|^2} \leq \prod_{i=1}^{d} \frac{\Gamma(N_i + 1)\Gamma(M_i + \rho)}{\Gamma(M_i + 1)\Gamma(N_i + \rho)}.$$

Furthermore, T_φ on $\mathcal{K}_\rho(\mathbb{B}^d)$ ($\rho \geq 1$) is not hyponormal.

We outline the plan of the paper. In section 2, we consider hyponormal operators related to weighted shifts. The matrix representations of those operators are banded matrices with four bands, and we give a necessary and sufficient condition for the hyponormality of those operators (Theorem 2.2 and Theorem 2.4). In section 3, we apply the results of section 2 to Toeplitz operators on weighted Hardy and Bergman spaces $\mathcal{K}_\rho(\mathbb{D})$ ($\rho > 1$). Among our results is an extension of the main result in [21] to weighted Hardy space (Theorem 3.1). The technical problem is to find the maximum or minimum of sequences. It turns out such a technical problem is rather difficult and in one case we are only able to resolve the problem when ρ is a natural number (Theorem 3.2). In section 4, we consider hyponormal operators related to a tuple of commuting weighted shifts. Some results (such as Theorem 4.3 and Theorem 4.5) are natural extensions of one variable results in Section 2, while other results (such as Theorem 4.6) require a number of new insights. In section 5, we consider hyponormality of Toeplitz operators on weighted Hardy space and Bergman spaces $\mathcal{K}_\rho(\mathbb{D}^d)$ and $\mathcal{K}_\rho(\mathbb{B}^d)$. On $\mathcal{K}_\rho(\mathbb{D}^d)$, exact calculations can be made (Theorem 5.1 and Theorem 5.2) thanks to results in previous sections, while on $\mathcal{K}_\rho(\mathbb{B}^d)$, only estimates are given (Proposition 5.4). However even those estimates are rather technical, so exact answers on $\mathcal{K}_\rho(\mathbb{B}^d)$ seems difficult to obtain.

2 Hyponormal operators close to weighted shifts

It is a simple and well-known fact that a weighted shift operator is hyponormal if and only if the absolute values of its weight sequence are increasing. In this section we show how to construct a class of hyponormal operators related to weighted shifts. This construction is based on the following lemma, which is an abstract version of some calculations done in [21].

Lemma 2.1 *Let a, b, c, d be four complex numbers such that $a\overline{b} = c\overline{d}$. Let $A, B \in B(H)$ be two commuting operators. Set $T = aA + bB + \overline{c}A^* + \overline{d}B^*$. Then*

$$[T^*, T] = \left(|a|^2 - |c|^2\right)[A^*, A] + \left(|b|^2 - |d|^2\right)[B^*, B].$$

Proof. Note that

$$\begin{aligned}
[T^*, T] &= (\overline{a}A^* + \overline{b}B^* + cA + dB)(aA + bB + \overline{c}A^* + \overline{d}B^*) \\
&\quad - (aA + bB + \overline{c}A^* + \overline{d}B^*)(\overline{a}A^* + \overline{b}B^* + cA + dB) \\
&= \left(|a|^2 - |c|^2\right)[A^*, A] + \left(|b|^2 - |d|^2\right)[B^*, B] \\
&\quad + \overline{a}bA^*B + \overline{a}\overline{d}A^*B^* + \overline{b}aB^*A + \overline{b}\overline{c}B^*A^* + \overline{c}\overline{b}AB + c\overline{d}AB^* + daBA + d\overline{c}BA^* \\
&\quad - a\overline{b}AB^* - adAB - b\overline{a}BA^* - bcBA - \overline{c}\overline{b}A^*B^* - \overline{c}dA^*B - \overline{d}\overline{a}B^*A^* - \overline{d}cB^*A \\
&= \left(|a|^2 - |c|^2\right)[A^*, A] + \left(|b|^2 - |d|^2\right)[B^*, B] \\
&\quad + (\overline{a}b - \overline{c}d)A^*B + \left(\overline{a}\overline{d} - \overline{b}\overline{c}\right)[A^*, B^*] + \left(\overline{b}a - \overline{d}c\right)B^*A \\
&\quad + \left(\overline{c}\overline{b} - da\right)[A, B] + \left(c\overline{d} - a\overline{b}\right)AB^* + (d\overline{c} - b\overline{a})BA^* \\
&= \left(|a|^2 - |c|^2\right)[A^*, A] + \left(|b|^2 - |d|^2\right)[B^*, B],
\end{aligned}$$

where in the last equality we use the assumption $a\overline{b} = c\overline{d}$ and $[A, B] = 0$. ■

If $|a| = |c|$ in Lemma 2.1, then T must be normal. To avoid the triviality, we will consider the case $|a| \neq |c|$ in the sequel.

Let l^2 be the Hilbert space consisting of square summable sequences. The standard basis for l^2 is $\{e_n : n \geq 0\}$. Let S be the weighted shift on l^2 with positive weight sequence $\{w_n\}$. That is,

$$\begin{aligned}
&Se_n = w_n e_{n+1} \quad (n \geq 0), \\
&S^*e_0 = 0 \text{ and } S^*e_{n+1} = w_n e_n \quad (n \geq 0).
\end{aligned}$$

Then for $k \geq 1$, $S^{*k}S^k$ and S^kS^{*k} are both diagonal operators such that

$$\begin{aligned}
&S^{*k}S^k e_n = w_n^2 \cdots w_{n+k-1}^2 e_n \quad (n \geq 0), \\
&S^kS^{*k} e_n = 0 \quad (0 \leq n < k), \\
&S^kS^{*k} e_n = w_{n-k}^2 \cdots w_{n-1}^2 e_n \quad (n \geq k). \qquad (1)
\end{aligned}$$

We first discuss the shift S which is hyponormal, that is, $\{w_n\}$ is increasing.

Theorem 2.2 *Let S be a hyponormal weighted shift with weight sequence $\{w_n\}$. Assume $a\overline{b} = c\overline{d}$. Let M and N be two positive integers with $M < N$ and $|a| \neq |c|$. Then*

$$T = aS^M + bS^N + \overline{c}S^{*M} + \overline{d}S^{*N}$$

is hyponormal if and only if one of the following two statements holds.
(i) *If $|a| > |c|$, then*

$$\frac{|a|^2 - |c|^2}{|d|^2 - |b|^2} \geq \max\{E_1, E_2, E_3\},$$

where

$$E_1 = w_{2M-1}^2 \cdots w_{M+N-2}^2, \tag{2}$$

$$E_2 = \max_{M \le n < N} \left\{ \frac{w_n^2 \cdots w_{n+N-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2} \right\}, \tag{3}$$

$$E_3 = \sup_{n \ge N} \left\{ \frac{w_n^2 \cdots w_{n+N-1}^2 - w_{n-N}^2 \cdots w_{n-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2} \right\}. \tag{4}$$

(ii) *If* $|a| < |c|$, *then*

$$\frac{|c|^2 - |a|^2}{|b|^2 - |d|^2} \le \min \{F_1, F_2, F_3\},$$

where

$$F_1 = w_M^2 \cdots w_{N-1}^2, \tag{5}$$

$$F_2 = \min_{M \le n < N} \left\{ \frac{w_n^2 \cdots w_{n+N-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2} \right\}, \tag{6}$$

$$F_3 = \inf_{n \ge N} \left\{ \frac{w_n^2 \cdots w_{n+N-1}^2 - w_{n-N}^2 \cdots w_{n-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2} \right\}. \tag{7}$$

Proof. By Lemma 2.1,

$$[T^*, T] = \left(|a|^2 - |c|^2\right) \left[S^{M*}, S^M\right] + \left(|b|^2 - |d|^2\right) \left[S^{N*}, S^N\right].$$

Set

$$\alpha = |a|^2 - |c|^2 \text{ and } \beta = |d|^2 - |b|^2.$$

By (1), $\left[S^{M*}, S^M\right]$ and $\left[S^{N*}, S^N\right]$ are diagonal operators with respect the standard bases $\{e_n\}$. Hence $[T^*, T] \ge 0$ if and only if for $0 \le n < M$,

$$\alpha w_n^2 \cdots w_{n+M-1}^2 - \beta w_n^2 \cdots w_{n+N-1}^2 \ge 0, \tag{8}$$

for $M \le n < N$,

$$\alpha \left[w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2\right] - \beta w_n^2 \cdots w_{n+N-1}^2 \ge 0, \tag{9}$$

and for $n \ge N$,

$$\alpha \left[w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2\right] - \beta \left[w_n^2 \cdots w_{n+N-1}^2 - w_{n-N}^2 \cdots w_{n-1}^2\right] \ge 0, \tag{10}$$

If $\alpha > 0$, then $\beta > 0$ since $a\bar{b} = c\bar{d}$. In this case, inequality (8) is the same as

$$\alpha - \beta w_{n+M}^2 \cdots w_{n+N-1}^2 \ge 0 \quad (0 \le n < M).$$

Since w_n is increasing, (8) holds if and only it holds at $n = M - 1$. That is,

$$\alpha - \beta w_{2M-1}^2 \cdots w_{M+N-2}^2 \ge 0 \quad \text{or} \quad \frac{\alpha}{\beta} \ge E_1 \tag{11}$$

Inequalities (9) and (10) are the same as

$$\frac{\alpha}{\beta} \geq E_2 \quad \text{and} \quad \frac{\alpha}{\beta} \geq E_3,$$

respectively. If $\alpha < 0$, then $\beta < 0$ since $a\overline{b} = c\overline{d}$. In this case, (8) holds if and only it holds at $n = 0$. That is,

$$\alpha - \beta w_M^2 \cdots w_{N-1}^2 \geq 0 \quad \text{or} \quad \frac{-\alpha}{-\beta} \leq F_1.$$

Inequalities (9) and (10) are the same as

$$\frac{-\alpha}{-\beta} \leq F_2 \quad \text{and} \quad \frac{-\alpha}{-\beta} \leq F_3,$$

respectively. ■

Remark 2.3 *In the definitions of E_2 and F_2, we adopt the convention that $1/0 = \infty$. In the definitions of E_3 and F_3, $0/0$ is deleted from the list of numbers for* sup *or* inf*.*

It turns out that $\min\{F_1, F_2, F_3\} = F_1$ which leads to the following theorem.

Theorem 2.4 *Let S be a hyponormal weighted shift with weight sequence $\{w_n\}$. Assume $a\overline{b} = c\overline{d}$. Let M and N be two positive integers with $M < N$ and $|a| < |c|$. Then*

$$T = aS^M + bS^N + \overline{c}S^{*M} + \overline{d}S^{*N}$$

is hyponormal if and only if

$$\frac{|c|^2 - |a|^2}{|b|^2 - |d|^2} \leq w_M^2 \cdots w_{N-1}^2.$$

Proof. Since S is a hyponormal weighted shift, we may assume that $0 \leq \omega_n \leq 1$. By Theorem 2.2, it suffices to show that

$$\min\{F_1, F_2, F_3\} = F_1.$$

For $n > N$, $m \in \mathbb{N}$, let

$$A(n,m) := w_n^2 \cdots w_{n+m-1}^2$$

and

$$\begin{aligned} W_n &\equiv \frac{w_n^2 \cdots w_{n+N-1}^2 - w_{n-N}^2 \cdots w_{n-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2} \\ &= \frac{A(n,M)A(n+M,N-M) - A(n-N,N-M)A(n-M,M)}{A(n,M) - A(n-M,M)} \\ &= \frac{A(n,M) - A(n-M,M)\frac{A(n-N,N-M)}{A(n+M,N-M)}}{\frac{A(n,M)-A(n-M,M)}{A(n+M,N-M)}}. \end{aligned}$$

Since ω_n is increasing, it follows that

$$\frac{A(n-N,N-M)}{A(n+M,N-M)} \leq 1 \text{ for all } n \geq N.$$

Thus we have

$$W_n \geq \frac{A(n,M) - A(n-M,M)}{\frac{A(n,M)-A(n-M,M)}{A(n+M,N-M)}} = A(n+M,N-M) \geq A(M,N-M),$$

which implies that

$$F_3 = \inf_{n \geq N} \{W_n\} \geq A(M,N-M) = F_1. \tag{12}$$

On the other hand, since $\omega_n \leq 1$, $A(i,j) \leq 1$ for all $i, j \in \mathbb{N}$. Thus for $M \leq n < N$,

$$\begin{aligned}\frac{w_n^2 \cdots w_{n+N-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2} &= \frac{A(n,N)}{A(n,M) - A(n-M,M)} \\ &\geq \frac{A(n,N)}{A(n,M)} = A(n+M,N-M).\end{aligned}$$

Thus

$$F_2 \geq A(2M,N-M) \geq A(M,N-M) = F_1. \tag{13}$$

It thus follows from (12) and(13) that

$$\min\{F_1, F_2, F_3\} = F_1.$$

This completes the proof. ■

The assumption on S of course can be replaced by more general conditions. For example, we have the following theorem. But it seems hard to find natural examples where $E_3 < F_1$ in the case $\{w_n\}$ is decreasing.

Theorem 2.5 *Let S be a weighted shift whose weight sequence $\{w_n\}$ is decreasing. Assume $a\bar{b} = c\bar{d}$. Let M and N be two positive integers with $M < N$ and $|a| \neq |c|$. Let E_3 and F_1 be defined by (4) and (5). Then*

$$T = aS^M + bS^N + \bar{c}S^{*M} + \bar{d}S^{*N}$$

is hyponormal if and only if $|a| < |c|$ and

$$E_3 \leq \frac{|c|^2 - |a|^2}{|b|^2 - |d|^2} \leq F_1.$$

Proof. We use the same notations as in the proof of Theorem 2.2. If $\alpha > 0$ and $\beta > 0$, then inequality (9) is impossible because $\{w_n\}$ is decreasing. In the case $\alpha < 0$ and $\beta < 0$, inequality (8) is equivalent to

$$\frac{-\alpha}{-\beta} \leq F_1.$$

Inequality (9) is automatically satisfied. Inequality (10) is the same as

$$\frac{-\alpha}{-\beta} \geq E_3.$$

The proof is complete. ■

Let S be the bilateral weighted shift with positive weight sequence $\{w_n\}_{n=-\infty}^{\infty}$. That is,

$$\begin{aligned} Se_n &= w_n e_{n+1} \quad (-\infty < n < \infty), \\ S^* e_{n+1} &= w_n e_n \quad (-\infty < n < \infty). \end{aligned} \tag{14}$$

We have the following simpler result.

Theorem 2.6 *Let S be a hyponormal bilateral weighted shift with weight sequence $\{w_n\}$. Assume $a\bar{b} = c\bar{d}$. Let M and N be two positive integers with $M < N$. Then*

$$T = aS^M + bS^N + \bar{c}S^{*M} + \bar{d}S^{*N}$$

is hyponormal if and only if one of the following two statement holds.
(i) *If $|a| > |c|$, then*

$$\frac{|a|^2 - |c|^2}{|d|^2 - |b|^2} \geq \max_{-\infty<n<\infty} \left\{ \frac{w_n^2 \cdots w_{n+N-1}^2 - w_{n-N}^2 \cdots w_{n-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2} \right\}.$$

(ii) *If $|a| < |c|$, then*

$$\frac{|c|^2 - |a|^2}{|b|^2 - |d|^2} \leq \min_{-\infty<n<\infty} \left\{ \frac{w_n^2 \cdots w_{n+N-1}^2 - w_{n-N}^2 \cdots w_{n-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2} \right\}.$$

Proof. By Lemma 2.1,

$$[T^*, T] = \left(|a|^2 - |c|^2\right) \left[S^{M*}, S^M\right] + \left(|b|^2 - |d|^2\right) \left[S^{N*}, S^N\right].$$

Set

$$\alpha = |a|^2 - |c|^2 \text{ and } \beta = |d|^2 - |b|^2.$$

Since $\left[S^{M*}, S^M\right]$ and $\left[S^{N*}, S^N\right]$ are diagonal operators by (14), $[T^*, T] \geq 0$ if and only if for $-\infty < n < \infty$,

$$\alpha \left[w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2\right] - \beta \left[w_n^2 \cdots w_{n+N-1}^2 - w_{n-N}^2 \cdots w_{n-1}^2\right] \geq 0.$$

The result now follows. ■

3 Hyponormal Toeplitz operators

We apply Theorem 2.2 on weighted Hardy spaces $\mathcal{K}_\rho(\mathbb{D})$ ($\rho > 1$). The space $\mathcal{K}_\rho(\mathbb{D})$ has reproducing kernel

$$K(z, \lambda) = \frac{1}{(1 - z\bar{\lambda})^\rho},$$

$$w_n^2 = \frac{n+1}{n+\rho} \quad (n \geq 0),$$

where $\rho > 1$. It is clear that w_n is increasing in n and the multiplication operator M_z or the shift S on $\mathcal{K}_\rho(\mathbb{D})$ is hyponormal. When $\rho = 1$, we have the Hardy space. When $\rho = 2$, we have the Bergman space. When $\rho \geq 2$, we have weighted Bergman spaces, which are the Hilbert spaces of analytic functions on $\mathbb{D}$ with norm

$$\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 (\rho - 1)(1 - |z|^2)^{\rho-2} dA(z),$$

where $dA(z)$ is the normalized area measure on $\mathbb{D}$. For $1 < \rho < 2$, we have weighted Hardy spaces. Note that

$$w_n^2 \cdots w_{n+\ell}^2 = \frac{\Gamma(n+\ell+2)\Gamma(n+\rho)}{\Gamma(n+1)\Gamma(n+\ell+\rho+1)},$$

The following result follows immediately from Theorem 2.4.

Theorem 3.1 *Assume $a\bar{b} = c\bar{d}$. Let M and N be two positive integers with $M < N$ and $|a| < |c|$. For $\varphi = az^M + bz^N + \overline{cz}^M + \overline{d}\overline{z}^N$, T_φ on $\mathcal{K}_\rho(\mathbb{D})$ is hyponormal if and only if*

$$\frac{|c|^2 - |a|^2}{|b|^2 - |d|^2} \le \frac{\Gamma(N+1)\Gamma(M+\rho)}{\Gamma(M+1)\Gamma(N+\rho)}.$$

If $M = N - 1$, then the right-hand side of the above inequality reduces to $N/(N + \rho - 1)$, which recaptures Theorem 4 in [21]. We next prove that $\max\{E_1, E_2, E_3\} = E_3 = N^2/M^2$ if ρ is a positive integer which leads to the following theorem.

Theorem 3.2 *Assume $a\bar{b} = c\bar{d}$. Let M and N be two positive integers with $M < N$ and $|a| > |c|$. If ρ is an positive integer, then for $\varphi = az^M + bz^N + \overline{cz}^M + \overline{d}\overline{z}^N$, T_φ on $\mathcal{K}_\rho(\mathbb{D})$ is hyponormal if and only if*

$$\frac{|a|^2 - |c|^2}{|d|^2 - |b|^2} \ge \frac{N^2}{M^2}. \tag{15}$$

Proof. We first note in general

$$E_1 \le E_2.$$

Now let us look at the weighted Bergman space $\mathcal{K}_\rho(\mathbb{D})$:

$$w_n^2 = \frac{n+1}{n+\rho} \quad (n \ge 0).$$

Assume ρ is a positive integer ≥ 2. Then

$$w_n^2 \cdots w_{n+\ell}^2 = \prod_{i=1}^{\rho-1} \frac{n+i}{n+l+1+i},$$

We will prove

$$\max\{E_2, E_3\} = E_3 = \frac{N^2}{M^2}.$$

Let us look at E_3 more closely:

$$E_3 = \sup\left\{c_n := \frac{w_n^2 \cdots w_{n+N-1}^2 - w_{n-N}^2 \cdots w_{n-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2}, n \ge N\right\}, \tag{16}$$

$$c_n = \left\{\frac{\prod_{i=1}^{\rho-1}(n+i)}{\prod_{i=1}^{\rho-1}(n+N+i)} - \frac{\prod_{i=1}^{\rho-1}(n-N+i)}{\prod_{i=1}^{\rho-1}(n+i)}\right\} \div \left\{\frac{\prod_{i=1}^{\rho-1}(n+i)}{\prod_{i=1}^{\rho-1}(n+M+i)} - \frac{\prod_{i=1}^{\rho-1}(n-M+i)}{\prod_{i=1}^{\rho-1}(n+i)}\right\}.$$

Note that

$$\begin{aligned}\frac{\prod_{i=1}^{\rho-1}(n+i)}{\prod_{i=1}^{\rho-1}(n+N+i)} - \frac{\prod_{i=1}^{\rho-1}(n-N+i)}{\prod_{i=1}^{\rho-1}(n+i)} &= \frac{\prod_{i=1}^{\rho-1}(n+i)^2 - \prod_{i=1}^{\rho-1}\left[(n+i)^2 - N^2\right]}{\prod_{i=1}^{\rho-1}(n+N+i)\prod_{i=1}^{\rho-1}(n+i)} \\ &= \frac{N^2\sum_{k=1}^{\rho-1}\prod_{i=1}^{\rho-1-k}(n+i)^2\prod_{\rho-k+1}^{\rho-1}\left[(n+i)^2 - N^2\right]}{\prod_{i=1}^{\rho-1}(n+N+i)\prod_{i=1}^{\rho-1}(n+i)},\end{aligned}$$

where in the third equation above we use

$$\prod_{i=1}^{\rho-1}a_i - \prod_{i=1}^{\rho-1}b_i = \sum_{k=1}^{\rho-1}\left(\prod_{i=1}^{\rho-1-k}a_i\right)(a_k - b_k)\left(\prod_{i=\rho-k+1}^{\rho-1}b_i\right).$$

Therefore

$$\begin{aligned} c_n &= \frac{N^2}{M^2} \frac{\prod_{i=1}^{\rho-1}(n+M+i)}{\prod_{i=1}^{\rho-1}(n+N+i)} \frac{\sum_{k=1}^{\rho-1}\prod_{i=1}^{\rho-1-k}(n+i)^2 \prod_{\rho-k+1}^{\rho-1}\left[(n+i)^2-N^2\right]}{\sum_{k=1}^{\rho-1}\prod_{i=1}^{\rho-1-k}(n+i)^2 \prod_{\rho-k+1}^{\rho-1}\left[(n+i)^2-M^2\right]} \\ &\leq \frac{N^2}{M^2} \cdot 1 \cdot 1. \end{aligned}$$

More precisely, c_n is increasing in n and

$$\lim_{n\to\infty} c_n = \frac{N^2}{M^2}.$$

Next we need to show

$$E_2 \leq \frac{N^2}{M^2}.$$

For $M \leq n < N$, let

$$\begin{aligned} d_n &:= \frac{w_n^2 \cdots w_{n+N-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2} \\ &= \left\{ \frac{\prod_{i=1}^{\rho-1}(n+i)}{\prod_{i=1}^{\rho-1}(n+N+i)} \right\} \div \left\{ \frac{\prod_{i=1}^{\rho-1}(n+i)}{\prod_{i=1}^{\rho-1}(n+M+i)} - \frac{\prod_{i=1}^{\rho-1}(n-M+i)}{\prod_{i=1}^{\rho-1}(n+i)} \right\}. \end{aligned}$$

Then

$$d_n = \frac{\prod_{i=1}^{\rho-1}(n+M+i)}{\prod_{i=1}^{\rho-1}(n+N+i)} \cdot \frac{\prod_{i=1}^{\rho-1}(n+i)^2}{\prod_{i=1}^{\rho-1}(n+i)^2 - \prod_{i=1}^{\rho-1}\left[(n+i)^2 - M^2\right]}.$$

Note that

$$\begin{aligned} \prod_{i=1}^{\rho-1}(n+i)^2 - \prod_{i=1}^{\rho-1}\left[(n+i)^2-M^2\right] &\geq \prod_{i=1}^{\rho-1}(n+i)^2 - \left[(n+1)^2-M^2\right]\prod_{i=2}^{\rho-1}(n+i)^2 \\ &= M^2 \prod_{i=2}^{\rho-1}(n+i)^2. \end{aligned}$$

So

$$d_n \leq \frac{\prod_{i=1}^{\rho-1}(n+M+i)}{\prod_{i=1}^{\rho-1}(n+N+i)} \cdot \frac{\prod_{i=1}^{\rho-1}(n+i)^2}{M^2\prod_{i=2}^{\rho-1}(n+i)^2} \leq \frac{(n+1)^2}{M^2} \leq \frac{N^2}{M^2}.$$

The proof is complete. ■

In the case ρ ($\rho > 1$) is not a positive integer, we can show the above condition is a necessary condition for the hyponormality of T_φ on $\mathcal{K}_\rho(\mathbb{D})$. Namely, it can be shown that

$$\lim_{n\to\infty} c_n = \frac{N^2}{M^2}.$$

This necessary condition on the Bergman space ($\rho = 2$) also follows from a necessary condition for a general hyponormal Toeplitz operator on the Bergman space [2] [29]. But it seems difficult to show that c_n is increasing in n for general $\rho > 1$, even though a numerical computation suggests this is true for big ρ. In fact, for ρ close to 1 and $M \leq n < N$, we have

$$\begin{aligned} d_n &:= \frac{w_n^2 \cdots w_{n+N-1}^2}{w_n^2 \cdots w_{n+M-1}^2 - w_{n-M}^2 \cdots w_{n-1}^2} \\ &= \left\{ \prod_{i=0}^{N-1} \frac{n+i+1}{n+i+\rho} \right\} \div \left\{ \prod_{i=1}^{M-1} \frac{n+i+1}{n+i+\rho} - \prod_{i=1}^{M-1} \frac{n-M+i+1}{n-M+i+\rho} \right\} \\ &\approx \frac{C}{\rho-1} \text{ or } \lim_{\rho\to 1^+} (\rho-1)\, d_n = C \end{aligned}$$

for some constant C since

$$\frac{n+i+1}{n+i+\rho}-\frac{n-M+i+1}{n-M+i+\rho}=\frac{M(\rho-1)}{(n+i+\rho)\,(n-M+i+\rho)}.$$

Thus, for ρ close to 1, $E_2 > N^2/M^2$ and the necessary condition (15) is not sufficient.

Therefore, we conjecture that there exists ρ_0 such that for $\rho \geq \rho_0$, (15) is a necessary and sufficient condition for the hyponormality of T_φ on $\mathcal{K}_\rho(\mathbb{D})$, and for $1<\rho<\rho_0$, (15) is only a necessary condition.

4 Hyponormal operators close to several weighted shifts

We first introduce a tuple of d-variables unilateral weighted shifts as in [22]. Let Z_+ be the set of nonnegative integers and

$$Z_+^d = \{\alpha = (\alpha_1, \cdots \alpha_d) : \alpha_i \in Z_+, 1 \leq i \leq d\}.$$

We write $\alpha \geq 0$ if $\alpha \in Z_+^d$. Let $\varepsilon_i = (0, \cdots, 1, \cdots, 0)$ be the multi-index having $\alpha_j = 1$ or 0 according as $j = i$ or otherwise and 0 be the multi-index $(0, 0, , \cdots, 0)$. For two multi-indexes α and β, $\alpha \geq \beta$ means $\alpha - \beta \geq 0$, and $\alpha > \beta$ means $\alpha \geq \beta$ but $\alpha \neq \beta$. Let $l^2(Z_+^d)$ be the complex Hilbert spaces with standard bases $\{e_\alpha : \alpha \in Z_+^d\}$, respectively. Let $\{w_{\alpha,i} : \alpha \in Z_+^d, i = 1, \cdots, d\}$ be a bounded set of positive numbers such that

$$w_{\alpha,i}w_{\alpha+\varepsilon_i,j} = w_{\alpha,j}w_{\alpha+\varepsilon_j,i} \quad (\alpha \in Z_+^d,\ 1 \leq i, j \leq d). \tag{17}$$

Definition 4.1 *A tuple of d-variables unilateral weighted shifts is a family of d bounded operators on $l^2(Z_+^d)$, $S = (S_1, \cdots, S_d)$ defined by*

$$S_i e_\alpha = w_{\alpha,i} e_{\alpha+\varepsilon_i} \quad (\alpha \in Z_+^d,\ i = 1, \cdots, d). \tag{18}$$

Note that the condition (17) on $w_{\alpha,i}$ implies that S is a tuple of commuting operators. Note also

$$\begin{aligned} &S_i^* e_\alpha = w_{\alpha-\varepsilon_i,i} e_{\alpha-\varepsilon_i} \text{ if } \alpha_i \geq 1 \quad (i = 1, \cdots, d),\\ &S_i^* e_\alpha = 0 \text{ if } \alpha_i = 0 \quad (i = 1, \cdots, d). \end{aligned}$$

For a multi-index $N = (N_1, \cdots N_d)$,

$$S^N = S_1^{N_1} S_2^{N_2} \cdots S_d^{N_d} \quad \text{and} \quad S^{*N} = S_1^{*N_1} S_2^{*N_2} \cdots S_d^{*N_d}.$$

Set

$$u_{\alpha,N} = \prod_{1\leq i\leq d}\prod_{0\leq k_i<N_i} w^2_{\alpha+k_i\varepsilon_i,i} \quad (\alpha \geq 0), \tag{19}$$

$$v_{\alpha,N} = \prod_{1\leq i\leq d}\prod_{1\leq k_i\leq N_i} w^2_{\alpha-k_i\varepsilon_i,i} \quad (\alpha \geq N). \tag{20}$$

Then

$$
\begin{aligned}
&S^{*N}S^{N}e_{\alpha}=u_{\alpha,N}e_{\alpha}\quad(\alpha\geq 0),\\
&S^{N}S^{*N}e_{\alpha}=0\quad(\alpha\geq 0\text{ and }\alpha\ngeq N),\\
&S^{N}S^{*N}e_{\alpha}=v_{\alpha,N}e_{\alpha}\quad(\alpha\geq N).
\end{aligned}\tag{21}
$$

For multi-index α, the set of $\{\alpha : \alpha \ngeq N\}$ is much larger than the set $\{\alpha : \alpha < N\}$ since $\{\alpha : \alpha \ngeq N\}$ is an infinite set while $\{\alpha : \alpha < N\}$ is a finite set.

Definition 4.2 *We say S is a tuple of hyponormal weighted shifts if each S_i is hyponormal for $i = 1, \cdots, d$.*

It is clear that S_i is hyponormal if and only if for each $\alpha \geq 0$, the sequence $\{w_{\alpha+n\varepsilon_i,i}\}_{n=0}^{\infty}$ is increasing. The following theorem is the multivariable version of Theorem 2.2. To avoid triviality, for the multi-index M, we shall assume $(M_1, \cdots, M_d) \geq (1, \cdots, 1)$.

Theorem 4.3 *Let S be a tuple of hyponormal weighted shifts. Assume $a\overline{b} = c\overline{d}$. Let M and N be two multi-indexes with $M < N$. Then*

$$T = aS^M + bS^N + \overline{c}S^{*M} + \overline{d}S^{*N}$$

is hyponormal if and only if one of the following two statements holds.
(i) *If $|a| > |c|$, then*

$$\frac{|a|^2 - |c|^2}{|d|^2 - |b|^2} \geq \max\{E_1, E_2, E_3\},$$

where

$$E_1 = \sup\left\{\frac{u_{\alpha,N}}{u_{\alpha,M}} : \alpha \ngeq M\right\},\tag{22}$$

$$E_2 = \sup\left\{\frac{u_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}} : \alpha \geq M \text{ and } \alpha \ngeq N\right\},\tag{23}$$

$$E_3 = \sup_{\alpha\geq N}\left\{\frac{u_{\alpha,N} - v_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}}\right\}.\tag{24}$$

(ii) *If $|a| < |c|$, then*

$$\frac{|c|^2 - |a|^2}{|b|^2 - |d|^2} \leq \min\{F_1, F_2, F_3\},$$

where

$$F_1 = \inf\left\{\frac{u_{\alpha,N}}{u_{\alpha,M}} : \alpha \ngeq M\right\},\tag{25}$$

$$F_2 = \inf\left\{\frac{u_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}} : \alpha \geq M \text{ and } \alpha \ngeq N\right\},\tag{26}$$

$$F_3 = \inf_{\alpha\geq N}\left\{\frac{u_{\alpha,N} - v_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}}\right\}.\tag{27}$$

Proof. The proof is similar by using (21) and Lemma 2.1, and by noting that $u_{\alpha,M} - v_{\alpha,M} \geq 0$ and $u_{\alpha,N} - v_{\alpha,N} \geq 0$. ■

Remark 4.4 *In the case S is a single hyponormal weighted shift, the assumption $M < N$ is not a restriction by symmetry. In the case S is a tuple of hyponormal weighted shifts, the assumption $M < N$ is a restriction.*

The next result is the multivariable version of Theorem 2.4 with an additional assumption (28) on the weight sequences $\left\{w_{\alpha,i} : \alpha \in Z_+^d, i = 1, \cdots, d\right\}$ which is automatically satisfied in the case $d = 1$. Without this assumption, the result may not hold (see Proposition 5.3 below).

Theorem 4.5 *Assume*

$$w_{\alpha,i} \geq w_{\beta,i} \text{ whenever } \alpha \geq \beta \text{ for } i = 1, \cdots, d. \tag{28}$$

Then S is a tuple of hyponormal weighted shifts. Assume $a\bar{b} = c\bar{d}$. Let M and N be two multi-indexes with $M < N$ and $|a| < |c|$. Then

$$T = aS^M + bS^N + \bar{c}S^{*M} + \bar{d}S^{*N}$$

is hyponormal if and only if

$$\frac{|c|^2 - |a|^2}{|b|^2 - |d|^2} \leq \frac{u_{0,N}}{u_{0,M}}.$$

Proof. The proof is similar but slightly more complicated than the proof of Theorem 2.4 because of multi-indexes, the additional assumption (28) is needed for this proof. We include the details for clarity. By (19) and the assumption (28),

$$\frac{u_{\alpha,N}}{u_{\alpha,M}} = \prod_{1 \leq i \leq d} \prod_{M_i \leq k_i < N_i} w^2_{\alpha + k_i \varepsilon_i, i} \geq \prod_{1 \leq i \leq d} \prod_{M \leq k_i < N_i} w^2_{k_i \varepsilon_i, i} = \frac{u_{0,N}}{u_{0,M}}.$$

That is,

$$F_1 = \inf\left\{\frac{u_{\alpha,N}}{u_{\alpha,M}} : \alpha \ngeq M\right\} = \frac{u_{0,N}}{u_{0,M}}.$$

Next we will show $\min\{F_1, F_2, F_3\} = F_1$. Note that

$$\begin{aligned} F_2 &= \inf\left\{\frac{u_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}} : \alpha \geq M \text{ and } \alpha \ngeq N\right\} \\ &\geq \inf\left\{\frac{u_{\alpha,N}}{u_{\alpha,M}} : \alpha \geq M \text{ and } \alpha \ngeq N\right\} \geq \frac{u_{0,N}}{u_{0,M}}, \end{aligned}$$

where the last inequality again follows from (28). For $\alpha \geq N$, we write

$$\begin{aligned} c_\alpha &:= \frac{u_{\alpha,N} - v_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}} = \frac{u_{\alpha,M} u_{\alpha+M,N-M} - v_{\alpha,M} v_{\alpha-M,N-M}}{u_{\alpha,M} - v_{\alpha,M}} \\ &= \left(u_{\alpha,M} - v_{\alpha,M} \frac{v_{\alpha-M,N-M}}{u_{\alpha+M,N-M}}\right) \div \left(\frac{u_{\alpha,M} - v_{\alpha,M}}{u_{\alpha+M,N-M}}\right) \end{aligned}$$

By (19), (20) and (28),

$$\frac{v_{\alpha-M,N-M}}{u_{\alpha+M,N-M}} \leq 1.$$

Therefore

$$\begin{aligned} c_\alpha &\geq (u_{\alpha,M} - v_{\alpha,M}) \div \left(\frac{u_{\alpha,M} - v_{\alpha,M}}{u_{\alpha+M,N-M}} \right) \\ &= u_{\alpha+M,N-M} = \frac{u_{\alpha,N}}{u_{\alpha,M}} \geq \frac{u_{0,N}}{u_{0,M}}. \end{aligned}$$

This proves $\min\{F_1, F_2, F_3\} = \frac{u_{0,N}}{u_{0,M}}$. ■

When S is double commuting, we have the following more explicit condition for the hyponormality of T in the case $|a| > |c|$. We first introduce some notation. We will assume

$$w_{\alpha,i} = \eta^{(i)}_{\alpha_i},$$

where for each $1 \leq i \leq d$, $\eta^{(i)}_{\alpha_i}$ is a sequence indexed by $\alpha_i = 0, 1, 2, \cdots$. The S is commuting since for $i \neq j$,

$$w_{\alpha,i} w_{\alpha+\varepsilon_i,j} = \eta^{(i)}_{\alpha_i} \eta^{(j)}_{\alpha_j} \text{ and } w_{\alpha,j} w_{\alpha+\varepsilon_j,i} = \eta^{(j)}_{\alpha_j} \eta^{(i)}_{\alpha_i}.$$

The S is double commuting since for $i \neq j, \alpha_j \geq 1$,

$$\begin{aligned} S_j^* S_i e_\alpha &= S_j^* w_{\alpha,i} e_{\alpha+\varepsilon_i} = w_{\alpha,i} w_{\alpha+\varepsilon_i-\varepsilon_j,j} e_{\alpha+\varepsilon_i-\varepsilon_j} = \eta^{(i)}_{\alpha_i} \eta^{(j)}_{\alpha_j-1} e_{\alpha+\varepsilon_i-\varepsilon_j}, \\ S_i S_j^* e_\alpha &= S_i w_{\alpha-\varepsilon_j,j} e_{\alpha-\varepsilon_j} = w_{\alpha-\varepsilon_j,j} w_{\alpha-\varepsilon_j,i} e_{\alpha-\varepsilon_j+\varepsilon_i} = \eta^{(j)}_{\alpha_j-1} \eta^{(i)}_{\alpha_i} e_{\alpha-\varepsilon_j+\varepsilon_i}, \end{aligned}$$

and when $\alpha_j = 0$, $S_j^* S_i e_\alpha = S_i S_j^* e_\alpha = 0$. Furthermore for $\alpha \geq 0$,

$$u_{\alpha,N} = \xi^{(1)}_{\alpha_1,N_1} \cdots \xi^{(d)}_{\alpha_d,N_d}, \quad \text{where } \xi^{(i)}_{\alpha_i,N_i} = \prod_{j=0}^{N_i-1} \left(\eta^{(i)}_{\alpha_i+j} \right)^2$$

and for $\alpha \geq N$,

$$v_{\alpha,N} = \theta^{(1)}_{\alpha_1,N_1} \cdots \theta^{(d)}_{\alpha_d,N_d}, \quad \text{where } \theta^{(i)}_{\alpha_i,N_i} = \prod_{j=\alpha_i-N_i}^{\alpha_i-1} \left(\eta^{(i)}_j \right)^2.$$

Theorem 4.6 *Let S be a tuple of hyponormal weighted shifts. Assume that*
(i) $w_{\alpha,i} = \eta^{(i)}_{\alpha_i}$ *for some sequences* $\eta^{(i)}_{\alpha_i}$.
(ii) $\lim_{n\to\infty} \eta^{(i)}_n = 1 \quad (i = 1, \cdots, d)$.
Then S is double commuting. If $|a| > |c|$, then T is hyponormal if and only if

$$\frac{|a|^2 - |c|^2}{|d|^2 - |b|^2} \geq \max_{1\leq i\leq d} \{G_i\}, \tag{29}$$

where

$$E_{1i} = \sup\left\{ \frac{\xi^{(i)}_{\alpha_i,N_i}}{\xi^{(i)}_{\alpha_i,M_i}} : 0 \le \alpha_i < M_i \right\},$$

$$E_{2i} = \sup\left\{ \frac{\xi^{(i)}_{\alpha_i,N_i}}{\xi^{(i)}_{\alpha_i,M_i} - \theta^{(i)}_{\alpha_i,M_i}} : M_i \le \alpha_i < N_i \right\},$$

$$E_{3i} = \sup_{\alpha_i \ge N_i}\left\{ \frac{\xi^{(i)}_{\alpha_i,N_i} - \theta^{(i)}_{\alpha_i,N_i}}{\xi^{(i)}_{\alpha_i,M_i} - \theta^{(i)}_{\alpha_i,M_i}} \right\},$$

$$G_i = \max\{E_{1i}, E_{2i}, E_{3i}\} = \max\{E_{2i}, E_{3i}\}.$$

Proof. We prove the result for $d = 2$ since the proof for $d > 2$ is similar or by induction. We first note that for any $\alpha \ngeq M$, there is a $\beta \ge M$ and $\beta \ngeq N$ such that $\beta \ge \alpha$. Thus

$$\frac{u_{\beta,N}}{u_{\beta,M} - v_{\beta,M}} \ge \frac{u_{\beta,N}}{u_{\beta,M}} \ge \frac{u_{\alpha,N}}{u_{\alpha,M}},$$

where the last inequality follows from the assumption $N > M$ and $\eta^{(i)}_{\alpha_i}$ is increasing in α_i. Therefore

$$E_1 = \sup\left\{ \frac{u_{\alpha,N}}{u_{\alpha,M}} : \alpha \ngeq M \right\} \le E_2 = \sup\left\{ \frac{u_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}} : \alpha \ge M \text{ and } \alpha \ngeq N \right\}$$

and we need to study only $\max\{E_2, E_3\}$.

We now prove the necessity of (29). We first prove $E_2 \ge \max\{E_{21}, E_{22}\}$. There are two cases for $\alpha \ge M$ and $\alpha \ngeq N$.

Case 1: Suppose $M_1 \le \alpha_1 < N_1$ and $\alpha_2 \ge M_2$. Note that

$$\begin{aligned}\frac{u_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}} &= \frac{\xi^{(1)}_{\alpha_1,N_1}\xi^{(2)}_{\alpha_2,N_2}}{\xi^{(1)}_{\alpha_1,M_1}\xi^{(2)}_{\alpha_2,M_2} - \theta^{(1)}_{\alpha_1,M_1}\theta^{(2)}_{\alpha_2,M_2}} \\ &= \frac{\xi^{(1)}_{\alpha_1,N_1}}{\xi^{(1)}_{\alpha_1,M_1}\left(\xi^{(2)}_{\alpha_2,M_2}/\xi^{(2)}_{\alpha_2,N_2}\right) - \theta^{(1)}_{\alpha_1,M_1}\left(\theta^{(2)}_{\alpha_2,M_2}/\xi^{(2)}_{\alpha_2,N_2}\right)}\end{aligned}$$

and

$$\lim_{\alpha_2\to\infty} \frac{\xi^{(2)}_{\alpha_2,M_2}}{\xi^{(2)}_{\alpha_2,N_2}} = \lim_{\alpha_2\to\infty} \frac{1}{\left(\eta^{(2)}_{\alpha_2+M_2}\right)^2 \cdots \left(\eta^{(2)}_{\alpha_2+N_2-1}\right)^2} = 1, \quad \lim_{\alpha_2\to\infty} \frac{\theta^{(2)}_{\alpha_2,M_2}}{\xi^{(2)}_{\alpha_2,N_2}} = 1.$$

Therefore

$$\lim_{\alpha_2\to\infty} \frac{u_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}} = \frac{\xi^{(1)}_{\alpha_1,N_1}}{\xi^{(1)}_{\alpha_1,M_1} - \theta^{(1)}_{\alpha_1,M_1}}$$

and

$$E_2 \ge E_{21}.$$

Case 2: Suppose $M_2 \le \alpha_2 < N_2$ and $\alpha_1 \ge M_1$. Then, similarly, we also have

$$E_2 \ge E_{22.}$$

The proof of $E_3 \geq \max\{E_{31}, E_{32}\}$ is also similar.

To prove the sufficiency of (29), we need to show $\max\{E_{2i}, E_{3i} : 1 \leq i \leq d\} = \max\{E_2, E_3\}$. A direct proof of this fact seems rather difficult. Instead, assume (29) holds, we will use Lemma 2.1 to prove T is hyponormal. By Lemma 2.1,

$$[T^*, T] = \delta\left[S^{M*}, S^M\right] - \beta\left[S^{N*}, S^N\right],$$

where

$$\delta = |a|^2 - |c|^2 \text{ and } \beta = |d|^2 - |b|^2.$$

Note that, since S is double commuting,

$$\begin{aligned}\left[S^{M*}, S^M\right] &= S_1^{*M_1} S_1^{M_1} S_2^{*M_2} S_2^{M_2} - S_1^{M_1} S_1^{*M_1} S_2^{M_2} S_2^{*M_2} \\ &= S_1^{*M_1} S_1^{M_1} S_2^{*M_2} S_2^{M_2} - S_1^{M_1} S_1^{*M_1} S_2^{*M_2} S_2^{M_2} \\ &+ S_1^{M_1} S_1^{*M_1} S_2^{*M_2} S_2^{M_2} - S_1^{M_1} S_1^{*M_1} S_2^{M_2} S_2^{*M_2} \\ &= \left[S_1^{*M_1}, S_1^{M_1}\right] S_2^{*M_2} S_2^{M_2} + S_1^{M_1} S_1^{*M_1} \left[S_2^{*M_2}, S_2^{M_2}\right].\end{aligned}$$

Thus

$$\begin{aligned}[T^*, T] = {} & \delta\left[S_1^{*M_1}, S_1^{M_1}\right] S_2^{*M_2} S_2^{M_2} - \beta\left[S_1^{*N_1}, S_1^{N_1}\right] S_2^{*N_2} S_2^{N_2} \\ & + \delta\left[S_2^{*M_2}, S_2^{M_2}\right] S_1^{M_1} S_1^{*M_1} - \beta\left[S_2^{*N_2}, S_2^{N_2}\right] S_1^{N_1} S_1^{*N_1}.\end{aligned}$$

By assumption, $M < N$ and S_1 and S_2 are contractions,

$$S_2^{*M_2} S_2^{M_2} \geq S_2^{*N_2} S_2^{N_2} \text{ and } S_1^{M_1} S_1^{*M_1} \geq S_1^{N_1} S_1^{*N_1}.$$

By condition (29) and Theorem 2.2 (i), we have

$$\delta\left[S_1^{*M_1}, S_1^{M_1}\right] \geq \beta\left[S_1^{*N_1}, S_1^{N_1}\right] \text{ and } \delta\left[S_2^{*M_2}, S_2^{M_2}\right] \geq \beta\left[S_2^{*N_2}, S_2^{N_2}\right].$$

Since S is double commuting, it follows that

$$\delta\left[S_1^{*M_1}, S_1^{M_1}\right] S_2^{*M_2} S_2^{M_2} \geq \beta\left[S_1^{*N_1}, S_1^{N_1}\right] S_2^{*N_2} S_2^{N_2},$$

and similarly,

$$\delta\left[S_2^{*M_2}, S_2^{M_2}\right] S_1^{M_1} S_1^{*M_1} \geq \beta\left[S_2^{*N_2}, S_2^{N_2}\right] S_1^{N_1} S_1^{*N_1}.$$

Therefore, $[T^*, T] \geq 0$ and T is hyponormal. ∎

Remark 4.7 *The assumption* $\lim_{n\to\infty} \eta_n^{(i)} = 1$ *is not restrictive. Since* S_i *is hyponormal,* $\eta_n^{(i)}$ *is increasing in* n*, and* $\lim_{n\to\infty} \eta_n^{(i)} = a_i$ *for a positive number* a_i *and we study the hyponormality of* $T/(a_1 \cdots a_d)$.

5 Hyponormal Toeplitz operators of several variables

We apply Theorem 4.3 on weighted Bergman spaces of the polydisk and the unit ball. Even on the Hardy space of the unit ball, the result seems to be interesting. On the Hardy space of the polydisk, the result seems to be trivial.

Let $\gamma = \{\gamma_\alpha : \alpha \geq 0\}$ be a set of positive numbers. Let $\mathbb{C}$ denote the set of complex numbers viewed as an one dimensional Hilbert space. Let H^2_γ be the weighted Hardy space as in [22]:

$$H^2_\gamma = \left\{ f(z) = \sum_{\alpha \geq 0} f_\alpha z^\alpha : f_\alpha \in \mathbb{C}, \|f(z)\|^2 = \sum_{\alpha \geq 0} \gamma_\alpha |f_\alpha|^2 < \infty \right\}. \tag{30}$$

Then $S_i = M_{z_i}$ is a weighted shift with

$$w_{\alpha,i} = \left\{ \sqrt{\frac{\gamma_{\alpha+\varepsilon_i}}{\gamma_\alpha}} : \alpha \geq 0 \right\} \quad (1 \leq i \leq d).$$

We first discuss the weighted Bergman space on the polydisk. Let $\mathbb{D}$ be the unit disk and $\mathbb{D}^d$ be the polydisk. We use $\mathcal{K}_\rho(\mathbb{D}^d)$ (for $\rho > 1$) to denote the Hilbert space of analytic functions on the polydisk $\mathbb{D}^d$ with reproducing kernel

$$K(z, \lambda) = \frac{1}{\prod_{i=1}^d \left(1 - z_i \overline{\lambda_i}\right)^\rho}.$$

This scale of spaces contains the Bergman space $L^2_a(\mathbb{D}^d)$ ($\rho = 2$) and Hardy space $H^2(\mathbb{D}^d)$ ($\rho = 1$). When $\rho > 2$, $\mathcal{K}_\rho(\mathbb{D}^d)$ is often called the weighted Bergman space on the polydisk. By the expansion formula,

$$\begin{aligned} K(z, \lambda) = \frac{1}{\prod_{i=1}^d \left(1 - z_i \overline{\lambda_i}\right)^\rho} &= \prod_{i=1}^d \left(\sum_{\alpha_i=0}^\infty \frac{\Gamma(\rho + \alpha_i)}{\alpha_i! \Gamma(\rho)} \left(z_i \overline{\lambda_i}\right)^{\alpha_i} \right) \\ &= \sum_{\alpha \geq 0} \prod_{i=1}^d \frac{\Gamma(\rho + \alpha_i)}{\alpha_i! \Gamma(\rho)} z^\alpha \overline{\lambda}^\alpha. \end{aligned}$$

Therefore

$$\mathcal{K}_\rho(\mathbb{D}^d) = H^2_\gamma \quad \text{with } \gamma = \left\{ \gamma_\alpha = \prod_{i=1}^d \frac{\alpha_i! \Gamma(\rho)}{\Gamma(\rho + \alpha_i)} : \alpha \geq 0 \right\}. \tag{31}$$

In this case

$$w^2_{\alpha,i} = \frac{\alpha_i + 1}{\alpha_i + \rho}.$$

So the results on weighted Bergman spaces of the polydisk follows from theorems of last section and the one variable case.

Theorem 5.1 *Assume $a\bar{b} = c\bar{d}$. Let $M = (M_1, \cdots, M_d)$ and $N = (N_1, \cdots, N_d)$ be two multi-indexes with $M < N$ and $|a| > |c|$. For $\varphi = az^M + bz^N + \overline{cz}^M + \overline{dz}^N$, T_φ on $\mathcal{K}_\rho(\mathbb{D}^d)$ ($\rho > 1$) is hyponormal if and only if*

$$\frac{|c|^2 - |a|^2}{|b|^2 - |d|^2} \leq \prod_{i=1}^d \frac{\Gamma(N_i + 1)\Gamma(M_i + \rho)}{\Gamma(M_i + 1)\Gamma(N_i + \rho)}.$$

Proof. The result follows from Theorem 4.5. ■

Theorem 5.2 *Assume $a\bar{b} = c\bar{d}$. Let $M = (M_1, \cdots, M_d)$ and $N = (N_1, \cdots, N_d)$ be two multi-indexes with $M < N$ and $|a| < |c|$. For $\varphi = az^M + bz^N + \overline{cz}^M + \overline{dz}^N$, T_φ on $\mathcal{K}_\rho(\mathbb{D}^d)$ ($\rho \geq 2$ is a positive integer) is hyponormal if and only if*

$$\frac{|a|^2 - |c|^2}{|d|^2 - |b|^2} \geq \max_{1 \leq i \leq d} \frac{N_i^2}{M_i^2}.$$

Proof. The result follows from Theorem 4.6 and the proof of Theorem 3.2. ■

We now discuss the weighted Bergman space on the unit ball. Let $\mathbb{B}^d$ be the unit ball of $\mathbb{C}^d$:

$$\mathbb{B}^d = \left\{ z = (z_1, \cdots, z_d) \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 < 1 \right\}.$$

Let $w = (w_1, \cdots, w_d) \in \mathbb{B}^d$ and $\langle z, w \rangle$ be the inner product defined by

$$\langle z, w \rangle = \sum_{i=1}^{d} z_i \overline{w_i}.$$

Let $\mathcal{K}_\rho(\mathbb{B}^d)$ ($\rho \geq 1$) be the Hilbert space of analytic functions on $\mathbb{B}^d$ with reproducing kernel

$$K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^\rho}.$$

This scale of spaces contains the Bergman space $L_a^2(\mathbb{B}^d)$ ($\rho = d + 1$), the Hardy space $H^2(\mathbb{B}^d)$ ($\rho = d$), and the Drury-Arveson space $H_d^2(\mathbb{B}^d)$ ($\rho = 1$). By the expansion formula,

$$\begin{aligned} K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^\rho} &= \sum_{i=0}^{\infty} \frac{\Gamma(\rho + i)}{i! \Gamma(\rho)} \langle z, w \rangle^i \\ &= \sum_{i=0}^{\infty} \frac{\Gamma(\rho + i)}{i! \Gamma(\rho)} \sum_{|\alpha| = i} \frac{i!}{\alpha!} z^\alpha \overline{w}^\alpha = \sum_{\alpha \geq 0} \frac{\Gamma(\rho + |\alpha|)}{\alpha! \Gamma(\rho)} z^\alpha \overline{w}^\alpha. \end{aligned}$$

Therefore

$$\mathcal{K}_\rho(\mathbb{B}^d) = H_\gamma^2 \text{ with } \gamma = \left\{ \gamma_\alpha = \frac{\alpha! \Gamma(\rho)}{\Gamma(\rho + |\alpha|)} : \alpha \geq 0 \right\}, \tag{32}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Thus the shift $S_i = M_{z_i}$ on $\mathcal{K}_\rho(\mathbb{B}^d)$ is a weighted shift with

$$w_{\alpha,i}^2 = \frac{\gamma_{\alpha+\varepsilon_i}}{\gamma_\alpha} = \frac{\alpha_i + 1}{|\alpha| + \rho} \ (\alpha \geq 0), \tag{33}$$

and S_i is hyponormal for $\rho \geq 1$, but $w_{\alpha,i}$ does not satisfy (28). Theorem 4.5 is not applicable. In fact, the following result shows that the assumption (28) cannot be removed in Theorem 4.5.

Proposition 5.3 *Assume $a\bar{b} = c\bar{d}$. Let $M = (M_1, \cdots, M_d)$ and $N = (N_1, \cdots, N_d)$ be two multi-indexes with $M < N$ and $|a| < |c|$. For $\varphi = az^M + bz^N + \overline{cz}^M + \overline{dz}^N$, T_φ on $\mathcal{K}_\rho(\mathbb{B}^d)$ ($\rho \geq 1$) is not hyponormal.*

Proof. By Theorem 4.3, T_φ on $\mathcal{K}_\rho(\mathbb{B}^d)$ $(\rho \geq 1)$ is hyponormal if and only if

$$\frac{|c|^2 - |a|^2}{|b|^2 - |d|^2} \leq \min\{F_1, F_2, F_3\}, \text{ where } F_1 = \inf\left\{\frac{u_{\alpha,N}}{u_{\alpha,M}} : \alpha \ngeq M\right\}.$$

Without loss of generality, assume $N_1 > M_1 \geq 1$. Note that

$$\frac{u_{\alpha,N}}{u_{\alpha,M}} = \prod_{1\leq i\leq d} \prod_{M_i \leq k_i < N_i} w_{\alpha+k_i\varepsilon_i,i} \leq \frac{\alpha_1 + M_1}{\alpha_1 + \alpha_2 + \cdots + \alpha_d + M_1 + \rho}. \tag{34}$$

Let $\alpha_1 = 0$ and $\alpha_2 \to \infty$. Then $\alpha \ngeq M$ and $\frac{u_{\alpha,N}}{u_{\alpha,M}} \to 0$. So $F_1 = 0$ and $|c|^2 - |a|^2 \leq 0$, which is a contradiction to $|a| < |c|$. ■

It is natural to ask in the case $|a| > |c|$ whether T_φ on $\mathcal{K}_\rho(\mathbb{B}^d)$ $(\rho \geq 1)$ could be hyponormal. The answer is yes. But Theorem 4.6 is not applicable because S on $\mathcal{K}_\rho(\mathbb{B}^d)$ $(\rho \geq 1)$ is not double commuting, so we only have the following (rough) estimation.

Proposition 5.4 *Assume $a\bar{b} = c\bar{d}$. Let $M = (M_1, \cdots, M_d)$ and $N = (N_1, \cdots, N_d)$ be two multi-indexes with $M < N$ and $|a| > |c|$. For $\varphi = az^M + bz^N + \overline{cz}^M + \overline{d}\overline{z}^N$, T_φ on $\mathcal{K}_\rho(\mathbb{B}^d)$ $(\rho \geq 1)$ is hyponormal if and only if*

$$\frac{|a|^2 - |c|^2}{|d|^2 - |b|^2} \geq \max\{E_1, E_2, E_3\},$$

where $\max\{E_1, E_2, E_3\}$ *is finite. More specifically,*

$$\max\{E_1, E_2, E_3\} \leq \max\left\{\max_{1\leq i\leq d} N_i^2,\ \sum_{i=1}^d (N_i - 1)\frac{(2N_i - 1)}{(2M_i - 1)}\frac{(|N| - M_i + \rho)}{(|N| - N_i + \rho)}\right\}.$$

Proof. By Theorem 4.3, we need to show each E_i is a finite number for $i = 1, 2, 3$. By (33) and (34),

$$E_1 = \sup\left\{\frac{u_{\alpha,N}}{u_{\alpha,M}} : \alpha \ngeq M\right\} \leq 1.$$

We next estimate E_2. Note that for $\alpha \geq M$ and $\alpha \ngeq N$,

$$c_\alpha := \frac{u_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}} = \frac{\frac{u_{\alpha,N}}{u_{\alpha,M}}}{1 - \frac{v_{\alpha,M}}{u_{\alpha,M}}} \leq \frac{1}{1 - \frac{v_{\alpha,M}}{u_{\alpha,M}}}.$$

We claim

$$\frac{v_{\alpha,M}}{u_{\alpha,M}} \leq \max_{1\leq i\leq d}\left\{1 - \frac{1}{N_i^2}\right\}. \tag{35}$$

Then

$$c_\alpha \leq \frac{1}{1 - \frac{v_{\alpha,M}}{u_{\alpha,M}}} \leq \max_{1\leq i\leq d}\left\{\frac{1}{1 - \left(1 - \frac{1}{N_i^2}\right)}\right\} = \max_{1\leq i\leq d} N_i^2.$$

Thus

$$E_2 = \sup\left\{\frac{u_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}} : \alpha \geq M \text{ and } \alpha \ngeq N\right\} \leq \max_{1\leq i\leq d} N_i^2.$$

To prove Claim (35), since $\alpha \ngeq N$, assume $\alpha_1 < N_1$. Since $w_{\alpha+n\varepsilon_i,i}$ is increasing in n,

$$\begin{aligned}
\frac{v_{\alpha,M}}{u_{\alpha,M}} &= \prod_{1\le i\le d}\prod_{1\le k_i\le M_i}\frac{w^2_{\alpha-k_i\varepsilon_i,i}}{w^2_{\alpha+(k_i-1)\varepsilon_i,i}} \le \frac{w^2_{\alpha-\varepsilon_1,1}}{w^2_{\alpha,1}} \\
&= \frac{\alpha_1}{|\alpha|+\rho-1}\cdot\frac{|\alpha|+\rho}{\alpha_1+1} = \left(1+\frac{1}{|\alpha|+\rho-1}\right)\left(1-\frac{1}{\alpha_1+1}\right) \\
&\le \left(1+\frac{1}{\alpha_1+1}\right)\left(1-\frac{1}{\alpha_1+1}\right) = 1-\frac{1}{(\alpha_1+1)^2} \le 1-\frac{1}{N_1^2},
\end{aligned}$$

where one inequality follows from

$$|\alpha|+\rho-1 \ge \alpha_1+1+\rho-1 \ge \alpha_1+1,$$

and the last inequality follows from $\alpha_1+1\le N_1$. This proves Claim (35).

We now estimate E_3. Let

$$\begin{aligned}
A_i &= \prod_{1\le k_i\le N_i} w^2_{\alpha+(k_i-1)\varepsilon_i,i}, \quad B_i = \prod_{1\le k_i\le N_i} w^2_{\alpha-k_i\varepsilon_i,i}, \\
C_i &= \prod_{1\le k_i\le M_i} w^2_{\alpha+(k_i-1)\varepsilon_i,i}, \quad D_i = \prod_{1\le k_i\le M_i} w^2_{\alpha-k_i\varepsilon_i,i} \quad (i=1,\cdots,d).
\end{aligned}$$

Note that by $w^2_{\alpha,i}\le 1$ for all $\alpha\ge 0$ and $1\le i\le d$, $N>M$,

$$\frac{A_i}{C_i}\le 1, \quad \frac{B_i}{D_i}\le 1, \quad C_i-D_i>0, \quad \text{and} \quad A_i-B_i>0.$$

Then

$$\begin{aligned}
\frac{u_{\alpha,N}-v_{\alpha,N}}{u_{\alpha,M}-v_{\alpha,M}} &= \frac{\prod\limits_{1\le i\le d}A_i-\prod\limits_{1\le i\le d}B_i}{\prod\limits_{1\le i\le d}C_i-\prod\limits_{1\le i\le d}D_i} \\
&= \frac{\sum_{j=1}^d\left(\prod_{i=1}^{j-1}A_i\right)(A_j-B_j)\left(\prod_{i=j+1}^dB_i\right)}{\sum_{j=1}^d\left(\prod_{i=1}^{j-1}C_i\right)(C_j-D_j)\left(\prod_{i=j+1}^dD_i\right)} \\
&\le \sum_{j=1}^d\left(\prod_{i=1}^{j-1}\frac{A_i}{C_i}\right)\left(\frac{A_j-B_j}{C_j-D_j}\right)\left(\prod_{i=j+1}^d\frac{B_i}{D_i}\right) \\
&\le \sum_{j=1}^d\left(\frac{A_j-B_j}{C_j-D_j}\right),
\end{aligned}$$

where $\prod_{i=1}^0A_i = 1 = \prod_{i=d+1}^dB_i$. Set

$$a_i = w^2_{\alpha+(i-1)\varepsilon_1,1} \quad \text{and} \quad b_i = w^2_{\alpha-i\varepsilon_1,1} \quad (1\le i\le N_1).$$

By a similar but more complicated argument (because $N_1 \neq M_1$),

$$\frac{A_1 - B_1}{C_1 - D_1} = \frac{\prod\limits_{1\le i\le N_1} a_i - \prod\limits_{1\le i\le N_1} b_i}{\prod\limits_{1\le i\le M_1} a_j - \prod\limits_{1\le i\le M_1} b_j} = \frac{\sum_{j=1}^{N_1}\left(\prod\limits_{1\le i\le j-1} a_i\right)(a_j - b_j)\left(\prod\limits_{j+1\le i\le N_1} b_j\right)}{\sum_{j=1}^{M_1}\left(\prod\limits_{1\le i\le j-1} a_i\right)(a_j - b_j)\left(\prod\limits_{j+1\le i\le M_1} b_j\right)}$$

$$\le \sum_{j=1}^{M_1}\frac{\left(\prod\limits_{1\le i\le j-1} a_i\right)(a_j - b_j)\left(\prod\limits_{j+1\le i\le N_1} b_j\right)}{\left(\prod\limits_{1\le i\le j-1} a_i\right)(a_j - b_j)\left(\prod\limits_{j+1\le i\le M_1} b_j\right)}$$

$$+ \sum_{j=M_1+1}^{N_1}\frac{\left(\prod\limits_{1\le i\le j-1} a_i\right)(a_j - b_j)\left(\prod\limits_{j+1\le i\le N_1} b_j\right)}{\sum_{j=1}^{M_1}\left(\prod\limits_{1\le i\le j-1} a_i\right)(a_j - b_j)\left(\prod\limits_{j+1\le i\le M_1} b_j\right)}$$

$$\le M_1 + \sum_{j=M_1+1}^{N_1}\frac{\left(\prod\limits_{1\le i\le j-1} a_i\right)(a_j - b_j)\left(\prod\limits_{j+1\le i\le N_1} b_j\right)}{\left(\prod\limits_{1\le i\le M_1-1} a_i\right)(a_{M_1} - b_{M_1})}$$

$$\le M_1 + \sum_{j=M_1+1}^{N_1}\frac{(a_j - b_j)}{(a_{M_1} - b_{M_1})},$$

where the fact $w_{\alpha,i}^2 \le 1$ $(a_i, b_i \le 1)$ is used several times. By a direct computation,

$$\begin{aligned} a_j - b_j &= w^2_{\alpha+(j-1)\varepsilon_1,1} - w^2_{\alpha-j\varepsilon_1,1} \\ &= \frac{\alpha_1 + j - 1 + 1}{|\alpha| + j - 1 + \rho} - \frac{\alpha_1 - j + 1}{|\alpha| - j + \rho} = \frac{(2j-1)(|\alpha| - \alpha_1 + \rho - 1)}{(|\alpha| + j - 1 + \rho)(|\alpha| - j + \rho)} \\ &\le \frac{(2N_1-1)(|\alpha| - \alpha_1 + \rho - 1)}{(|\alpha| + N_1 - 1 + \rho)(|\alpha| - N_1 + \rho)} \quad (M_1 + 1 \le j \le N_1), \\ a_{M_1} - b_{M_1} &= \frac{(2M_1-1)(|\alpha| - \alpha_1 + \rho - 1)}{(|\alpha| + M_1 - 1 + \rho)(|\alpha| - M_1 + \rho)}. \end{aligned}$$

Therefore for $\alpha \ge N$, we have

$$\begin{aligned} \sum_{j=M_1+1}^{N_1}\frac{(a_j - b_j)}{(a_{M_1} - b_{M_1})} &= (N_1 - M_1 - 1)\frac{(2N_1-1)}{(2M_1-1)}\frac{(|\alpha| + (M_1 - 1) + \rho)(|\alpha| - M_1 + \rho)}{(|\alpha| + N_1 - 1 + \rho)(|\alpha| - N_1 + \rho)} \\ &\le (N_1 - M_1 - 1)\frac{(2N_1-1)}{(2M_1-1)}\frac{(|N| - M_1 + \rho)}{(|N| - N_1 + \rho)}. \end{aligned}$$

Summarizing,

$$\begin{aligned} \frac{A_1 - B_1}{C_1 - D_1} &\le M_1 + (N_1 - M_1 - 1)\frac{(2N_1-1)}{(2M_1-1)}\frac{(|N| - M_1 + \rho)}{(|N| - N_1 + \rho)} \\ &\le (N_1 - 1)\frac{(2N_1-1)}{(2M_1-1)}\frac{(|N| - M_1 + \rho)}{(|N| - N_1 + \rho)}. \end{aligned}$$

Therefore,

$$E_3 = \sup_{\alpha \geq N} \left\{ \frac{u_{\alpha,N} - v_{\alpha,N}}{u_{\alpha,M} - v_{\alpha,M}} \right\} \leq \sum_{j=1}^{d} (N_i - 1) \frac{(2N_i - 1)}{(2M_i - 1)} \frac{(|N| - M_i + \rho)}{(|N| - N_i + \rho)}.$$

The proof is complete. ■

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