Operator-valued rational functions

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Abstract. We show that every inner divisor of the operator-valued coordinate function, zI_E, is a Blaschke-Potapov factor. We also introduce a notion of operator-valued "rational" function and then show that Δ is two-sided inner and rational if and only if it can be represented as a finite Blaschke-Potapov product; this extends to operator-valued functions the well-known result proved by V.P. Potapov for matrix-valued functions.

1 Introduction

Many properties of matrix-valued functions may not be transferred to operator-valued functions, since some properties of finite matrices are destined to fail for infinite matrices. For example, if A is an $n \times n$ matrix of H^2-functions and B is an $n \times n$ diagonal constant inner function, i.e., $B = \text{diag}\,(\theta, \cdots, \theta)$ (for θ an inner function), then A and B are left coprime if and only if A and B are right coprime; in other words, left-coprimeness and right-coprimeness coincide for A and B (cf. [7, Lemma C.14]). However this is no longer true for operator-valued functions. For example, if $A(z) := S$ (the shift on ℓ^2) and $B(z) := \theta I$ (where I is the identity operator on ℓ^2), then A and B are right coprime, but not left coprime (cf. [7, Example C.12]).

In this paper we consider a question on left inner divisors of the "operator-valued coordinate" function zI_E (where E is a Hilbert space). We consider the well-known result, proved by V.P. Potapov [16], that every rational inner $n \times n$ matrix-valued function can be written as a finite Blaschke-Potapov product. We extend this result to the case of operator-valued functions.

Let X be a complex Banach space and $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$. For $1 \leq p < \infty$, let $L^p(\mathbb{T}, X)$ be the linear space of all (equivalence classes of) strongly measurable functions $f : \mathbb{T} \to X$ for which

$$\int_{\mathbb{T}} ||f||^p dm < \infty,$$

where m is the normalized Lebesgue measure on $\mathbb{T}$. We define $L^\infty(\mathbb{T}, X)$ as the linear space of all (equivalence classes of) strongly measurable functions $f : \mathbb{T} \to X$ for which

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there exists $r > 0$ such that $m\big(\{z \in \mathbb{T} : ||f(z)|| > r\}\big) = 0$. Endowed with the norms

$$||f||_{L^p(\mathbb{T},X)} := \left(\int_{\mathbb{T}} ||f||^p dm\right)^{\frac{1}{p}}$$

and

$$||f||_{L^\infty(\mathbb{T},X)} := \inf\big\{r > 0 : m(\{z \in \mathbb{T} : ||f(z)|| > r\}) = 0\big\},$$

the spaces $L^p(\mathbb{T}, X)$ are complex Banach spaces ($1 \le p \le \infty$). For $f \in L^1(\mathbb{T}, X)$, the n-th Fourier coefficient of f, denoted by $\widehat{f}(n)$, is defined by

$$\widehat{f}(n) := \int_{\mathbb{T}} \overline{z}^n f(z)\, dm(z) \quad \text{for each } n \in \mathbb{Z},$$

where the integral is understood in the sense of the Bochner integral. For $1 \le p \le \infty$, define

$$H^p(\mathbb{T}, X) := \big\{f \in L^p(\mathbb{T}, X) : \widehat{f}(n) = 0 \ \text{ for } n < 0\big\}.$$

Let $\mathcal{B}(D, E)$ denote the set of all bounded linear operators between complex Hilbert spaces D and E and abbreviate $B(E, E)$ to $B(E)$. For $1 \le p \le \infty$, let $L^p_s(\mathbb{T}, \mathcal{B}(D, E))$ be the set of all (SOT-measurable) $\mathcal{B}(D, E)$-valued functions Φ on $\mathbb{T}$ such that $\Phi(\cdot)x \in L^p(\mathbb{T}, E)$ for each $x \in D$. A function $\Phi \in L^p_s(\mathbb{T}, \mathcal{B}(D, E))$ is called a *strong L^p-function*. We can see that every function in $L^p(\mathbb{T}, B(D, E))$ is a strong L^p-function. The notion of strong L^2-function was introduced by V. Peller in [15]; the formal theory of strong L^2-functions was developed in [7].

If $\Phi \in L^1_s(\mathbb{T}, \mathcal{B}(D, E))$ and $x \in D$, then $\Phi(\cdot)x \in L^1_E$. The n-th Fourier coefficient of $\Phi \in L^1_s(\mathbb{T}, \mathcal{B}(D, E))$, denoted by $\widehat{\Phi}(n)$, is given by

$$\widehat{\Phi}(n)x := \widehat{\Phi(\cdot)x}(n) \quad (n \in \mathbb{Z},\ x \in D).$$

We define

$$H^p_s(\mathbb{T}, \mathcal{B}(D, E)) := \big\{\Phi \in L^p_s(\mathbb{T}, \mathcal{B}(D, E)) :\ \widehat{\Phi}(n) = 0 \text{ for } n < 0\big\}.$$

Let $L^\infty(\mathbb{T}, \mathcal{B}(D, E))$ be the space of all bounded SOT-measurable $\mathcal{B}(D, E)$-valued functions on $\mathbb{T}$. Then we can easily see that

$$L^\infty(\mathbb{T}, \mathcal{B}(D, E)) = L^\infty_s(\mathbb{T}, \mathcal{B}(D, E)). \tag{1}$$

Indeed, obviously $L^\infty(\mathbb{T}, \mathcal{B}(D, E)) \subseteq L^\infty_s(\mathbb{T}, \mathcal{B}(D, E))$. For the reverse inclusion, suppose $\Phi \in L^\infty_s(\mathbb{T}, \mathcal{B}(D, E))$. Then $\Lambda : x \mapsto \Phi(\cdot)x$ is a closed linear transform from X into $L^\infty(\mathbb{T}, E)$, so that, by the closed graph theorem, it is bounded. Thus for almost all $z \in \mathbb{T}$,

$$||\Phi(z)||_{\mathcal{B}(D,E))} = \sup_{||x||=1} ||\Phi(z)x||_E \le ||\Lambda||,$$

which implies $\Phi \in L^\infty(\mathbb{T}, \mathcal{B}(D, E))$. This proves (1). We will also write $H^\infty(\mathbb{T}, \mathcal{B}(D, E)) \equiv H^\infty_s(\mathbb{T}, \mathcal{B}(D, E))$.

Write I_E for the identity operator acting on E. Write $M_{m\times n}$ for the set of $m \times n$ complex matrices and abbreviate $M_{n\times n}$ to M_n. Also we abbreviate I_{M_n} to I_n. For a $B(D,E)$-valued function Φ, write

$$\breve{\Phi}(z) := \Phi(\overline{z}) \quad \text{and} \quad \widetilde{\Phi} := \breve{\Phi}^*.$$

We say that a function $\Delta \in H^\infty(\mathbb{T}, \mathcal{B}(D,E))$ is an *inner* function if

$$\Delta^*\Delta = I_D \ \text{ a.e. on } \mathbb{T}$$

and that Δ is a *two-sided inner* function if $\Delta\Delta^* = I_E$ a.e. on $\mathbb{T}$ and $\Delta^*\Delta = I_D$ a.e. on $\mathbb{T}$. For a function $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D,E))$, we say that an inner function $\Delta \in H^\infty(\mathbb{T}, \mathcal{B}(D',E))$ is a *left inner divisor* of Φ if $\Phi = \Delta A$ for some $A \in H^\infty(\mathbb{T}, \mathcal{B}(D,D'))$ and that $\Omega \in H^\infty(\mathbb{T}, \mathcal{B}(D,E'))$ is an *right inner divisor* of Φ if $\Phi = B\Omega$ for some $B \in H^\infty(\mathbb{T}, \mathcal{B}(E',E))$. A function Δ is an *inner divisor* of Φ if it is both a left and a right inner divisor of Φ. As customarily done, we say that two inner functions $A, B \in H^\infty(\mathbb{T}, B(E))$ are equal if they are equal up to a unitary constant right factor, i.e., there exists a unitary (constant) operator $V \in \mathcal{B}(E)$ such that $A = BV$.

For $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D_1,E))$ and $\Psi \in H^\infty(\mathbb{T}, \mathcal{B}(D_2,E))$, we say that Φ and Ψ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary operator. Also, we say that Φ and Ψ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. The determination of left or right coprime-ness seems to be a somewhat delicate problem. For matrix-valued functions, left and right coprime-ness was developed in [4], [5], [6], [7] and [8].

Note that if V is a unitary operator in $B(E)$, then for every $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(E))$,

$$\Phi = V(V^*\Phi) = (\Phi V^*)V,$$

which implies that V is an inner divisor of Φ.

For a function $\Phi \in H^\infty(\mathbb{T}, B(E))$, we say that a function $\Delta \in H^\infty(\mathbb{T}, B(E))$ is a *nontrivial left* (resp. *right*) *inner divisor* of Φ if Δ is a non-unitary operator and is a left (res. right) inner divisor of Φ.

For $\alpha \in \mathbb{D}$, write

$$b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z},$$

which is called a *Blaschke factor*. If M is a closed subspace of a Hilbert space E, then a function of the form

$$b_\alpha P_M + (I_E - P_M)$$

is called a (*operator-valued*) *Blaschke-Potapov factor*, where P_M is the orthogonal projection of E onto M. A function D is called a (*operator-valued*) *finite Blaschke-Potapov product* if D is of the form

$$D = V \prod_{m=1}^{r} \Big(b_m P_m + (I - P_m)\Big),$$

where V is a unitary operator, b_m is a Blaschke factor, and P_m is an orthogonal projection in E for each $m = 1, \cdots, r$. In particular, a scalar-valued function D reduces to a finite

Blaschke product $D = \nu \prod_{m=1}^{r} b_m$, where $\nu = e^{i\omega}$. It is known (cf. [16]) that an $n \times n$ matrix function D is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product.

On the other hand, we may ask a question: What is a left inner divisor of zI_n? For this question, we may guess that each left inner divisor of zI_n is a Blaschke-Potapov factor. More specifically, we wonder if a left inner divisor of $\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right] \equiv zI_2$ should be of the following form up to a unitary constant right factor (also up to unitary equivalence):

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}, \quad \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}.$$

For example, $A \equiv \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & -z \\ 1 & z \end{smallmatrix}\right]$ is a left inner divisor of $\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right] \equiv zI_2$: indeed,

$$A \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} z & z \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}.$$

In this case, if we take a unitary operator $V := \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right]$, then

$$\begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} = V \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -z \\ 1 & z \end{bmatrix} = \left[V \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -z \\ 1 & z \end{bmatrix} \cdot V^* \right] \cdot V.$$

In fact, it was shown in [4, Lemma 2.5] that

$$\text{every left inner divisor of } zI_n \in H^\infty(\mathbb{T}, M_n) \text{ is a Blaschke-Potapov factor.} \tag{2}$$

This fact is useful for the study of coprime-ness of functions (cf. [6]). In [7, p.23], the authors asked:

Question 1.1. Is the statement in (2) still true for operator-valued functions?

We will call a function of the form zI_E the operator-valued coordinate function. This allows us to rephrase Question 1.1 as follows: Is every left inner divisor of the operator-valued coordinate function a Blaschke-Potapov factor? In Section 2, we give an affirmative answer to this question. In Section 3, we introduce a notion of operator-valued "rational" function and then show that Δ is two-sided inner and rational if and only if it can be represented as a finite Blaschke-Potapov product, which extends the well-known result for the matrix-valued functions due to V.P. Potapov [16]. In Section 4, we consider coprime operator-valued rational functions. Lastly, in Section 5, we take a glance at right coprime-ness and subnormality of Toeplitz operators, aiming at shedding new light on the differences between matrix-valued functions and operator-valued functions.

To proceed, we give an elementary observation.

If $\dim E < \infty$ and $\Theta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ is a two-sided inner function, then any left inner divisor of Θ is two-sided inner (cf. [6, Lemma 4.10]). We can say more:

Lemma 1.2. (cf. [7, Lemma 2.2]) Let $\Theta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ be a two-sided inner function. If $\Delta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ is a left inner divisor of Θ, then Δ is two-sided inner.

Proof. If Δ is a left inner divisor of Θ, we may write

$$\Theta = \Delta\Omega \quad \text{for some } \Omega \in H^\infty(\mathbb{T}, \mathcal{B}(E)).$$

Thus Δ is surjective, and hence unitary a.e. on $\mathbb{T}$. Thus Δ is two-sided inner. □

In particular, if $\Theta := \theta I_E$ (for θ a scalar inner function) then every left inner divisor of Θ is an inner divisor of Θ. However, in general, a left inner divisor of a two-sided inner function need not be its right inner divisor. To see this, we first observe:

Lemma 1.3. Let $\Theta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ be a two-sided inner function and $\Delta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ be a left inner divisor of Θ. Then Δ is an inner divisor of Θ if and only if $\Theta\Delta^* \in H^\infty(\mathbb{T}, \mathcal{B}(E))$.

Proof. Let $\Theta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ be a two-sided inner function and $\Delta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ be a left inner divisor of Θ. Then, by Lemma 1.2, Δ is two-sided inner and we may write

$$\Theta = \Delta\Omega \quad (\Omega \in H^\infty(\mathbb{T}, \mathcal{B}(E))).$$

Suppose that Δ is an inner divisor of Θ. Then we can also write $\Theta = \Psi\Delta$ for some $\Psi \in H^\infty(\mathbb{T}, \mathcal{B}(E)))$. Thus we have that $\Theta\Delta^* = \Psi\Delta\Delta^* = \Psi \in H^\infty(\mathbb{T}, \mathcal{B}(E))$. The converse is obvious. □

We then have:

Example 1.4. Let $\{e_n : n \in \mathbb{Z}\}$ be the canonical orthonormal basis for $L^2(\mathbb{T})$. Define Δ and Θ in $H^\infty(\mathbb{T}, \mathcal{B}(L^2(\mathbb{T})))$ by

$$\Delta(z)e_n := \begin{cases} e_{n+1}z & \text{if } n \geq 0 \\ e_{n+1} & \text{if } n < 0 \end{cases} \qquad \text{and} \qquad \Theta(z)e_n := \begin{cases} e_{-n+1}z^2 & \text{if } n \leq 1 \\ e_{-n+1} & \text{if } n > 1. \end{cases}$$

Then Θ and Δ are two-sided inner. Observe that

$$\Delta^*(z)e_n = \begin{cases} e_{n-1}z^{-1} & \text{if } n \geq 1 \\ e_{n-1} & \text{if } n < 1 \end{cases} \quad \text{and hence} \quad \Delta^*(z)\Theta(z)e_n = \begin{cases} e_{-n}z & \text{if } n < 1 \\ e_{-1}z^2 & \text{if } n = 1 \\ e_{-n} & \text{if } n > 1. \end{cases}$$

Thus Δ is a left inner divisor of Θ. Since $\Theta(z)\Delta^*(z)e_3 = e_{-1}z^{-1}$, it follows from Lemma 1.3 that Δ is not a right inner divisor of Θ.

On the other hand, Lemma 1.2 may fail if "left" is replaced by "right."

Example 1.5. Let S be the shift operator on $H^2(\mathbb{T})$ defined by

$$(Sf)(z) := zf(z) \quad (f \in H^2(\mathbb{T}),\ z \in \mathbb{T})$$

and let $\Delta(z) := S \in H^\infty(\mathbb{T}, \mathcal{B}(H^2(\mathbb{T})))$. Then

$$\Delta(z)^*\Delta(z) = S^*S = I,$$

which implies that Δ is a right inner divisor of (a two-sided inner function) I. But Δ is not two-sided inner. By Lemma 1.2, Δ is not a left inner divisor of I.

2 Inner divisors of the operator-valued coordinate functions

For a complex Banach space X and an open subset G of $\mathbb{C}$, a function $A : G \to X$ is called holomorphic if, in any sufficiently small disc $D(\lambda, r) = \{\zeta : |\zeta - \lambda| < r\} \subset G$, A is the sum of convergent power series

$$A(\zeta) = \sum_{n=0}^{\infty} \widehat{A}(n)(\zeta - \lambda)^n \quad (\widehat{A}(n) \in X).$$

Denote by $\text{Hol}(G, X)$ the space of all holomorphic functions $A : G \to X$. Let us associate to any function f on $\mathbb{D}$ a family of function f_r on $\mathbb{T}$, defined by

$$f_r(z) := f(rz) \quad (0 \leq r < 1).$$

For $1 \leq p < \infty$, let $H^p(\mathbb{D}, X)$ be the set of all functions $f \in \text{Hol}(\mathbb{D}, X)$ satisfying

$$||f||_{H^p(\mathbb{D},X)} := \left(\sup_{0<r<1} \int_{\mathbb{T}} ||f_r||^p dm \right)^{\frac{1}{p}} < \infty.$$

We define $H^\infty(\mathbb{D}, X)$ be the set of all bounded functions $f \in \text{Hol}(\mathbb{D}, X)$.

We recall that the Poisson kernel is defined by

$$P_r(t) := \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} = \frac{1 - r^2}{1 - 2r\cos t + r^2} \quad (0 \leq r < 1, \ t \in \mathbb{R}).$$

If $\Phi \in L^\infty(\mathbb{T}, \mathcal{B}(D, E))$ and $\zeta = re^{i\theta} \in \mathbb{D}$, let $P[\Phi]$ denote the Poisson integral of f defined by

$$P[\Phi](\zeta)x := \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\Phi(e^{it})x \, dt \quad (x \in D; \, 0 \leq r < 1). \tag{3}$$

It is known (cf. [10, Lemma 2.1]) that if $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E))$ and $\Psi \in H^\infty(\mathbb{T}, \mathcal{B}(D', D))$, then

$$P[\Phi\Psi] = P[\Phi]P[\Psi]. \tag{4}$$

We now consider Question 1.1. We actually wish to study a more general case, and we first observe that if A is an inner divisor of $z^N I_E$, then there exists a function $\Omega \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ such that

$$A\Omega = \Omega A = z^N I_E \quad \text{a.e. on } \mathbb{T},$$

so that

$$A^* z^N I_E = \Omega \in H^\infty(\mathbb{T}, \mathcal{B}(E)),$$

which implies that A is a polynomial of degree at most N.

We are ready for:

Theorem 2.1. Let A be a polynomial of degree N. Then the following are equivalent:

(a) A is an inner divisor of $z^N I_E$;

(b) $\widehat{A}(n)$ and $\widehat{A}(n)^*$ are partial isometries for each $n = 0, 1, \cdots, N$, and

$$E = \bigoplus_{n=0}^{N} \operatorname{ran} \widehat{A}(n) = \bigoplus_{n=0}^{N} \operatorname{ran} \widehat{A}(n)^*;$$

(c) A is a finite Blaschke-Potapov product of the form

$$A(z) = V \prod_{m=1}^{N} \Big(zP_m + (I - P_m) \Big),$$

where P_m is the orthogonal projection from E onto $\bigoplus_{n=m}^{N} \operatorname{ran} \widehat{A}(n)^*$, and $V := \operatorname{diag}\big(\widehat{A}(0)|_{\operatorname{ran} \widehat{A}(0)^*}, \widehat{A}(1)|_{\operatorname{ran} \widehat{A}(1)^*}, \cdots, \widehat{A}(N)|_{\operatorname{ran} \widehat{A}(N)^*}\big)$.

Proof. (a) $\Rightarrow$ (b): Suppose that A is an inner divisor of $z^N I_E$. Without loss of generality we may assume that A_0 and A_N are nonzero. Write

$$A^{(1)}(z) := \sum_{n=1}^{N} \widehat{A}(n) z^{n-1}.$$

Then

$$A(z) = \widehat{A}(0) + zA^{(1)}(z). \tag{5}$$

By Lemma 1.2, A is two-sided inner. Thus, for almost all $z \in \mathbb{T}$,

$$A^{(1)}(z)^* \widehat{A}(0) = 0 \quad \text{and} \quad \widehat{A}(0)\widehat{A}(0)^* + A^{(1)}(z)A^{(1)}(z)^* = I_E, \tag{6}$$

so that

$$\widehat{A}(0)\widehat{A}(0)^*\widehat{A}(0) = \widehat{A}(0) \quad \text{and} \quad A^{(1)}(z)A^{(1)}(z)^*A^{(1)}(z) = A^{(1)}(z),$$

which implies that $\widehat{A}(0)$ and $A^{(1)}(z)$ are partial isometries for almost all $z \in \mathbb{T}$ (cf. [12]). Since A is inner, it follows from (5) that, for almost all $z \in \mathbb{T}$,

$$\ker \widehat{A}(0) \bigcap \ker A^{(1)}(z) = \{0\}.$$

Since $(H \cap K)^\perp = H^\perp \bigvee K^\perp$ for closed subspaces H, K of E, it follows that

$$\operatorname{ran} \widehat{A}(0)^* \bigvee \operatorname{ran} A^{(1)}(z)^* = E. \tag{7}$$

On the other hand, it follows from (5) that $\operatorname{ran} \widehat{A}(0)^* \subseteq \big(\operatorname{cl} \operatorname{ran} A^{(1)}(z)^*\big)^\perp$ and, by (7), we have that

$$\operatorname{ran} \widehat{A}(0)^* \overset{\perp}{\bigoplus} \operatorname{ran} A^{(1)}(z)^* = E.$$

Thus, for almost all $z \in \mathbb{T}$,

$$\big(\ker A^{(1)}(z)\big)^{\perp} = \operatorname{ran} A^{(1)}(z)^* = \ker \widehat{A}(0).$$

Similarly, we can show that, for almost all $z \in \mathbb{T}$,

$$\operatorname{ran} A^{(1)}(z) = \ker \widehat{A}(0)^* = \big(\operatorname{ran} \widehat{A}(0)\big)^{\perp}. \tag{8}$$

Since $A^{(1)}(z)$ is a partial isometry, $A^{(1)}|_{\ker \widehat{A}(0)} : \mathbb{T} \to \mathcal{B}(\ker \widehat{A}(0), (\operatorname{ran} \widehat{A}(0))^{\perp})$ is two-sided inner. Now we will show that

$$\operatorname{ran} \widehat{A}(1) \subseteq \big(\operatorname{ran} \widehat{A}(0)\big)^{\perp}. \tag{9}$$

For $x \in E$, it follows from (8) that

$$\begin{aligned} A^{(1)}(z)x = \sum_{n=1}^{N} \widehat{A}(n)xz^{n-1} \in \big(\operatorname{ran} \widehat{A}(0)\big)^{\perp} \text{ for almost all } z \in \mathbb{T} \\ \Longrightarrow \big\langle \widehat{A}(n)x,\ \widehat{A}(0)y \big\rangle = 0 \quad \text{for all } y \in E \\ \Longrightarrow P[A^{(1)}](\zeta)x = \sum_{n=1}^{N} \widehat{A}(n)x\zeta^{n-1} \in \big(\operatorname{ran} \widehat{A}(0)\big)^{\perp} \text{ for all } \zeta \in \mathbb{D} \\ \Longrightarrow \widehat{A}(1)x = P[A^{(1)}](0)x \in \big(\operatorname{ran} \widehat{A}(0)\big)^{\perp}, \end{aligned}$$

which proves (9). Put $A^{(2)}(z) := \sum_{n=2}^{N} \widehat{A}(n)z^{n-2}$. Then, by the same argument, we have that

(i) $\widehat{A}(1)$ is a partial isometry;

(ii) $(\ker A^{(2)}(z))^{\perp} = \ker \widehat{A}(1)$ and $\operatorname{ran} A^{(2)}(z) = \big(\operatorname{ran} \widehat{A}(0) \bigoplus \operatorname{ran} \widehat{A}(1)\big)^{\perp}$;

(iii) $A^{(2)}|_{\ker \widehat{A}(1)} : \mathbb{T} \to \mathcal{B}(\ker \widehat{A}(1), (\operatorname{ran} \widehat{A}(1))^{\perp})$ is two-sided inner;

(iv) $\operatorname{ran} \widehat{A}(2) \subseteq \big(\operatorname{ran} \widehat{A}(0) \bigoplus \operatorname{ran} \widehat{A}(1)\big)^{\perp}$.

Continuing this process, we have that $\widehat{A}(n)$ is a partial isometry for each $n = 0, 1, \cdots, N$, and

$$\operatorname{ran} \widehat{A}(N) \subseteq \left(\bigoplus_{n=0}^{N-1} \operatorname{ran} \widehat{A}(n) \right)^{\perp}.$$

But since $A(z)$ is unitary for almost all $z \in \mathbb{T}$, we have that

$$\bigoplus_{n=0}^{N} \operatorname{ran} \widehat{A}(n) = E.$$

Similarly, we also have that $\widehat{A}(n)^*$ is a partial isometry for each $n = 0, 1, \cdots, N$, and

$$E = \bigoplus_{n=0}^{N} \operatorname{ran} \widehat{A}(n)^*.$$

(b) $\Rightarrow$ (c): Suppose $\widehat{A}(n)$ and $\widehat{A}(n)^*$ are partial isometries for each $n = 0, 1, \cdots, N$, and

$$E = \bigoplus_{n=0}^{N} \operatorname{ran} \widehat{A}(n) = \bigoplus_{n=0}^{N} \operatorname{ran} \widehat{A}(n)^*.$$

Write $E_n := \operatorname{ran} \widehat{A}(n)$ and $F_n := \operatorname{ran} \widehat{A}(n)^*$. Then we can write

$$A(z) = \begin{bmatrix} \widehat{A}(0)|_{F_0} & 0 & 0 & \cdots & 0 \\ 0 & (\widehat{A}(1)|_{F_1})z & 0 & \cdots & 0 \\ \vdots & 0 & (\widehat{A}(2)|_{F_2})z^2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & (\widehat{A}(N)|_{F_N})z^N \end{bmatrix} : \bigoplus_{n=0}^{N} F_n \to \bigoplus_{n=0}^{N} E_n.$$

Let P_m be the orthogonal projection from E onto $\bigoplus_{k=m}^{N} F_k$ $(m = 1, 2, \cdots, N)$. Then

$$A(z) = V \prod_{m=1}^{N} \Big(zP_m + (I - P_m) \Big),$$

where $V := \operatorname{diag}(\widehat{A}(0), \widehat{A}(1), \cdots, \widehat{A}(N))$ is unitary.

(c) $\Rightarrow$ (a): Obvious.

This completes the proof. □

The following corollary gives an affirmative answer to Question 1.1.

Corollary 2.2. If $\Delta \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ is a left inner divisor of zI_E, then Δ is a Blaschke-Potapov factor.

Proof. Immediate from Theorem 2.1. □

We recall:

Lemma 2.3. [14, Theorem 3.11.10] Let $A \in H^\infty(\mathbb{D}, \mathcal{B}(D, E))$. Then the nontangential SOT limit

$$\lim_{r \to 1} A(rz) \equiv bA(z)$$

exists for almost all $z \in \mathbb{T}$, such that $bA \in H^\infty(\mathbb{T}, \mathcal{B}(D, E))$ and

$$A(\zeta)x = P[bA](\zeta)x,$$

for $x \in D$ and $\zeta \in \mathbb{D}$. The Taylor coefficients of A coincide with the nonnegatively-indexed Fourier coefficients of bA. Moreover, the mapping $b : A \longrightarrow bA$ is an isometric bijection from $H^\infty(\mathbb{D}, \mathcal{B}(D, E))$ onto $H^\infty(\mathbb{T}, \mathcal{B}(D, E))$.

For $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ and $\alpha \in \mathbb{D}$, write

$$\Phi_\alpha := \Phi \circ b_\alpha.$$

Then we can easily check the following (cf. [10]):

(a) $\Phi_\alpha \in H^\infty(\mathbb{T}, \mathcal{B}(E))$;

(b) If Δ is an inner function with values in $\mathcal{B}(E)$, then so is Δ_α.

We then have:

Corollary 2.4. Let $A \in H^\infty(\mathbb{T}, \mathcal{B}(E))$. Then the following are equivalent:

(a) A is an inner divisor of $b_\alpha^N I_E$;

(b) A is a polynomial in b_α of degree at most N, of the form

$$A = \sum_{n=0}^{N} A_n b_\alpha^n,$$

where A_n and A_n^* are partial isometries for each $n = 0, 1, \cdots, N$, and

$$E = \bigoplus_{n=0}^{N} \operatorname{ran} A_n = \bigoplus_{n=0}^{N} \operatorname{ran} A_n^*;$$

(c) A is a finite Blaschke-Potapov product of the form

$$A = V \prod_{m=1}^{N} \Big(b_\alpha P_m + (I - P_m) \Big),$$

where P_m is the orthogonal projection from E onto $\bigoplus_{n=m}^{N} \operatorname{ran} A_n^*$, and $V := \operatorname{diag}(A_0|_{\operatorname{ran} A_0^*}, A_1|_{\operatorname{ran} A_1^*}, \cdots, A_N|_{\operatorname{ran} A_N^*})$.

Proof. Observe that

(i) $A(z) = \sum_{n=0}^{N} A_n b_\alpha(z)^n \Longleftrightarrow A_{-\alpha}(z) = \sum_{n=0}^{N} A_n z^n$;

(ii) A is an inner divisor of $b_\alpha^N I_E \Longleftrightarrow A_{-\alpha}$ is an inner divisor of $z^N I_E$;

(iii) $A(z) = V \prod_{m=1}^{N} \Big(b_\alpha P_m + (I - P_m) \Big) \Longleftrightarrow A_{-\alpha}(z) = V \prod_{m=1}^{N} \Big(z P_m + (I - P_m) \Big)$.

Thus the result follows at once from Theorem 2.1. $\square$

Corollary 2.5. Let $A \in H^\infty(\mathbb{T}, \mathcal{B}(E))$. If A is a nontrivial inner divisor of $b_\alpha^N I_E$, then $\widehat{A_{-\alpha}}(n) \neq 0$ for some $n = 1, 2, \cdots, N$.

Proof. Immediate from Corollary 2.4. $\square$

3 Operator-valued rational functions

In this section we will introduce the notion of operator-valued "rational" function. Recall that a matrix-valued function is rational if its entries are rational functions. But this definition is not appropriate for operator-valued functions, in particular H^∞-functions, even though the terminology of "entry" may be properly interpreted. Thus, the new idea should capture and encapsulate a definition of operator-valued rational function which is equivalent to the condition that each entry is rational when the function is matrix-valued. In the sequel, we give a formal definition of operator-valued rational function.

To do so, we first recall Toeplitz and Hankel operators. For $\Phi \in L^\infty(\mathbb{T}, \mathcal{B}(D,E))$, the Hankel operator $H_\Phi : H^2(\mathbb{T}, D) \to H^2(\mathbb{T}, E)$ is defined by

$$H_\Phi f := JP_-(\Phi f) \quad (f \in H^2(\mathbb{T}, D)),$$

where J denotes the unitary operator from $L^2(\mathbb{T}, E)$ to $L^2(\mathbb{T}, E)$ given by $(Jg)(z) := \overline{z}g(\overline{z})$ for $g \in L^2(\mathbb{T}, E)$ and P_- is the orthogonal projection from $L^2(\mathbb{T}, E)$ onto $L^2(\mathbb{T}, E) \ominus H^2(\mathbb{T}, E)$. Also a Toeplitz operator $T_\Phi : H^2(\mathbb{T}, D) \to H^2(\mathbb{T}, E)$ is defined by

$$T_\Phi f := P_+(\Phi f) \quad (f \in H^2(\mathbb{T}, D)),$$

where P_+ is the orthogonal projection from $L^2(\mathbb{T}, E)$ onto $H^2(\mathbb{T}, E)$.

As usual, a *shift* operator S_E on $H^2(\mathbb{T}, E)$ is defined by

$$(S_E f)(z) := zf(z) \quad \text{for each } f \in H^2(\mathbb{T}, E).$$

The following theorem is a fundamental result in modern operator theory.

The Beurling-Lax-Halmos Theorem. [7], [12], [8], [15] A subspace M of $H^2(\mathbb{T}, E)$ is invariant for the shift operator S_E on $H^2(\mathbb{T}, E)$ if and only if

$$M = \Delta H^2(\mathbb{T}, E'),$$

where E' is a subspace of E and Δ is an inner function with values in $\mathcal{B}(E', E)$. Furthermore, Δ is unique up to a unitary constant right factor, i.e., if $M = \Theta H^2(\mathbb{T}, E'')$, where Θ is an inner function with values in $\mathcal{B}(E'', E)$, then $\Delta = \Theta V$, where V is a unitary operator from E' onto E''.

If $\Phi \in L^\infty(\mathbb{T}, \mathcal{B}(D,E))$ then, by the Beurling-Lax-Halmos Theorem,

$$\ker H_{\Phi^*} = \Delta H^2(\mathbb{T}, E')$$

for some inner function Δ with values in $\mathcal{B}(E', E)$. We note that E' may be the zero space and Δ need not be two-sided inner.

We recall:

Definition 3.1. [7] A function $\Phi \in L^2_s(\mathbb{T}, B(D,E))$ is said to be of *bounded type* if $\ker H^*_\Phi = \Theta H^2(\mathbb{T}, E)$ for some two-sided inner function Θ with values in $\mathcal{B}(E)$.

It is known that if $\phi \equiv (\phi_{ij})$ is a matrix-valued function of bounded type then each entry ϕ_{ij} is of bounded type, i.e., ϕ_{ij} is a quotient of two bounded analytic functions. In particular, it is also known ([1], [4]) that if $\phi \in L^2$ is of bounded type then ϕ can be written as

$$\phi = \overline{\theta} a \quad \text{(where } \theta \text{ is an inner function and } a \in H^2). \tag{10}$$

Lemma 3.2. [7, Lemma 2.4] If $\Phi \in L^\infty(\mathbb{T}, \mathcal{B}(D,E))$ and Δ is a two-sided inner function with values in $\mathcal{B}(E)$, then the following are equivalent:

(a) $\breve{\Phi}$ is of bounded type, i.e., $\ker H_{\Phi^*} = \Delta H^2(\mathbb{T}, E)$;

(b) $\Phi = \Delta A^*$, where $A \in H^\infty(\mathbb{T}, \mathcal{B}(E,D))$ is such that Δ and A are right coprime.

We now introduce:

Definition 3.3. A function $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D,E))$ is said to be *rational* if

$$\theta H^2(\mathbb{T}, E) \subseteq \ker H_{\Phi^*} \tag{11}$$

for some finite Blaschke product θ.

Observe that if $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D,E))$, then

$$\Phi \text{ is rational} \Longrightarrow \breve{\Phi} \text{ is of bounded type.} \tag{12}$$

To see this, suppose Φ is rational. By definition and the Beurling-Lax-Halmos Theorem there exist a finite Blaschke product θ and an inner function $\Delta \in H^\infty(\mathbb{T}, \mathcal{B}(E', E)))$ such that

$$\theta H^2(\mathbb{T}, E) \subseteq \ker H_{\Phi^*} = \Delta H^2(\mathbb{T}, E'),$$

which implies that Δ is a left inner divisor of θI_E. Thus Δ is two-sided inner, so that by Lemma 3.2, $\breve{\Phi}$ is of bounded type, which proves (12).

Also, if $\Phi \equiv (\phi_{ij}) \in H^\infty(\mathbb{T}, M_{m\times n})$ is a rational function in the sense of Definition 3.3, then each entry ϕ_{ij} is rational. To see this suppose a matrix-valued function Φ satisfies the condition (11). Put $A := \theta\Phi^*$. Then $A \in H^\infty(\mathbb{T}, M_{n\times m})$ and $\Phi = \theta A^*$. Thus ϕ_{ij} can be written as

$$\phi_{ij} = \theta \overline{a_{ij}} \quad (a_{ij} \in H^\infty).$$

Via Kronecker's Lemma [13, p.183], we can see that

$$\phi_{ij} \text{ is rational} \Longleftrightarrow \phi_{ij} = \theta \overline{a_{ij}} \text{ with a finite Blaschke product } \theta, \tag{13}$$

which says that each ϕ_{ij} is rational.

In particular, if $\theta = z^n$ in (11), Φ becomes an operator-valued polynomial.

In 1955, V.P. Potapov [16] proved that an $n \times n$ matrix-valued function Φ is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product. In this section, we extend this result to operator-valued functions. To so so, we first observe:

Lemma 3.4. Suppose that θ is a finite Blaschke product of the form

$$\theta = \prod_{n=1}^{r} b_{\alpha_n} \quad (\alpha_n \in \mathbb{D})$$

and Δ is an inner divisor of $\Theta = \theta I_E$. Let $\Omega := \Theta\Delta^*$ and P_n be the orthogonal projection of E onto $\mathrm{cl\,ran}\, P[\Omega](\alpha_n)$. Then $\Delta\big(b_{\alpha_n} P_n + (I_E - P_n)\big)^*$ is an inner divisor of $\Theta_n := \theta \overline{b_{\alpha_n}} I_E$ for each $n = 1, 2, \cdots, r$.

Proof. Write

$$A := P[\Delta] \quad \text{and} \quad C := P[\Omega].$$

Then it follows from (4) that

$$P[\Theta] = AC = CA.$$

Since $0 = P[\Theta](\alpha_n) = A(\alpha_n)C(\alpha_n)$, we have that $\mathrm{cl\,ran}\, C(\alpha_n) \subseteq \ker A(\alpha_n)$. Thus we may write

$$A(\alpha_n) = A(\alpha_n)\big(b_{\alpha_n} P_n + (I_E - P_n)\big), \tag{14}$$

where P_n is the orthogonal projection of E onto $\mathrm{cl\,ran}\, C(\alpha_n)$. On the other hand, we can write

$$A - A(\alpha_n) = A_n(b_{\alpha_n} I_E) \quad \text{for some } A_n \in H^\infty(\mathbb{D}, \mathcal{B}(E)).$$

It thus follows from (14) that

$$\begin{aligned} A =& A(\alpha_n)\big(b_{\alpha_n} P_n + (I_E - P_n)\big) \\ &+ A_n\big(P_n + b_{\alpha_n}(I_E - P_n)\big)\big(b_{\alpha_n} P_n + (I_E - P_n)\big) \\ =& \Big[A(\alpha_n) + A_n\big(P_n + b_{\alpha_n}(I_E - P_n)\big)\Big]\big(b_{\alpha_n} P_n + (I_E - P_n)\big). \end{aligned}$$

Since $A(\alpha_n) + A_n\big(P_n + b_{\alpha_n}(I_E - P_n)\big) \in H^\infty(\mathbb{D}, \mathcal{B}(E))$, it follows from Lemma 2.3 that

$$\Delta_n := b\big(A(\alpha_n) + A_n\big(P_n + b_{\alpha_n}(I_E - P_n)\big)\big) \in H^\infty(\mathbb{T}, \mathcal{B}(E))$$

and by (4),

$$\Delta = b(A) = \Delta_n\big(b_{\alpha_n} P_n + (I_E - P_n)\big). \tag{15}$$

Now write $B_n := b_{\alpha_n} P_n + (I_E - P_n)$. Then $(B_n C)(\alpha_n) = (I_E - P_n)C(\alpha_n) = 0$. Thus we can write

$$B_n C = (b_{\alpha_n} I_E) F \quad \text{for some } F \in H^\infty(\mathbb{D}, \mathcal{B}(E)).$$

Thus $P[\Theta] = AC = AB_n^*(b_{\alpha_n} I_E)F$ and hence, by (4) and (15), we have

$$\Theta = \Delta_n (b_{\alpha_n} I_E) b(F),$$

so that

$$\theta \overline{b_{\alpha_n}} I_E = \Delta_n b(F).$$

This completes the proof. □

We then have:

Theorem 3.5. Let θ be a finite Blaschke product. If Δ is an inner divisor of $\Theta = \theta I_E$, then Δ is a finite Blaschke-Potapov product.

Proof. Suppose θ is a finite Blaschke product of the form

$$\theta = \prod_{n=1}^{r} b_{\alpha_n}.$$

If Δ is an inner divisor of $\Theta = \theta I_E$, then $\Omega := \Theta\Delta^*$ is also an inner divisor of Θ. Let P_n be the orthogonal projection of E onto cl ran $P[\Omega](\alpha_n)$. Then it follows from Lemma 3.4 that $\Delta_1 := \Delta\big(b_{\alpha_r} P_r + (I_E - P_r)\big)^*$ is an inner divisor of $\theta\overline{b_{\alpha_r}} I_E$. By the same argument we have that

$$\begin{aligned}\Delta_2 : &= \Delta_1\big(b_{\alpha_{r-1}} P_{r-1} + (I_E - P_{r-1})\big)^* \\ &= \Delta\big(b_{\alpha_r} P_r + (I_E - P_r)\big)^*\big(b_{\alpha_{r-1}} P_{r-1} + (I_E - P_{r-1})\big)^*\end{aligned}$$

is an inner divisor of $\theta\overline{b_{\alpha_{r-1}} b_{\alpha_r}} I_E$. Continuing this process, we have that

$$\Delta_r := \Delta \prod_{n=0}^{r-1} \big(b_{\alpha_{r-n}} P_{r-n} + (I_E - P_{r-n})\big)^*$$

is an inner divisor of I_E. Thus $V \equiv \Delta_r$ is a unitary operator, and hence

$$\Delta = V \prod_{n=1}^{r} \big(b_{\alpha_n} P_n + (I_E - P_n)\big).$$

is a finite Blaschke-Potapov product. This completes the proof. $\square$

The following corollary is an operator-valued version of Kronecker's Lemma (cf. (13)).

Corollary 3.6. A function $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E))$ is rational if and only if

$$\Phi = \Delta A^*, \tag{16}$$

where Δ is a finite Blaschke-Potapov product and $A \in H^\infty(\mathbb{T}, \mathcal{B}(E, D))$ is such that Δ and A are right coprime. In this case, Δ is obtained from the equation $\ker H_{\Phi^*} = \Delta H^2(\mathbb{T}, E)$. In particular,

$$\Phi \text{ is a polynomial} \Longrightarrow A \text{ is a polynomial.} \tag{17}$$

Proof. The first and the second assertions follow at once from Lemma 3.2, (12) and Theorem 3.5. For the implication (17), suppose Φ is a polynomial. Then for some $N \geq 0$,

$$z^N H^2(\mathbb{T}, E) \subseteq \ker H_{\Phi^*} = \Delta H^2(\mathbb{T}, E),$$

which implies that $\Delta\Omega = z^N I_E$ for some two-sided inner function $\Omega \in H^\infty(\mathbb{T}, \mathcal{B}(E))$. Thus, $\Phi = \Delta\Omega\Omega^* A^* = z^N\Omega^* A^* = \Omega^* A^* z^N$, so that

$$A^* z^N H^2(\mathbb{T}, D) = \Omega\Phi H^2(\mathbb{T}, D) \subseteq H^2(\mathbb{T}, E).$$

Therefore, $z^N H^2(\mathbb{T}, D) \subseteq \ker\, H_{A^*}$, which says that A is a polynomial. $\square$

The converse of (17) is not true in general: for example, take $A = 1$. On the other hand, A in the decomposition (16) need not be a polynomial in general even for scalar-valued rational functions Φ. For example, if

$$\Phi(z) = \frac{z - \frac{1}{2}}{z - 3},$$

then the decomposition (16) is given by

$$\Delta = b_{\frac{1}{3}} \equiv \frac{z - \frac{1}{3}}{1 - \frac{1}{3}z} \quad \text{and} \quad A = \frac{1 - \frac{1}{2}z}{z - 3},$$

where Δ and A are coprime.

We are ready for:

Corollary 3.7. A two-sided inner function $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ is rational if and only if it can be represented as a finite Blaschke-Potapov product.

Proof. Suppose that Φ is rational and two-sided inner. Then it follows from Corollary 3.6 that

$$\Phi = \Delta A^*, \tag{18}$$

where Δ is a finite Blaschke-Potapov product and $A \in H^\infty(\mathbb{T}, \mathcal{B}(E))$. Since Φ and Δ are two-sided inner, so is A. Thus, by (18), Φ is a left inner divisor of Δ, and hence the result follows from Theorem 3.5. The converse is clear. This completes the proof. □

4 Coprime operator-valued rational functions

In this section we consider coprime operator-valued rational functions.

Lemma 4.1. Let $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(E))$. If $\operatorname{ran} P[\Phi](\alpha)$ is not dense for $\alpha \in \mathbb{D}$, then

$$P := b_\alpha P_M + (I - P_M) \quad (M := \ker P[\Phi](\alpha)^*)$$

is a nontrivial left inner divisor of Φ.

Proof. Write $A := P[\Phi]$; that is, A is the Poisson integral of Φ, defined by (3). Suppose that the range of $A(\alpha)$ is not dense. Then $M := \ker A(\alpha)^* = \big(\text{cl ran}\, A(\alpha)\big)^\perp \neq \{0\}$. Put $P := b_\alpha P_M + (I - P_M)$. Then $(P^* b_\alpha I_E A)(\alpha) = 0$, and hence we can write

$$P^* b_\alpha I_E A = b_\alpha I_E A_1 \quad \text{for some } A_1 \in H^\infty(\mathbb{D}, \mathcal{B}(E)),$$

which implies that $A = P A_1$. This completes the proof. □

For an inner function θ, let $\mathcal{Z}(\theta)$ be the set of all zeros of θ. Then we have:

Theorem 4.2. Let $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ and $\Theta := \theta I_E$ with a finite Blaschke product θ. Then the following statements are equivalent:

(a) $\operatorname{ran} P[\Phi](\alpha)$ is dense for each $\alpha \in \mathcal{Z}(\theta)$;

(b) Φ and Θ are left coprime.

Proof. (a) $\Rightarrow$ (b): Suppose that Φ and Θ are not left coprime. Then by Theorem 3.5, there exist $\alpha_0 \in \mathcal{Z}(\theta)$ and a nonzero subspace M of E such that

$$\Phi = \big(b_{\alpha_0} P_M + (I - P_M)\big)\Omega,$$

where $\Omega \in H^\infty(\mathbb{T}, \mathcal{B}(E))$. Thus $\operatorname{cl} \operatorname{ran} P[\Phi](\alpha_0) \subseteq M^\perp \neq E$.

(b) $\Rightarrow$ (a): This follows from at once form Lemma 4.1. □

Lemma 4.3. If $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E))$, then $\widetilde{\Phi} \in H^\infty(\mathbb{T}, \mathcal{B}(E, D))$. In this case,

$$P[\widetilde{\Phi}] = \widetilde{P[\Phi]}$$

Proof. Since $\widehat{\widetilde{\Phi}}(n) = \widehat{\Phi}(n)^*$ for all $n = 0, 1, 2, \cdots$, it follows that

$$P[\widetilde{\Phi}](\zeta) = \sum_{n=0}^{\infty} \widehat{\Phi}(n)^* \zeta^n = \widetilde{P[\Phi]}(\zeta).$$

□

Corollary 4.4. Let $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(E))$. If $P[\Phi](\alpha)$ is not injective for $\alpha \in \mathbb{D}$, then

$$P := b_\alpha P_M + (I - P_M) \quad (M := \ker P[\Phi](\alpha))$$

is a nontrivial right inner divisor of Φ.

Proof. Suppose that $P[\Phi](\alpha)$ is not injective. Then, by Lemma 4.3, $\operatorname{ran} (P[\Phi](\alpha))^* = \operatorname{ran} P[\widetilde{\Phi}](\overline{\alpha})$ is not dense. Let

$$Q := b_{\overline{\alpha}} P_M + (I - P_M),$$

where $M := \ker P[\Phi](\alpha) = \ker P[\widetilde{\Phi}](\overline{\alpha})^* \neq \{0\}$. Then it follow from Lemma 4.1 that Q is a nontrivial left inner divisor of $\widetilde{\Phi}$. But since Q is two-sided inner, it follows that $P = \widetilde{Q}$ is a nontrivial right inner divisor of Φ. This completes the proof. □

We also have:

Corollary 4.5. Let $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ and $\Theta := \theta I_E$ with a finite Blaschke product θ. Then the following statements are equivalent:

(a) $P[\Phi](\alpha)$ is injective for each $\alpha \in \mathcal{Z}(\theta)$;

(b) Φ and Θ are right coprime.

Proof. Immediate from Theorem 4.2 and Lemma 4.3. □

Corollary 4.6. Let $\Phi \in H^\infty(\mathbb{T}, M_n)$ and $\Theta := \theta I_n$ with a finite Blaschke product θ. Then the following statements are equivalent:

(a) $P[\Phi](\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$;

(b) Φ and Θ are right coprime;

(c) Φ and Θ are left coprime.

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from Theorem 4.2 and Corollary 4.5 together with matrix theory. The equivalence (b) $\Leftrightarrow$ (c) comes from [5, Lemma 3.3]. □

The equivalence (b) $\Leftrightarrow$ (c) of Corollary 4.6 may fail for operator-valued functions. For example, if we take $E = \ell^2(\mathbb{Z}_+)$, then S_E and zI_E are right coprime, but not left coprime (cf. [7]).

5 Miscellany

In this section, we establish some key differences between matrix-valued functions and operator-valued functions.

5.1 A glance at right coprime-ness

If Φ and Ψ are not left coprime then there exists a common nontrivial left inner divisor Δ of both Φ and Ψ. However we don't guarantee that this is still true for right coprime-ness. In other words, if Φ and Ψ are not right coprime then by definition $\Phi = A\Delta$ and $\Psi = B\Delta$ for some nontrivial inner function $\widetilde{\Delta}$. However we need not expect that Δ is inner.

We here give a sufficient condition for the existence of a common nontrivial right inner divisor of two functions when they are not right coprime.

To see this, we first recall that a function $F \in H^\infty(\mathbb{T}, \mathcal{B}(E', E))$ is called *outer* if $\mathrm{cl}\, FH^2(\mathbb{T}, E') = H^2(\mathbb{T}, E)$. We then have an analogue of the scalar factorization theorem (called the *inner-outer factorization*):

The inner-outer factorization [13]. If $A \in H^\infty(\mathbb{T}, \mathcal{B}(D, E))$, then we can write

$$A = A^i A^e \quad \text{(inner-outer factorization)},$$

where E' is a subspace of E, $A^i \in H^\infty(\mathbb{T}, \mathcal{B}(E', E))$ is an inner function, and $A^e \in H^\infty(\mathbb{T}, \mathcal{B}(D, E'))$ is an outer function.

The following lemma is a characterization of functions of bounded type.

Lemma 5.1. [7, Corollary 2.25.] Let Ω be an inner function with values in $\mathcal{B}(D,E)$. Then

$$\breve{\Omega} \text{ is of bounded type} \Longleftrightarrow [\Omega, \Omega_c] \text{ is two-sided inner},$$

where Ω_c is the complementary factor of Ω, i.e., $\ker H_{\Omega^*} = [\Omega, \Omega_c]\, H^2_{D\oplus D'}$ for some Hilbert space D', and $[\Omega, \Omega_c]$ denotes the 1×2 matrix whose entries are Ω and Ω_c.

We then have:

Theorem 5.2. Suppose that $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E_1))$ and $\Psi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E_2))$ are not right coprime. If there exists a nontrivial left inner divisor Ω of $\Delta :=$ left-g.c.d.$(\widetilde{\Phi}, \widetilde{\Psi})$ and $\breve{\Omega}$ is of bounded type, then $\widetilde{[\Omega, \Omega_c]}$ is a common nontrivial right inner divisor of both Φ and Ψ.

Proof. Since $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E_1))$ and $\Psi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E_2))$ are not right coprime, $\Delta :=$ left-g.c.d.$(\widetilde{\Phi}, \widetilde{\Psi}) \in H^\infty(\mathbb{T}, \mathcal{B}(D_1, D))$ is not a unitary operator and we can write

$$\widetilde{\Phi} = \Delta\widetilde{\Phi}_1 \quad \text{and} \quad \widetilde{\Psi} = \Delta\widetilde{\Psi}_1, \tag{19}$$

where $\widetilde{\Phi}_1 \in H^\infty(\mathbb{T}, \mathcal{B}(E_1, D_1))$ and $\widetilde{\Psi}_1 \in H^\infty(\mathbb{T}, \mathcal{B}(E_2, D_1))$. Since Ω is a left inner divisor of Δ, we can write

$$\Delta = \Omega\Delta_1 \quad (\Delta_1 \in H^\infty(\mathbb{T}, \mathcal{B}(D_1, D_2)) \quad (\Omega \in H^\infty(\mathbb{T}, \mathcal{B}(D_2, D))). \tag{20}$$

Since $\breve{\Omega}$ is not a unitary operator and is of bounded type, by Lemma 5.1, $\Omega_0 \equiv [\Omega, \Omega_c]$ is not a unitary operator and a two-sided inner function. Note that Ω_c is an inner function with values in $\mathcal{B}(D_3, D)$ for some Hilbert space D_3. Thus we can write

$$\Delta = \Omega\Delta_1 = [\Omega, \Omega_c]\begin{bmatrix}\Delta_1 \\ \mathbf{0}\end{bmatrix} \equiv \Omega_0\Delta_0 \quad (\text{where } \mathbf{0} : D_4 \to D_3).$$

It thus follows from (19) and(20) that

$$\Phi = \Phi_1\widetilde{\Delta}_0\widetilde{\Omega}_0 \quad \text{and} \quad \Psi = \Psi_1\widetilde{\Delta}_0\widetilde{\Omega}_0.$$

But since Ω_0 is two-sided inner, we have that $\widetilde{\Omega}_0$ is (two-sided) inner. This completes the proof. □

Corollary 5.3. Let $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E_1))$, $\Psi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E_2))$ and $\Delta :=$ left-g.c.d.$(\widetilde{\Phi}, \widetilde{\Psi})$. If $\breve{\Delta}$ is of bounded type, then $\widetilde{[\Delta, \Delta_c]}$ is a common nontrivial right inner divisor of both Φ and Ψ.

Proof. Immediate from Theorem 5.2. □

5.2 Subnormality of Toeplitz operators

In 1970, P.R. Halmos posed the following problem, listed as Problem 5 in his series of lectures, "Ten problems in Hilbert space" [11]:

Is every subnormal Toeplitz operator either normal or analytic ?

Halmos' Problem 5 has been partially answered in the affirmative by many authors. However, in 1984, Halmos' Problem 5 was answered in the negative by C. Cowen and J. Long [2]. Despite considerable efforts, to date researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. Thus we have:

Halmos' Problem 5 reformulated. *Which Toeplitz operators are subnormal ?*

For cases of matrix-valued symbols, the subnormality of Toeplitz operators was studied in [5], in which it was shown that if the matrix-valued symbol Φ satisfies a general condition on coprime factorization and T_Φ is subnormal then it is either normal or analytic. Also in [3], it was conjectured that every subnormal Toeplitz operator with matrix-valued rational symbol is unitarily equivalent to a direct sum of a normal operator and a Toeplitz operator with analytic symbol. In fact, if an $n \times n$ matrix-valued function Φ is analytic then the normal extension of T_Φ is the multiplication operator M_Φ, so clearly T_Φ is subnormal. However, this is not the case for the operator-valued symbols. In this section we will give an example (see Example 5.5 below). On the other hand, if Φ is matrix-valued and T_Φ is subnormal (even hyponormal), then Φ should be normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$ a.e. on $\mathbb{T}$ (cf. [9]). However this may also fail for operator-valued symbols.

Example 5.4. Let $S := T_z$ on $H^2(\mathbb{T})$ and $\Phi(z) = Sz^n \in H^\infty(\mathbb{T}, \mathcal{B}(H^2(\mathbb{T})))$ $(n \geq 0)$. Then

$$T_\Phi^* T_\Phi = T_{S^*S} = I_{H^2(\mathbb{T}, H^2(\mathbb{T}))},$$

so that T_Φ is quasinormal and hence subnormal. However,

$$\Phi(z)\Phi^*(z) = SS^* \neq S^*S = \Phi^*(z)\Phi(z) \quad \text{for all } z \in \mathbb{T},$$

which implies that Φ is not normal. Here we don't need to expect that the multiplication operator $M_\Phi : L^2(\mathbb{T}, \mathcal{B}(H^2(\mathbb{T}))) \to L^2(\mathbb{T}, \mathcal{B}(H^2(\mathbb{T})))$ is a normal extension of T_Φ. Indeed, it is easy to show that M_Φ is not normal, and hence M_Φ can never be a normal extension of T_Φ. What is a normal extension of T_Φ ? Let $B := M_z$ on $L^2(\mathbb{T})$ and $\Psi(z) := Bz^n \in H^\infty(\mathbb{T}, \mathcal{B}(L^2(\mathbb{T})))$. Then a straightforward calculation shows that the multiplication operator $M_\Psi : L^2(\mathbb{T}, \mathcal{B}(L^2(\mathbb{T}))) \to L^2(\mathbb{T}, \mathcal{B}(L^2(\mathbb{T})))$ is a normal extension of T_Φ.

The following simple example shows that analytic Toeplitz operators with operator-valued symbols need not be even hyponormal.

Example 5.5. Let $\Phi(z) = S^* \in H^\infty(\mathbb{T}, \mathcal{B}(H^2(\mathbb{T})))$ and e_0 be the costant function $\mathbf{1} \in H^2(\mathbb{T})$. If $f(z) = e_0 z$, then

$$\langle (T_{\Phi^*}T_\Phi - T_\Phi T_{\Phi^*})f,\ f\rangle = \langle -e_0 z, e_0 z\rangle = -1 < 0,$$

which implies that T_Φ is not hyponormal and hence not subnormal even though Φ is analytic.

We would like to pose:

Question 5.6. Which analytic Toeplitz operators with operator-valued symbols are subnormal ?

For a sufficient condition, one may be tempted to conjecture that if $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(H^2(\mathbb{T})))$ and if $\Phi(z)$ is subnormal for almost all $z \in \mathbb{T}$, then T_Φ is subnormal. We have not been able to decide whether this is true.

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