Recent developments in the interplay between function theory and operator theory for block Toeplitz, Hankel, and model operators

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Abstract. This is a semi-expository paper on some recent developments in the interplay between function theory and operator theory in the context of Toeplitz, Hankel, and model operators. We place special emphasis on the connections with the Beurling-Lax-Halmos Theorem, which characterizes the shift-invariant subspaces of the vector-valued Hardy space.

Contents

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1 Introduction

Over the last several years, the authors have studied a natural interplay between function theory and operator theory in the context of Toeplitz, Hankel, and model operators. For a long time, operator theory has had deep connections with function theory. The intensive and fruitful study of Toeplitz and Hankel operators has contributed in great measure to this synergy, including the study of the spectral properties of Toeplitz and Hankel operators, which are intrinsically determined by their symbols, i.e., by functions defined on the Hardy space. Also, the model operator for contractions, introduced by B. Sz.-Nagy and C. Foiaş, has been a focus in the study of this field. In fact, the model operator is a truncated backward shift operator on the model space constructed from an inner function called the characteristic function of the model operator. Since the spectral theory of the model operator is naturally determined by properties of its characteristic function, one is drawn to the study the inner functions. In particular, we pay close attention to the Beurling-Lax-Halmos Theorem, which characterizes the invariant subspaces of the shift operator acting on the vector-valued Hardy space. There are several interesting questions emerging from the Beurling-Lax-Halmos Theorem. A study of those questions invites a detailed analysis of matrix-valued functions and operator-valued functions.

In this semi-expository paper, we combine a brief survey of recent developments in function theory associated with Toeplitz, Hankel, and model operators with some new results on this subject.

For this study, we need some new notions; e.g., strong L^2-functions, complementary factors, and degree of non-cyclicity. In Section 2, we give a brief sketch of these notions. Section 3 is devoted to the matrix-valued function theory associated with Toeplitz, Hankel, and model operators. The first subject of Section 3 deals with the product of Hankel operators. It is well known that if the product of two (classical) Hankel operators is zero then one of them must be zero. However, this is not the case for matrix-valued symbols. To get an affirmative answer for matrix-valued symbols, we introduce a notion of tensored-scalar singularity and then prove a new result under such a condition. The second subject of Section 3 is the spectral multiplicity of the model operator with a matrix-valued characteristic function. Here we introduce the Beurling degree of an inner matrix function and then obtain an elegant formula: the spectral multiplicity of the model operator is equal to the Beurling degree of its characteristic function. The third subject of this section is Halmos' Problem 5: Is every subnormal Toeplitz operator either normal or analytic? Abrahamse's theorem gave a general sufficient condition for the answer to be affirmative. However, Abrahamse's theorem may fail for Toeplitz operators with matrix-valued symbols. Despite this, one can obtain a matrix-valued version under the constraint of tensored-scalar singularity of the symbol. The last subject of Section 3 is an H^∞-functional calculus for compressions of the shift operator. We review this functional calculus and then extend it to an $\overline{H^\infty} + H^\infty$-functional calculus.

Section 4 is devoted to operator-valued function theory. Firstly, we review meromorphic pseudo-continuations of bounded type and give an application to C_0-contractions. Secondly, we consider a canonical decomposition of strong L^2-functions, which generalizes the Douglas-Shapiro-Shields factorization for functions of bounded type. This idea provides a description of a set F such that

complementary factors, the Beurling degree, tensored-scalar singularity, meromorphic pseudo-continuation of bounded type, model operators, Hankel operators, Toeplitz operators.

given a model space, i.e., a backward shift-invariant subspace, the smallest invariant subspace of the backward shift operator containing F is equal to the model space. Thirdly, we examine a question on the spectrum of the model operator. In fact, if the characteristic function of the model operator is two-sided inner, then by the operator-valued version of the Livšic-Moeller Theorem, the spectrum of the model operator is computed from the spectrum of the characteristic function. However, this is not the case for general model operators. We give a partial answer to this question for general model operators using the complementary factor of the characteristic function. Fourthly, we introduce operator-valued rational functions and give an operator-valued extension of Potapov's matrix-valued factorization theorem. Lastly, we pose some unsolved problems on hyponormality and subnormality of Toeplitz operators with operator-valued symbols.

2 Preliminaries and basic theory

In this section we provide the notation, basic notions and basic results which will be used in this paper. For instance, we introduce the notions of strong L^2-functions, the Beurling-Lax-Halmos Theorem, the Douglas-Shapiro-Shields factorization, the complementary factor of an inner function, and the degree of non-cyclicity.

2.1 Basic notions

Throughout the paper, we suppose that D and E are separable complex Hilbert spaces. We write $\mathcal{B}(D,E)$ for the set of all bounded linear operators from D to E and abbreviate $\mathcal{B}(E,E)$ as $\mathcal{B}(E)$. For $A, B \in \mathcal{B}(E)$, we let $[A,B] := AB - BA$. An operator $T \in \mathcal{B}(E)$ is said to be normal if $[T^*,T] = 0$ and hyponormal if $[T^*,T] \geq 0$. For an operator $T \in \mathcal{B}(E)$, we write $\ker T$ and $\operatorname{ran} T$ for the kernel and the range of T, respectively. For a subset $\mathcal{M} \subseteq E$, $\operatorname{cl}\mathcal{M}$ and $\mathcal{M}^\perp$ denote the closure and the orthogonal complement of $\mathcal{M}$, respectively.

If $A : D \to E$ is a linear operator whose domain is a subspace of D, then A is also a linear operator from the closure of the domain of A into E. So we will only consider those A such that the domain of A is dense in D. Such an operator A is said to be densely defined. If $A : D \to E$ is densely defined, write $\operatorname{dom} A^* = \big\{e \in E : d \mapsto \langle Ad, e\rangle$ is a bounded linear functional on $\operatorname{dom} A\big\}$. If $e \in \operatorname{dom} A^*$, then there exists a unique $f \in E$ such that $\langle Ad, e\rangle = \langle d, f\rangle$ for all $d \in \operatorname{dom} A$. Denote this unique vector f by $f \equiv A^*e$. Thus $\langle Ad, e\rangle = \langle d, A^*e\rangle$ for all $d \in \operatorname{dom} A$ and $e \in \operatorname{dom} A^*$. We call A^* the adjoint of A. It is well known from unbounded operator theory (cf. [Go], [Con]) that if A is densely defined, then $\ker A^* = (\operatorname{ran} A)^\perp$, so that $\ker A^*$ is closed even though $\ker A$ may not be closed.

We write $\mathbb{D}$ for the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ for the unit circle in $\mathbb{C}$. For $\phi \in L^2$, write

$$\breve{\phi}(z) := \phi(\overline{z}) \quad \text{and} \quad \widetilde{\phi}(z) := \overline{\phi(\overline{z})}.$$

For $\phi \in L^2$, write

$$\phi_+ := P_+\phi \quad \text{and} \quad \breve{\phi}_- := P_-\phi,$$

where P_+ and P_- are the orthogonal projections from L^2 onto H^2 and $L^2 \ominus H^2$, respectively. Thus, we may write $\phi = \breve{\phi}_- + \phi_+$. We recall ([Ab], [Co2], [GHR], [Ni1]) that a meromorphic function

$\phi : \mathbb{D} \to \mathbb{C}$ is said to be of *bounded type* (or in the Nevanlinna class $\mathcal{N}$) if there are functions $\psi_1, \psi_2 \in H^\infty$ such that $\phi(z) = \frac{\psi_1(z)}{\psi_2(z)}$ for almost all $z \in \mathbb{T}$. We can easily check that if $\phi \in L^2$ is of bounded type, then ϕ can be written as

$$\phi = \overline{\theta} a, \tag{1}$$

where θ is inner, $a \in H^2$ and θ and a are coprime. Write $\mathbb{D}^e := \{z : 1 < |z| \le \infty\}$. For a function $g : \mathbb{D}^e \to \mathbb{C}$, define a function $g_{\mathbb{D}} : \mathbb{D} \to \mathbb{C}$ by $g_{\mathbb{D}}(\zeta) := \overline{g(1/\overline{\zeta})}$ $(\zeta \in \mathbb{D})$. For a function $g : \mathbb{D}^e \to \mathbb{C}$, we say that g belongs to $H^p(\mathbb{D}^e)$ if $g_{\mathbb{D}} \in H^p$ $(1 \le p \le \infty)$. A function $g : \mathbb{D}^e \to \mathbb{C}$ is said to be of *bounded type* if $g_{\mathbb{D}}$ is of bounded type. If $f \in H^2$, then the function $\hat{f}$ defined in $\mathbb{D}^e$ is called a *pseudo-continuation* of f if $\hat{f}$ is a function of bounded type and $\hat{f}(z) = f(z)$ for almost all $z \in \mathbb{T}$ (cf. [BoB], [Ni1], [Sh]). Then we can easily show that $\breve{f}$ is of bounded type if and only if f has a pseudo-continuation $\hat{f}$. In this case, $\hat{f}_{\mathbb{D}}(z) = \overline{f(z)}$ for almost all $z \in \mathbb{T}$. In particular,

$$\phi \equiv \breve{\phi}_- + \phi_+ \in L^2 \text{ is of bounded type } \iff \phi_- \text{ has a pseudo-continuation.} \tag{2}$$

Let m denote the normalized Lebesgue measure on $\mathbb{T}$. For a complex Banach space X and $1 \le p \le \infty$, let

$$L^p_X \equiv L^p(\mathbb{T}, X) := \big\{ f : \mathbb{T} \to X : f \text{ is strongly measurable and } ||f||_p < \infty \big\},$$

where

$$||f||_p \equiv ||f||_{L^p_X} := \begin{cases} \left(\int_{\mathbb{T}} ||f(z)||^p_X dm(z) \right)^{\frac{1}{p}} & (1 \le p < \infty); \\ \text{ess sup}_{z \in \mathbb{T}} ||f(z)||_X & (p = \infty). \end{cases}$$

Then we can see that L^p_X forms a Banach space. For $f \in L^1_X$, the n-th Fourier coefficient of f, denoted by $\widehat{f}(n)$, is defined by

$$\widehat{f}(n) := \int_{\mathbb{T}} \overline{z}^n f(z)\, dm(z) \quad \text{for each } n \in \mathbb{Z}.$$

Also, $H^p_X \equiv H^p(\mathbb{T}, X)$ is defined by the set of $f \in L^p_X$ with $\widehat{f}(n) = 0$ for $n < 0$.

We write $M_{n\times m}$ for the set of $n \times m$ complex matrices, and abbreviate $M_{n\times n}$ to M_n. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$. For a matrix-valued function $\Phi \equiv \big(\varphi_{ij}\big) \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry φ_{ij} is of bounded type, and we say that Φ is *rational* if each entry φ_{ij} is a rational function. Let $\Phi \equiv \big(\varphi_{ij}\big) \in L^\infty_{M_n}$ be such that Φ^* is of bounded type. Then each $\overline{\varphi}_{ij}$ is of bounded type. Thus in view of (1), we may write $\varphi_{ij} = \theta_{ij} \bar{b}_{ij}$, where θ_{ij} is inner and θ_{ij} and b_{ij} are coprime; in other words, there does not exist a nonconstant common inner divisor of θ_{ij} and b_{ij}. Thus if θ is the least common multiple of $\{\theta_{ij} : i, j = 1, 2, \cdots, n\}$, then we may write

$$\Phi = \big(\varphi_{ij}\big) = \big(\theta_{ij}\bar{b}_{ij}\big) = \big(\theta\overline{a}_{ij}\big) \equiv \theta A^* \quad (\text{where } A \equiv \big(a_{ji}\big) \in H^2_{M_n}). \tag{3}$$

In particular, $A(\alpha)$ is nonzero whenever $\theta(\alpha) = 0$ and $|\alpha| < 1$.

2.2 Strong L^2-functions

We often need to consider operator-valued functions defined on the unit circle constructed by arranging the vectors in a given set $F \subseteq H^2_s(\mathcal{B}(D,E))$ as their column vectors. Using this viewpoint, we consider operator-valued functions whose "column" vectors are L^p-functions. Note that (bounded linear) operators between separable Hilbert spaces may be represented as infinite matrices, so that column vectors of operators are well justified. This viewpoint leads us to define (operator-valued) strong L^p-functions.

For $1 \le p < \infty$, we define the class $L^p_s(\mathcal{B}(D,E)) \equiv L^p_s(\mathbb{T},\mathcal{B}(D,E))$ as the set of all (WOT) measurable $\mathcal{B}(D,E)$-valued functions Φ on $\mathbb{T}$ such that $\Phi(\cdot)x \in L^p_E$ for each $x \in D$. A function $\Phi \in L^p_s(\mathcal{B}(D,E))$ is called a *strong L^p-function.* Thus if $\Phi \in L^1_s(\mathcal{B}(D,E))$ and $x \in D$, then $\Phi(\cdot)x \in L^1_E$, so that the n-th Fourier coefficient $\widehat{\Phi(\cdot)x}(n)$ of $\Phi(\cdot)x$ is given by $\widehat{\Phi(\cdot)x}(n) = \int_{\mathbb{T}} \overline{z}^n \Phi(z)x\, dm(z)$. Now define the n-th Fourier coefficient of $\Phi \in L^1_s(\mathcal{B}(D,E))$, denoted by $\widehat{\Phi}(n)$, by

$$\widehat{\Phi}(n)x := \widehat{\Phi(\cdot)x}(n) \quad (n \in \mathbb{Z},\ x \in D).$$

Define

$$H^p_s(\mathcal{B}(D,E)) \equiv H^p_s(\mathbb{T},\mathcal{B}(D,E)) := \big\{\Phi \in L^p_s(\mathcal{B}(D,E)) :\ \widehat{\Phi}(n) = 0 \text{ for } n < 0\big\},$$

or equivalently, $H^p_s(\mathcal{B}(D,E))$ is the set of all (WOT) measurable functions Φ on $\mathbb{T}$ such that $\Phi(\cdot)x \in H^p_E$ for each $x \in D$. The terminology of a "strong H^2-function" is reserved for the operator-valued functions on the unit disk $\mathbb{D}$, following N.K. Nikolskii [Ni1]: A function $\Phi : \mathbb{D} \to \mathcal{B}(D,E)$ is called a *strong H^2-function* if $\Phi(\cdot)x \in H^2(\mathbb{D},E)$ for each $x \in D$. Let $L^\infty(\mathcal{B}(D,E))$ be the space of all bounded (WOT) measurable $\mathcal{B}(D,E)$-valued functions on $\mathbb{T}$ and let

$$H^\infty(\mathcal{B}(D,E)) := \big\{\Phi \in L^\infty(\mathcal{B}(D,E)) :\ \widehat{\Phi}(n) = 0 \text{ for } n < 0\big\}.$$

We can show that (cf. [CHL4, Appendix A])

(a) If $\dim D < \infty$, then $L^2_s(\mathcal{B}(D,E)) = L^2_{\mathcal{B}(D,E)}$ and $H^2_s(\mathcal{B}(D,E)) = H^2_{\mathcal{B}(D,E)}$.

(b) $L^\infty_{\mathcal{B}(D,E)} \subseteq L^\infty(\mathcal{B}(D,E)) \subseteq L^p_s(\mathcal{B}(D,E))$.

A function $\Delta \in H^\infty(\mathcal{B}(D,E))$ is called an *inner* function with values in $\mathcal{B}(D,E)$ if $\Delta(z)$ is an isometric operator from D into E for almost all $z \in \mathbb{T}$, i.e., $\Delta^*\Delta = I_D$ a.e. on $\mathbb{T}$. Also, Δ is called a *two-sided inner* function if $\Delta\Delta^* = I_E$ a.e. on $\mathbb{T}$ and $\Delta^*\Delta = I_D$ a.e. on $\mathbb{T}$. If Δ is an inner function with values in $\mathcal{B}(D,E)$, we may assume that D is a subspace of E, and if further Δ is two-sided inner then we may assume that $D = E$. We write $\mathcal{P}_D$ for the set of all polynomials with values in D, i.e., $p(z) = \sum_{k=0}^n \widehat{p}(k)z^k$, where $\widehat{p}(k) \in D$. If $F \in H^2_s(B(D,E))$, then the function Fp belongs to H^2_E for all $p \in \mathcal{P}_D$. A function $F \in H^2_s(B(D,E))$ is called *outer* if $\operatorname{cl} F\mathcal{P}_D = H^2_E$. We then have an analogue of the scalar factorization theorem:

Inner-Outer Factorization for H^2_s-functions. If $F \in H^2_s(B(D,E))$, then F can be expressed in the form $F = F^iF^e$, where F^e is an outer function with values in $\mathcal{B}(D,E')$ and F^i is an inner function with values in $\mathcal{B}(E',E)$ for some subspace E' of E.

The proof of the above inner-outer factorization for H^2_s-functions is identical to the proof for strong H^2-function (cf. [Ni1, Corollary I.9]).

For a function $\Phi : \mathbb{T} \to \mathcal{B}(D, E)$, write

$$\breve{\Phi}(z) := \Phi(\overline{z}), \quad \widetilde{\Phi} := \breve{\Phi}^*.$$

We call $\breve{\Phi}$ the *flip* of Φ. For $\Phi \in L^2_s(\mathcal{B}(D, E))$, we denote by $\breve{\Phi}_- \equiv \mathbb{P}_-\Phi$ and $\Phi_+ \equiv \mathbb{P}_+\Phi$ the functions

$$((\mathbb{P}_-\Phi)(\cdot))x := P_-(\Phi(\cdot)x) \quad \text{a.e. on } \mathbb{T} \quad (x \in D);$$
$$((\mathbb{P}_+\Phi)(\cdot))x := P_+(\Phi(\cdot)x) \quad \text{a.e. on } \mathbb{T} \quad (x \in D),$$

where P_+ and P_- are the orthogonal projections from L^2_E onto H^2_E and $L^2_E \ominus H^2_E$, respectively. Then we may write $\Phi \equiv \breve{\Phi}_- + \Phi_+$. Note that if $\Phi \in L^2_s(\mathcal{B}(D, E))$, then $\Phi_+,\ \Phi_- \in H^2_s(\mathcal{B}(D, E))$.

For a function $\Phi \in H^2_s(\mathcal{B}(D, E))$, we say that an inner function Δ with values in $\mathcal{B}(D', E)$ is a *left inner divisor* of Φ if $\Phi = \Delta A$ for $A \in H^2_s(\mathcal{B}(D, D'))$. For $\Phi \in H^2_s(\mathcal{B}(D_1, E))$ and $\Psi \in H^2_s(\mathcal{B}(D_2, E))$, we say that Φ and Ψ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary operator. Also, we say that Φ and Ψ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. The determination of left or right coprime-ness seems to be a somewhat delicate problem. For matrix-valued functions, left and right coprime-ness was developed in [CHKL], [CHL1], [CHL2], [CHL3] and [FF].

If θ is an inner function, write $I_\theta := \theta I$ (where I is the identity operator). We also write left-g.c.d.$(\cdot)$ and left-l.c.m.$(\cdot)$ for the greatest common left inner divisor and the least common left inner multiple, respectively. By contrast with scalar-valued functions, in (3), I_θ and A need not be (right) coprime. If $\Omega = \text{left-g.c.d.}\,\{I_\theta, A\}$ in the representation (3), that is, $\Phi = \theta A^*$, then $I_\theta = \Omega\Omega_\ell$ and $A = \Omega A_\ell$ for some inner matrix Ω_ℓ (where $\Omega_\ell \in H^2_{M_n}$ because $\det(I_\theta) \neq 0$) and some $A_l \in H^2_{M_n}$. Therefore if $\Phi^* \in L^\infty_{M_n}$ is of bounded type then we can write

$$\Phi = A_\ell{}^*\Omega_\ell, \quad \text{where } A_\ell \text{ and } \Omega_\ell \text{ are left coprime.} \tag{4}$$

In this case, $A^*_\ell\Omega_\ell$ is called the *left coprime factorization* of Φ. Similarly, we can write

$$\Phi = \Omega_r A^*_r, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime.} \tag{5}$$

In this case, $\Omega_r A^*_r$ is called the *right coprime factorization* of Φ. We also say that Ω_ℓ and Ω_r are called the *inner parts* of those factorizations.

For an inner function $\Delta \in H^\infty(\mathcal{B}(D, E))$, $\mathcal{H}(\Delta)$ denotes the orthogonal complement of the subspace ΔH^2_D in H^2_E, i.e.,

$$\mathcal{H}(\Delta) := H^2_E \ominus \Delta H^2_D.$$

The space $\mathcal{H}(\Delta)$ is often called a *model space* or a *de Branges-Rovnyak space* (cf. [dBR], [Sa], [SFBK]). The name "model space" comes from the model theory of Sz.-Nagy and Foiaş contractions (see § 2.3).

2.3 The Douglas-Shapiro-Shields factorization

We first review the definition of (vectorial) Toeplitz operators and (vectorial) Hankel operators, and for that we will use [BS], [Do1], [Do2], [Ni1], [Ni2], and [Pe] for general references. For $\Phi \in L^2_s(\mathcal{B}(D,E))$, the Hankel operator $H_\Phi : H^2_D \to H^2_E$ is a densely defined operator given by

$$H_\Phi p := JP_-(\Phi p) \quad (p \in \mathcal{P}_D),$$

where J denotes the unitary operator from L^2_E to L^2_E given by $(Jg)(z) := \overline{z} g(\overline{z})$ for $g \in L^2_E$. Also a Toeplitz operator $T_\Phi : H^2_D \to H^2_E$ is a densely defined operator defined by

$$T_\Phi p := P_+(\Phi p) \quad (p \in \mathcal{P}_D).$$

The following basic properties can be easily derived: For $\Phi \in L^\infty(\mathcal{B}(D,E))$ and $\Psi \in L^\infty(\mathcal{B}(D',D))$

$$T^*_\Phi = T_{\Phi^*}, \ H^*_\Phi = H_{\widetilde{\Phi}}; \tag{6}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi. \tag{7}$$

The *shift* operator on H^2_E is defined by

$$(S_E f)(z) := zf(z) \quad \text{for each } f \in H^2_E.$$

The following theorem characterizes the invariant subspaces for the shift operator.

The Beurling-Lax-Halmos Theorem. [Be], [La], [Ha1], [FF], [Pe] A subspace M of H^2_E is invariant for the shift operator S_E on H^2_E if and only if

$$M = \Delta H^2_{E'},$$

where E' is a subspace of E and Δ is an inner function with values in $\mathcal{B}(E', E)$. Furthermore, Δ is unique up to a unitary constant right factor, i.e., if $M = \Theta H^2_{E''}$, where Θ is an inner function with values in $\mathcal{B}(E'', E)$, then $\Delta = \Theta V$, where V is a unitary operator from E' onto E''.

As customarily done, we say that two inner functions $A, B \in H^\infty(\mathcal{B}(D,E))$ are *equal* if they are equal up to a unitary constant right factor.

By the Beurling-Lax-Halmos Theorem, $\mathcal{H}(\Delta)$ is an invariant subspace for the backward shift operator S^*_E. Thus the truncated backward shift $S^*_E|_{\mathcal{H}(\Delta)}$ is well-defined. We here recall the Model Theorem ([Ni1], [SFBK]): If $T \in \mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ is a contraction (i.e., $||T|| \leq 1$) satisfying

$$\lim_{n\to\infty} T^n x = 0 \quad \text{for each } x \in \mathcal{H}, \tag{8}$$

then T is unitarily equivalent to a truncated backward shift $S^*_E|_{\mathcal{H}(\Delta)}$ for some inner function Δ with values in $\mathcal{B}(E', E)$. In this case, $S^*_E|_{\mathcal{H}(\Delta)}$ is called the *model operator* of T and Δ is called the *characteristic function* of T. We often write $T \in C_{0\boldsymbol{\cdot}}$ for a contraction $T \in \mathcal{B}(\mathcal{H})$ satisfying the condition (8).

On the other hand, if $\Phi \in L^\infty(\mathcal{B}(D,E))$, then $H_{\Phi^*} S_E = S^*_E H_{\Phi^*}$, which implies that the kernel of H_{Φ^*} is an invariant subspace of the shift operator S_E on H^2_E. Thus, by the Beurling-Lax-Halmos Theorem, $\ker H_{\Phi^*} = \Delta H^2_{E'}$ for some inner function Δ with values in $\mathcal{B}(E', E)$. We note that Δ need not be two-sided inner.

Theorem 2.1. ([CHL4, Lemma 2.4]) If $\Phi \in L^\infty(\mathcal{B}(D,E))$ and Δ is a two-sided inner function with values in $\mathcal{B}(E)$, then the following are equivalent:

(a) $\ker H_{\Phi^*} = \Delta H^2_E$;

(b) $\Phi = \Delta A^*$, where $A \in H^\infty(\mathcal{B}(E,D))$ is such that Δ and A are right coprime;

The factorization in Theorem 2.1(b) is called the *Douglas-Shapiro-Shields* (briefly, DSS) *factorization* of $\Phi \in L^\infty(\mathcal{B}(D,E))$ (see [DSS], [FB], [Fu2]; in particular, [Fu2] contains many important applications of the DSS factorization to linear system theory). Consequently, Theorem 2.1 may be rephrased as: If $\Phi \in L^\infty(\mathcal{B}(D,E))$, then the following are equivalent:

(a) Φ admits a DSS factorization;

(b) $\ker H_{\Phi^*} = \Delta H^2_E$ for some two-sided inner function $\Delta \in H^\infty(\mathcal{B}(E))$.

It is known (cf. [CHL2], [FB], [GHR]) that if $\Phi \in L^\infty(\mathcal{B}(D,E))$ admits a DSS factorization $\Phi = \Delta A^*$, then Δ can be obtained from the equation

$$\ker H_{\Phi^*} = \Delta H^2_E; \tag{9}$$

in this case, Δ and A are right coprime. The DSS factorization satisfying (9) is called *canonical*. Consequently, each function that admits a DSS factorization can be arranged in a canonical form. It is also known that if Φ is a matrix-valued function then (cf. [CHL4], [GHR])

$$\Phi^* \text{ is of bounded type } \iff \Phi \text{ admits a (canonical) DSS factorization.} \tag{10}$$

If the condition "Δ is two-sided" is dropped in a DSS factorization $\Phi = \Delta A^*$, what can we say about a DSS factorization ? More concretely, we may ask: If $\Phi \in L^\infty(\mathcal{B}(E',E))$ is expressed as $\Phi = \Delta A^*$, where $\Delta \in H^\infty(\mathcal{B}(D,E))$ is inner and $A \in H^\infty(\mathcal{B}(D,E'))$, does it follows that Δ can be obtained from the equation $\ker H_{\Phi^*} = \Delta H^2_E$?

An answer to this question is affirmative.

Theorem 2.2. ([GHL]) If $\Phi \in L^\infty(\mathcal{B}(E',E))$ is expressed as $\Phi = \Delta A^*$, where $\Delta \in H^\infty(\mathcal{B}(D,E))$ is inner and $A \in H^\infty(\mathcal{B}(D,E'))$, then we can write $\Phi = \Delta_A B_0^*$, where $B_0 \in H^\infty(\mathcal{B}(E_0,E'))$ and $\Delta_A \in H^\infty(\mathcal{B}(E_0,E))$ is an inner function which comes from the equation $\ker H_{\Phi^*} = \Delta_A H^2_{E_0}$ for some Hilbert space E_0. Moreover, Δ_A and B_0 are right coprime.

The expression $\Phi = \Delta A^*$ in Theorem 2.2 is called a *pseudo-DSS factorization* and the expression $\Phi = \Delta_A B_0^*$ is called a *canonical* pseudo-DSS factorization. Thus Theorem 2.2 says that if a function $\Phi \in L^\infty(\mathcal{B}(E',E))$ admits a pseudo-DSS factorization then we can always arrange the pseudo-DSS factorization of Φ in a canonical form.

2.4 Complementary factors

We now consider the following question: If Δ is an inner function with values in $\mathcal{B}(D,E)$, what is the kernel of H_{Δ^*} ? We are tempted to guess that the answer is ΔH^2_E, which is wrong in general. We examine an answer to the above question.

For $\Phi \in L^\infty(\mathcal{B}(D,E))$, we symbolically define the kernel of Φ by

$$\ker \Phi := \big\{ f \in H^2_D : \Phi(z) f(z) = 0 \text{ for almost all } z \in \mathbb{T} \big\}.$$

Note that the kernel of Φ consists of functions in H^2_D, but not in L^2_D, such that $\Phi f = 0$ a.e. on $\mathbb{T}$. Since $\ker \Phi$ is an invariant subspace for S_D, it follows from the Beurling-Lax-Halmos Theorem that $\ker \Phi = \Omega H^2_{D'}$, for some inner function $\Omega \in H^\infty(D', D)$.

We recall a notion from classical Banach space theory, about regarding a vector as an operator acting on the scalars. This notion is important as motivation for the study of strong L^2-functions. Let E be a separable complex Hilbert space. For a function $f : \mathbb{T} \to E$, define $[f] : \mathbb{T} \to \mathcal{B}(\mathbb{C}, E)$ by

$$[f](z)\alpha := \alpha f(z) \quad (\alpha \in \mathbb{C}). \tag{11}$$

Let Δ be an inner function with values in $\mathcal{B}(D,E)$. If $g \in \ker \Delta^*$, then $g \in H^2_E$, so that we can see that $[g] \in H^2_s(\mathcal{B}(\mathbb{C}, E))$. Write

$$[g] = [g]^i [g]^e \quad \text{(inner-outer factorization)},$$

where $[g]^e$ is an outer function with values in $\mathcal{B}(\mathbb{C}, E')$ and $[g]^i$ is an inner function with values in $\mathcal{B}(E', E)$ for some subspace E' of E. If $g \neq 0$, then $[g]^e$ is a nonzero outer function, so that $E' = \mathbb{C}$. Thus, $[g]^i \in H^\infty(\mathcal{B}(\mathbb{C}, E))$. If instead $g = 0$, then $E' = \{0\}$. Therefore, in this case, $[g]^i \in H^\infty(\mathcal{B}(\{0\}, E))$.

The following theorem provides a description of the kernels of H_{Δ^*} and Δ^* ([CHL4, Lemma 2.7])).

Theorem 2.3. Let Δ be an inner function with values in $\mathcal{B}(D,E)$. Then $\ker \Delta^* = \Delta_c H^2_{D'}$, where

$$\Delta_c := \text{left-g.c.d.} \big\{ [g]^i : g \in \ker \Delta^* \big\},$$

which is an inner function with values in $B(D', E)$. Moreover, $[\Delta, \Delta_c]$ is an inner function with values in $\mathcal{B}(D \oplus D', E)$ and

$$\ker H_{\Delta^*} = [\Delta, \Delta_c] H^2_{D \oplus D'} \equiv \Delta H^2_D \bigoplus \Delta_c H^2_{D'}. \tag{12}$$

The inner function Δ_c is called the *complementary factor* of the inner function Δ.

Example 2.4. If $\Delta = \begin{bmatrix} z \\ 0 \end{bmatrix}$, then Δ is inner. A straightforward calculation shows that $\Delta_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Indeed,

$$\ker H_{\Delta^*} = \ker \begin{bmatrix} H_{\overline{z}} & 0 \end{bmatrix} = zH^2 \oplus H^2 = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} H^2_{\mathbb{C}^2} = \begin{bmatrix} \Delta & \Delta_c \end{bmatrix} H^2_{\mathbb{C}^2}. \quad \square$$

On the other hand, it is known ([Ab, Lemma 4]) that if $\phi \in L^\infty$, then

$$\phi \text{ is of bounded type} \Longleftrightarrow \ker H_\phi \neq \{0\}. \tag{13}$$

The following corollary shows that (13) still holds for L^2-functions ([CHL4, Corollaries 2.18 and 2.19]).

Corollary 2.5. If $\phi \in L^2$, then ϕ is of bounded type if and only if $\ker H_\phi^* \neq \{0\}$. Moreover, if Δ is an inner matrix function then the following are equivalent:

(a) Δ^* is of bounded type;

(b) $\breve{\Delta}$ is of bounded type;

(c) $[\Delta, \Delta_c]$ is two-sided inner.

R.G. Douglas and J.W. Helton [DH] have considered a problem from engineering circuit theory called Darlington synthesis which mathematically translates to: given a contractive analytic operator-valued function S on the unit disk, can one embed S into a two-sided 2×2 inner matrix function $\Theta = \left[\begin{smallmatrix} S & A \\ B & C \end{smallmatrix}\right]$? The special case where $S = \Delta$ is inner and the second block-row is vacuous amounts to our problem of finding Ω so that $[\Delta, \Omega]$ is two-sided inner. Thus, Corollary 2.5 can be obtained from [DH, Theorem].

2.5 Degree of non-cyclicity

By the Beurling-Lax-Halmos Theorem, every invariant subspace for the backward shift operator S_E^* on H_E^2 is a model space $\mathcal{H}(\Delta)$ for some inner function Δ with values in $\mathcal{B}(D, E)$. For a subset F of H_E^2, let E_F^* denote the smallest S_E^*-invariant subspace containing F, i.e.,

$$E_F^* := \bigvee \big\{ S_E^{*n} F : \ n \geq 0 \big\}.$$

Then by the Beurling-Lax-Halmos Theorem, $E_F^* = \mathcal{H}(\Delta)$ for an inner function Δ with values in $\mathcal{B}(D, E)$. In general, if $\dim E = 1$, then every S_E^*-invariant subspace M admits a cyclic vector, i.e., $M = E_f^*$ for some $f \in H^2$. However, if $\dim E \geq 2$, then this is not such a case. For example, if $M = \mathcal{H}(\Delta)$ with $\Delta = \left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right]$, then M does not admit a cyclic vector, i.e., $M \neq E_f^*$ for any vector $f \in H^2_{\mathbb{C}^2}$.

We thus introduce:

Definition 2.6. Let $F \subseteq H_E^2$. The *degree of non-cyclicity* of F, denoted by $\mathrm{nc}(F)$, is defined by the number

$$\mathrm{nc}(F) := \sup_{\zeta \in \mathbb{D}} \dim \big\{ g(\zeta) : g \in H_E^2 \ominus E_F^* \big\}.$$

We will often refer to $\mathrm{nc}(F)$ as the nc-*number* of F. Since E_F^* is an invariant subspace for S_E^*, it follows from the Beurling-Lax-Halmos Theorem that $E_F^* = \mathcal{H}(\Delta)$ for some inner function Δ with values in $\mathcal{B}(D, E)$. Thus

$$\mathrm{nc}(F) = \sup_{\zeta \in \mathbb{D}} \dim \big\{ g(\zeta) : \ g \in \Delta H_D^2 \big\} = \dim D.$$

In particular, $\text{nc}(F) \le \dim E$. We note that $\text{nc}(F)$ may take the value ∞. So it is customary to make the following conventions: (i) if n is a natural number then $n + \infty = \infty$; (ii) $\infty + \infty = \infty$.

If $\dim E = r < \infty$, then $\text{nc}(F) \le r$ for every subset $F \subseteq H^2_E$. If $F \subseteq H^2_E$ and $\dim E = r < \infty$, then the *degree of cyclicity* of F, denoted by $\text{dc}(F)$, is defined by the number (cf. [VN]): $\text{dc}(F) := r - \text{nc}(F)$. In particular, if $E^*_F = \mathcal{H}(\Delta)$, then Δ is two-sided inner if and only if $\text{dc}(F) = 0$.

To understand E^*_F for a subset $F \subseteq H^2_E$, we need to consider the kernels of the adjoints of unbounded Hankel operators with strong L^2-symbols involved with F. Thus we deal with unbounded Hankel operators H_Φ with strong L^2-symbols Φ. However, the adjoint of an unbounded Hankel operator need not be a Hankel operator. But if Φ is an L^∞-function then $H_{\Phi^*} = H^*_{\breve{\Phi}}$. Thus for a bounded symbol Φ, we may use the notations H_{Φ^*} and $H^*_{\breve{\Phi}}$ interchangeably. By contrast, for a strong L^2-function Φ, H_{Φ^*} may not be equal to $H^*_{\breve{\Phi}}$ even when Φ^* is a strong L^2-function. In particular, the kernel of an unbounded Hankel operator H_{Φ^*} is likely to be trivial because it is defined on the dense subset of polynomials. From this viewpoint, to avoid potential technical issues in our arguments, we deal with the operator $H^*_{\breve{\Phi}}$ in place of H_{Φ^*}. In spite of this, and since the kernel of the adjoint of an unbounded operator is always closed, we can show that via the Beurling-Lax-Halmos Theorem, the kernel of $H^*_{\breve{\Phi}}$, with strong L^2-symbol Φ, is still of the form $\Delta H^2_{E'}$ (see [CHL4, Corollary 2.6]).

The next question arises naturally from the Beurling-Lax-Halmos Theorem.

Question 2.7. Recall that, by the Beurling-Lax-Halmos Theorem, $\ker H^*_{\breve{\Phi}} = \Theta H^2_{E'}$ for some inner function Θ. Which property of Φ determines the dimension of the space E'? In particular, if Φ is an $n \times m$ matrix-valued L^2-function and $\dim E' = r$, which property of Φ determines the number r?

If $\Phi \in H^2_s(\mathcal{B}(D,E))$ and $\{d_k\}_{k\ge 1}$ is an orthonormal basis for D, write

$$\phi_k := \Phi d_k \in H^2_E \cong H^2_s(\mathcal{B}(\mathbb{C}, E)).$$

We then define

$$\{\Phi\} := \{\phi_k\}_{k\ge 1} \subseteq H^2_E.$$

Hence, $\{\Phi\}$ may be regarded as the set of "column" vectors ϕ_k (in H^2_E) of Φ, in which case we may think of Φ as an infinite matrix-valued function. Then it was shown in [CHL4, Lemma 2.9] that for $\Phi \in H^2_s(\mathcal{B}(D,E))$,

$$E^*_{\{\Phi\}} = \text{cl ran}\, H_{\overline{z}\breve{\Phi}}. \tag{14}$$

By definition, $\{\Phi\}$ depends on the orthonormal basis of D. However, (14) shows that $E^*_{\{\Phi\}}$ is independent of a particular choice of the orthonormal basis of D because the right-hand side of (14) is independent of the orthonormal basis of D.

The following theorem gives an answer to Question 2.7 (cf. [CHL4, Theorem 2.13]).

Theorem 2.8. Let Φ be a strong L^2-function with values in $\mathcal{B}(D,E)$. In view of the Beurling-Lax-Halmos Theorem we may write $E^*_{\{\Phi_+\}} = \mathcal{H}(\Delta)$ and $\ker H^*_{\breve{\Phi}} = \Theta H^2_{E'}$ for some inner functions Δ and Θ with values in $\mathcal{B}(E'', E)$ and $\mathcal{B}(E', E)$, respectively. Then $\Delta = \Theta\Delta_1$ for some two-sided inner function Δ_1 with values in $\mathcal{B}(E'', E')$. Hence, in particular,

$$\ker H^*_{\breve{\Phi}} = \Theta H^2_{E'} \iff \text{nc}\{\Phi_+\} = \dim E'. \tag{15}$$

Example 2.9. If

$$\Phi := \begin{bmatrix} z & 0 \\ 0 & z \\ 0 & 0 \\ a & 0 \end{bmatrix} \quad \text{(with } a \in H^\infty \text{ not of bounded type)},$$

then a straightforward calculation shows that

$$\ker H^*_{\breve{\Phi}} = \ker \begin{bmatrix} H_{\overline{z}} & 0 & 0 & H_{\overline{a}} \\ 0 & H_{\overline{z}} & 0 & 0 \end{bmatrix} = zH^2 \oplus zH^2 \oplus H^2 \oplus \{0\} = \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} H^2_{\mathbb{C}^3},$$

which implies, by (15), $\text{nc}\{\Phi\} = 3$. □

On the other hand, if Φ is an $n \times m$ matrix L^2-function then Theorem 2.8 gives

$$\ker H^*_{\Phi} = \Delta H^2_{\mathbb{C}^n} \text{ for some two-sided inner matrix function } \Delta \Longleftrightarrow \text{dc}\,\{\widetilde{\Phi_-}\} = 0. \tag{16}$$

Also, by an argument of [Ni1, p.47],

$$\text{dc}\,\{\widetilde{\Phi_-}\} = 0 \Longleftrightarrow \Phi \text{ is of bounded type}. \tag{17}$$

Consequently, by (16) and (17), we can see that if $\Phi \in L^2_{M_{n\times m}}$, then

$$\Phi \text{ is of bounded type} \Longleftrightarrow \ker H^*_{\Phi} = \Delta H^2_{\mathbb{C}^n} \text{ for some two-sided inner matrix function } \Delta. \tag{18}$$

3 Matrix-valued function theory

In this section, we consider matrix-valued function theory associated with Toeplitz and Hankel operators. We focus on the notions of tensored-scalar singularity, Beurling degree of inner matrix functions, Abrahamse's Theorem for matrix-valued symbols, and the H^∞-functional calculus for the compressions of the shift.

3.1 Tensored-scalar singularities

We ask a question: How does one define a singularity for matrix functions? Conventionally, the singularity (or the existence of a pole) of matrix L^∞-functions is defined by a singularity (or a pole) of some entry of the matrix function (cf. [BGR], [BR]). However we propose another notion of singularity which is more useful for the study of Hankel and Toeplitz operators. To do so, we recall that if the product of two Hankel operators is zero then one of the operators must be zero. However, this is not such the case for Hankel operators with matrix-valued symbols. For example, if we take

$$\Psi = \begin{bmatrix} 1 & 0 & \overline{z} \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \Phi = \begin{bmatrix} \overline{z} & 0 \\ 0 & \overline{z} \\ 0 & 0 \end{bmatrix}, \tag{19}$$

then

$$H_\Psi H_\Phi = \begin{bmatrix} 0 & 0 & H_{\overline{z}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{\overline{z}} & 0 \\ 0 & H_{\overline{z}} \\ 0 & 0 \end{bmatrix} = 0,$$

but $H_\Phi \neq 0$ and $H_\Psi \neq 0$. In this section, we examine the following question: Under what conditions, does it follow that

$$H_\Psi H_\Phi = 0 \implies H_\Psi = 0 \quad \text{or} \quad H_\Phi = 0\,? \tag{20}$$

To examine this question, in [CHL3] the notion of "tensored-scalar singularity" was introduced for square matrix functions of bounded type. In [CHL3, Theorem 5.4], it was shown that the answer to question (20) is affirmative for bounded type matrix functions such that Φ or Ψ has a tensored-scalar singularity. In this subsection we extend the above result for general matrix-valued L^∞-functions having a tensored-scalar singularity. To see this, we introduce the notion of tensored-scalar singularity for general L^∞-matrix functions.

Definition 3.1. For $\Phi \in L^\infty_{n\times m}$, we say that Φ has a *tensored-scalar singularity* (*with respect to* θ) if there exists a nonconstant inner function θ such that

$$\{0\} \neq \ker H_\Phi \subseteq \theta H^2_{\mathbb{C}^m}\,.$$

Note 3.2. Every bounded type function $\varphi \in L^2$ has, trivially, a tensored-scalar singularity; for, if φ is of bounded type, then by Beurling's Theorem, $\ker H_\varphi = \theta H^2$ for an inner function $\theta \in H^\infty$. Also we can easily see that if $\Phi \in L^\infty_{M_n}$ is of bounded type and $\Phi = A\Theta^*$ (right coprime) then (cf. [CHL3, Lemma 5.2])

$$\Phi \text{ has a tensored-scalar singularity} \iff \Theta \text{ has a diagonal-constant inner divisor.} \tag{21}$$

From this viewpoint, if $\Phi \in L^\infty_{M_n}$ has the coprime factorization $\Phi_- = \theta B^*$ (coprime), then clearly, Φ has a tensored-scalar singularity.

Example 3.3. In Definition 3.1, we cannot conclude that $m \leq n$. To see this, let $h(z) := e^{\frac{1}{z-3}}$. Then $h \in H^\infty$ and $\overline{h}$ is not of bounded type. Let

$$f(z) := \frac{h(z)}{\sqrt{2}||h||_\infty}.$$

Clearly, $\overline{f}$ is not of bounded type. Let $h_1(z) := \sqrt{1 - |f(z)|^2}$. Then $h_1 \in L^\infty$ and $|h_1| \geq \frac{1}{\sqrt{2}}$. Thus there exists an outer function g such that $|h_1| = |g|$ a.e. on $\mathbb{T}$ (see [Do1, Corollary 6.25]). Let

$$\Delta := \begin{bmatrix} zf \\ zg \end{bmatrix}.$$

Then

$$\ker H_{\Delta^*} = \begin{bmatrix} zf \\ zg \end{bmatrix} \subseteq zH^2_{\mathbb{C}^2},$$

which implies Δ^* has a tensored-scalar singularity with respect to z.

Remark 3.4. Let $\Phi \in L^2_{n\times m}$. Then we may ask whether Φ has a tensored-scalar singularity if and only if $\widetilde{\Phi}$ has a tensored-scalar singularity. The answer is negative. For example, let

$$\Phi := \begin{bmatrix} \overline{z} & 0 \\ 0 & \overline{z} \\ 0 & 0 \end{bmatrix}.$$

Then a straightforward calculation shows that Φ has a tensored-scalar singularity with respect to z, but $\widetilde{\Phi}$ has no tensored-scalar singularity.

We now have:

Theorem 3.5. Let $\Phi \in L^\infty_{n\times m}$ and $\Psi \in L^\infty_{r\times n}$. If $\widetilde{\Phi}$ or Ψ has a tensored-scalar singularity then

$$H_\Psi H_\Phi = 0 \implies H_\Phi = 0 \text{ or } H_\Psi = 0.$$

Proof. Suppose that $\widetilde{\Phi}$ has a tensored-scalar singularity with respect to δ. Then by the Beurling-Lax-Halmos Theorem, there exists an inner matrix function $\Delta \in H^\infty_{n\times p}$ such that $\ker H^*_\Phi = \Delta H^2_{\mathbb{C}^p} \subseteq \delta H^2_{\mathbb{C}^n}$. If $H_\Psi H_\Phi = 0$, then we have

$$\mathcal{H}(I_\delta) \subseteq \text{cl ran}\, H_\Phi \subseteq \ker H_\Psi \equiv \Omega H^2_{\mathbb{C}^q}, \tag{22}$$

where $\Omega \equiv \big(\omega_{ij}\big)$ is an $n\times q$ inner function. Since δ is not constant, $\mathcal{H}(\delta)$ has at least an outer function g that is invertible in H^∞ (cf. [CHL1, Lemma 3.4]). Put

$$e_1 = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ g \\ \vdots \\ 0 \end{bmatrix}, \cdots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g \end{bmatrix}.$$

Then for each $j = 1, 2, \cdots, n$, $e_j \in \mathcal{H}(I_\delta)$ and hence by (22), $e_j \in \Omega H^2_{\mathbb{C}^q}$. Then

$$q = \max\big\{\text{rank}\, \Omega(\zeta) : \zeta \in \mathbb{D}\big\} \geq n,$$

which implies $n = q$. For all $j = 1, 2, \cdots, n$, it follows from (22) that $\Omega^* e_j \in H^2_{\mathbb{C}^n}$, which implies that

$$\overline{\omega}_{ij}\, g \in H^2 \quad \text{for each } i, j = 1, 2, \cdots, n,$$

so that $\overline{\omega}_{ij} \in \frac{1}{g} H^2 \subseteq H^2$ for each i, j. Therefore each ω_{ij} is constant and hence, Ω is a unitary matrix. Thus, again by (22), $\ker H_\Psi = H^2_{\mathbb{C}^q}$. Therefore $H_\Psi = 0$. Suppose that Ψ has a tensored-scalar singularity and $H_\Psi H_\Phi = 0$. Then $H_{\widetilde{\Phi}} H_{\widetilde{\Psi}} = 0$, so that $H_{\widetilde{\Phi}} = 0$ by what we proved just above, and hence $H_\Phi = 0$. This completes the proof. □

In Theorem 3.5, the condition "$\widetilde{\Phi}$ or Ψ has a tensored-scalar singularity" cannot be replaced by the condition "Φ or Ψ has a tensored-scalar singularity." For example, if Φ, Ψ are given in (19), then by Remark 3.4, Φ has a tensored-scalar singularity, but $\widetilde{\Phi}$ has no tensored-scalar singularity. Note that, in such a case, $H_\Psi H_\Phi = 0$, but $H_\Phi \neq 0$ and $H_\Psi \neq 0$.

On the other hand, we recall ([GHR, Theorem 3.3]) that if $\Phi \in L^\infty_{M_n}$ then

$$H^*_{\Phi^*} H_{\Phi^*} \geq H^*_\Phi H_\Phi \iff \exists\, K \in H^\infty_{M_n} \text{ such that } ||K||_\infty \leq 1 \text{ and } \Phi - K\Phi^* \in H^\infty_{M_n}. \tag{23}$$

The following theorem extends [CHL3, Theorem 5.5] for general matrix-valued L^∞-functions.

Theorem 3.6. Let $\Phi, \Psi \in L^\infty_{M_n}$. If Φ or Ψ has a tensored-scalar singularity then

$$H^*_\Phi H_\Phi = H^*_\Psi H_\Psi \iff \Phi - U\Psi \in H^\infty_{M_n}$$

for some unitary matrix $U \in M_n$.

Proof. Suppose $H^*_\Phi H_\Phi = H^*_\Psi H_\Psi$. Then $\ker H_\Psi = \ker H_\Phi$, which implies that Φ has a tensored-scalar singularity if and only if Ψ does. Thus, without loss of generality, we may assume that Φ has a tensored-scalar singularity with respect to θ. Then $\ker H_\Phi \subseteq \theta H^2_{\mathbb{C}^n}$, so that

$$\mathcal{H}(I_\theta) \subseteq \text{cl ran}\, H_{\widetilde{\Phi}}. \tag{24}$$

By our assumption together with (23), there exist $U, U' \in H^\infty_{M_n}$ with $||U||_\infty \leq 1$ and $||U'||_\infty \leq 1$ such that

$$\Phi^*_- - U\Psi^*_- \in H^2_{M_n} \quad \text{and} \quad \Psi^*_- - U'\Phi^*_- \in H^2_{M_n}, \tag{25}$$

which implies $H_{\Phi^*_-} - H_{UU'\Phi^*_-} = 0$, and hence $(I - T^*_{\widetilde{UU'}})H_\Phi = 0$. By the same argument as in [CHL3, Proof of Theorem 5.5], we can see that $UU' = I_n$. Therefore U is a unitary constant and by (25), $\Phi - U\Psi \in H^\infty_{M_n}$. The converse is clear. □

If $\Phi \in L^\infty_{M_n}$, then by (7),

$$[T^*_\Phi, T_\Phi] = H^*_{\Phi^*} H_{\Phi^*} - H^*_\Phi H_\Phi - T_{\Phi^*\Phi - \Phi\Phi^*}.$$

Thus if Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$ a.e. on $\mathbb{T}$, then it follows at once that

$$T_\Phi \text{ is normal} \iff H^*_{\Phi^*} H_{\Phi^*} = H^*_\Phi H_\Phi. \tag{26}$$

Therefore if $\Phi \in L^\infty_{M_n}$ is normal and has a tensored-scalar singularity then by Theorem 3.6 and (26), T_Φ is normal if and only if $\Phi - U\Phi^* \in H^\infty_{M_n}$ for some unitary matrix $U \in M_n$.

3.2 The Beurling degree of an inner matrix function

In this subsection we consider the following question: *If Δ is an $n \times r$ inner matrix function, does there exist a function $\Phi \in L^2_{M_{n\times m}}$ satisfying the equation*

$$\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}\ ? \tag{27}$$

To formulate an answer to question (27), we consider whether there exists an inner function Ω satisfying $\ker H_{\Omega^*} = \Delta H^2_{\mathbb{C}^r}$ when Δ is an $n \times r$ inner matrix function. In fact, the answer to this question is negative. Indeed, if $\ker H_{\Omega^*} = \Delta H^2_{\mathbb{C}^r}$ for some inner matrix function $\Omega \in H^\infty_{M_{n\times m}}$, then by Theorem 2.3, we have $[\Omega, \Omega_c] = \Delta$, and hence $\Delta_c = 0$. Conversely, if $\Delta_c = 0$ then again by

Theorem 2.3, we should have $\ker H_{\Delta^*} = \Delta H^2_{\mathbb{C}^r}$. Consequently, $\ker H_{\Omega^*} = \Delta H^2_{\mathbb{C}^r}$ for some inner function Ω if and only if $\Delta_c = 0$. Thus if $\Delta := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then there exists no inner function Ω such that $\ker H_{\Omega^*} = \Delta H^2$. On the other hand, we note that the solution Φ is not unique although there exists an inner function Φ satisfying the equation (27). For example, if $\Delta := \text{diag}\,(z, 1, 1)$, then the following Φ's are such solutions:

$$\Phi = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} z & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Delta.$$

The following theorem gives an affirmative answer to question (27): indeed, we can always find an analytic solution $\Phi \in H^\infty_{M_{n\times m}}$ ([CHL4, Corollary 4.2]).

Theorem 3.7. For a given $n\times r$ inner matrix function Δ, there exists at least a solution $\Phi \in H^\infty_{M_{n\times m}}$ (with $m \le r+1$) of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$.

Remark 3.8. In view of Theorem 3.7, it is reasonable to ask whether such a solution $\Phi \in L^2_{M_{n\times m}}$ of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$ (Δ an $n \times r$ inner matrix function) exists for each $m = 1, 2, \cdots$ even though it exists for some m. However, the answer is negative in general, i.e., a solution exists for some m, but may not exist for another $m_0 < m$. To see this, let

$$\Delta := \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in H^\infty_{M_{4\times 3}}. \tag{28}$$

Then Δ is inner. A straightforward calculation shows that there exists no solution Φ, in $L^2_{M_{4\times 1}}$ (the case $m = 1$), of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^3}$. By contrast, if $m = 2$, then we can find a solution $\Phi \in L^2_{M_{4\times 2}}$. Indeed, let

$$\Phi := \begin{bmatrix} z & 0 \\ 0 & z \\ 0 & 0 \\ a & 0 \end{bmatrix},$$

where $a \in H^\infty$ is such that $\overline{a}$ is not of bounded type. Then $\ker H_{\Phi^*} = \Delta H^2_{\mathbb{C}^3}$. Thus we obtain a solution for $m = 2$ although there exists no solution for $m = 1$.

Let Δ be an $n\times r$ inner matrix function. In view of Remark 3.8, we may ask how to determine a possible value m for which there exists a solution $\Phi \in L^2_{M_{n\times m}}$ of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$. In fact, if we have a solution $\Phi \in L^2_{M_{n\times m}}$ of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$, then a solution $\Psi \in L^2_{M_{n\times q}}$ also exists if $q \ge m$: indeed, if $\mathbf{0}$ denotes the zero operator in $M_{n\times(q-m)}$ and $\Psi := [\Phi, \mathbf{0}]$, then it follows that $\ker H^*_{\breve{\Phi}} = \ker H^*_{\breve{\Psi}}$. Thus we would like to ask what is the infimum of the set of positive integers m such that there exists a solution $\Phi \in L^2_{M_{n\times m}}$ of the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r}$. To answer this question, we introduce a notion of the "Beurling degree" for an inner function.

Definition 3.9. If Δ is an $n \times r$ inner matrix function then the *Beurling degree* of Δ, denoted by $\deg_B(\Delta)$, is defined by

$$\deg_B(\Delta) = \inf\bigl\{m : \ \ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^r} \text{ for some } \Phi \in L^2_{M_{n\times m}}\bigr\}. \tag{29}$$

We recall that the *spectral multiplicity* for $T \in \mathcal{B}(E)$ is defined by the number μ_T:

$$\mu_T := \inf \dim \bigvee \{f : f \in F\},$$

where $F \subseteq E$, the infimum being taken over all generating subsets F, i.e., subsets such that $\bigvee\{T^n F : n \geq 0\} = E$. It was shown in [CHL4, Theorem 4.6] that if Δ is the characteristic function of the model operator T with values in $M_{n\times r}$, then the spectral multiplicity of the model operator is equal to the Beurling degree of Δ.

Theorem 3.10. (The Beurling degree and the spectral multiplicity) Given an $n \times r$ inner matrix function Δ, let $T := S_E^*|_{\mathcal{H}(\Delta)}$. Then

$$\mu_T = \deg_B(\Delta). \tag{30}$$

Example 3.11. If Δ is given by (28), then by the preceding argument and (29), $\deg_B(\Delta) = 2$. Thus if $T := S^*_{\mathbb{C}^4}|_{\mathcal{H}(\Delta)}$, then by (30), $\mu_T = 2$. □

For an inner matrix function $\Delta \in H^\infty_{M_n}$, write

$$\mathbf{S}_\Delta := P_{\mathcal{H}(\Delta)} S_{\mathbb{C}^n}|_{\mathcal{H}(\Delta)}.$$

It is known (cf. [Ni1, Appendix]) that if $\Delta \in H^\infty_{M_n}$ is square-inner, then the spectral multiplicity of $\mathbf{S}_\Delta$ can be computed from its characteristic function Δ by using the Jordan model theory for C_0 contractions due to Sz.-Nagy and Foiaş (cf. [SFBK]) and the Moore–Nordgren theory of quasi-equivalence (cf. [MN],[No]). To see this, for an inner matrix function $\Delta \in H^\infty_{M_n}$, write

$$\delta_k := \text{g.c.d.}\bigl\{\text{all inner parts of the minors of order } n-k \text{ of } \Delta\bigr\} \tag{31}$$

for $k = 0, 1, 2, \cdots, n$. If $T := \mathbf{S}_\Delta$, then

$$\mu_T = \max\bigl\{k : \delta_{k-1} \neq \delta_k \text{ up to constant}\bigr\}. \tag{32}$$

It is also known ([Ni1, Lemma on the Function Θ^t, p.75]) that $S^*_{\mathbb{C}^n}|_{\mathcal{H}(\Delta)}$ is unitarily equivalent to $\mathbf{S}_{\widetilde{\Delta}}$. Moreover, if Δ is an inner matrix function in $H^\infty_{M_n}$ and δ_k is given in (31), then we can easily check that for $k = 0, 1, 2, \cdots, n$,

$$\widetilde{\delta}_k = \text{g.c.d.}\bigl\{\text{all inner parts of the minors of order } n-k \text{ of } \widetilde{\Delta}\bigr\}. \tag{33}$$

Therefore, by (32) and Theorem 3.10 together with this argument, we can get

Corollary 3.12. If Δ is an inner matrix function in $H^\infty_{M_n}$, then

$$\deg_B(\Delta) = \max\bigl\{k : \delta_{k-1} \neq \delta_k \text{ up to constant}\bigr\}. \tag{34}$$

We would like to pose:

Problem 3.13. Give a direct (matricial) proof for (34) without using the formula (32), which relies upon the Jordan model theory of Sz.-Nagy and Foiaş.

In general, we wish to ask:

Problem 3.14. If Δ is an $n \times r$ inner matrix function, describe $\deg_B(\Delta)$ in terms of entries of Δ (or matrices involving Δ).

3.3 Abrahamse's Theorem for matrix-valued symbols

In 1970, P.R. Halmos addressed the following problem, listed as Problem 5 in his lecture "Ten problems in Hilbert space" [Ha2], [Ha3]:

Is every subnormal Toeplitz operator either normal or analytic ?

Any analytic Toeplitz operator T_φ ($\varphi \in H^\infty$) is easily seen to be subnormal: indeed, $T_\varphi h = P(\varphi h) = \varphi h = M_\varphi h$ for $h \in H^2$, where M_φ is the normal operator of multiplication by φ on L^2. The question is natural because normal and analytic Toeplitz operators are fairly well understood, and they are both subnormal. In 1984, Halmos' Problem 5 was answered in the negative by C. Cowen and J. Long [CoL]. However, Cowen and Long's construction does not provide an intrinsic connection between subnormality and the theory of Toeplitz operators. Until now researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. Thus, we may reformulate Halmos' Problem 5: *Which Toeplitz operators are subnormal ?* The most interesting partial answer to Halmos' Problem 5 was given by M.B. Abrahamse [Ab], who gave a general sufficient condition for the answer to Halmos' Problem 5 to be affirmative.

Abrahamse's Theorem ([Ab]). Let $\varphi \in L^\infty$ be such that φ or $\overline{\varphi}$ is of bounded type. If T_φ is hyponormal and $\ker\,[T_\varphi^*, T_\varphi]$ is invariant under T_φ, then T_φ is normal or analytic.

Consequently, if $\varphi \in L^\infty$ is such that φ or $\overline{\varphi}$ is of bounded type, then every subnormal Toeplitz operator must be either normal or analytic, since $\ker\,[S^*, S]$ is invariant under S for every subnormal operator S. We say that a Toeplitz operator T_Φ with matrix-valued symbol Φ is *analytic* if $\Phi \in H^\infty_{M_n}$. Evidently, any analytic Toeplitz operator with a normal symbol is subnormal because the multiplication operator M_Φ is a normal extension of T_Φ. As a first inquiry in the above reformulation of Halmos' Problem 5 the following question can be raised: *Is Abrahamse's Theorem valid for Toeplitz operators with matrix-valued symbols ?* In general, a straightforward matrix-valued version of Abrahamse's Theorem is doomed to fail: for instance, if $\Phi := \left[\begin{smallmatrix} z+\overline{z} & 0 \\ 0 & z \end{smallmatrix}\right]$, then clearly, both Φ and Φ^* are of bounded type and $T_\Phi = \left[\begin{smallmatrix} U+U^* & 0 \\ 0 & U \end{smallmatrix}\right]$ (where U is the shift on H^2) is subnormal, but neither normal nor analytic.

We here extend the above result to the case of bounded type symbols: we shall say that T_Φ has a *bounded type symbol* if both Φ and Φ^* are of bounded type.

It was shown in [CHL1] that if T_Φ has a bounded type symbol with

$$\Phi_- = \theta B^* \quad \text{(coprime)} \tag{35}$$

and if T_Φ is hyponormal and $\ker [T_\varphi^*, T_\varphi]$ is invariant under T_φ, then T_Φ is normal or analytic. However, the condition (35) forces the inner part of the right coprime factorization (5) of Φ_- to be diagonal-constant. Also, it was shown in [CHKL] that if Φ is a matrix-valued *rational* function then the condition (35) can be weakened to the condition that the inner part of the right coprime factorization (5) of Φ_- has a nonconstant diagonal-constant inner divisor. We note that in view of (21), those conditions of [CHL1] and [CHKL] are special cases of the condition of "having a tensored-scalar singularity." Indeed, it was shown in [CHL3, Theorem 7.3] that for a bounded type symbols $\Phi \in L^\infty_{M_n}$, if Φ has a tensored-scalar singularity then we get a full-fledged matrix-valued version of Abrahamse's Theorem.

Theorem 3.15. (Abrahamse's Theorem for matrix-valued symbols) Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Assume Φ has a tensored-scalar singularity. If T_Φ is hyponormal and $\ker [T_\Phi^*, T_\Phi]$ is invariant under T_Φ, then T_Φ is normal. Hence, in particular, if T_Φ is subnormal then T_Φ is normal.

Remark 3.16. (a) We note that the assumption "Φ has a tensored-scalar singularity" is essential in Theorem 3.15. As we saw before, if $\Phi := \left[\begin{smallmatrix} \overline{z}+z & 0 \\ 0 & z \end{smallmatrix}\right]$, then T_Φ is neither normal nor analytic. But since $\ker H_\Phi = \ker H_{\left[\begin{smallmatrix} \overline{z} & 0 \\ 0 & 0 \end{smallmatrix}\right]} = \left[\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right] H^2_{\mathbb{C}^n}$, it follows that $\Theta \equiv \left[\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right]$ does not have any nonconstant diagonal-constant inner divisor, so that Φ does not have a tensored-scalar singularity.

(b) If $n = 1$, then $\Theta \equiv \theta \in H^\infty$ is vacuously diagonal-constant, so that Theorem 3.15 reduces to the original Abrahamse's Theorem.

3.4 H^∞-functional calculus for the compressions of the shift

It is well known that the functional calculus for polynomials of compressions of the shift results in the Hermite-Fejér matrix via the classical Hermite-Fejér Interpolation Problem. We now extend the polynomial calculus to an H^∞-functional calculus (so-called the Sz.-Nagy-Foiaş functional calculus) via the triangularization theorem, and then extend it further to an $\overline{H^\infty} + H^\infty$-functional calculus for compressions of the shift operator. Let

$$\mathbf{S}_\theta := P_{\mathcal{H}(\theta)} S|_{\mathcal{H}(\theta)}$$

be the compression of the shift operator S to $\mathcal{H}(\theta)$. We briefly review a functional calculus for polynomials of $\mathbf{S}_\theta$. (cf. [FF]).

Let θ be a finite Blaschke product of degree d. Let W be the unitary operator from $\bigoplus_1^d \mathbb{C}^n$ onto $\mathcal{H}(I_\theta)$. It is known [FF, Theorem X.1.5] that $\mathbf{S}_\theta$ is unitarily equivalent to the lower triangular matrix M on $\mathbb{C}^d$. Now let $P(z) \in H^\infty_{M_n}$ be a matrix polynomial of degree k. Then the matrix $P(M)$ on $\mathbb{C}^{n\times d}$ is defined by

$$P(M) := \sum_{i=0}^k P_i \otimes M^i, \quad \text{where } P(z) = \sum_{i=0}^k P_i z^i. \tag{36}$$

If the matrix $P(M)$ is the *Hermite-Fejér matrix* determined by (36) and if $(T_P)_\Theta := P_{\mathcal{H}(\Theta)} T_P|_{\mathcal{H}(\Theta)}$ is the compression of T_P to $\mathcal{H}(\Theta)$ (where $\Theta := I_\theta$ for an inner function θ), then it is known [FF, Theorem X.5.6] that

$$W^*(T_P)_\Theta W = P(M), \tag{37}$$

which says that $P(M)$ is a matrix representation for $(T_P)_\Theta$.

We now extend the representation (37) to the case of matrix H^∞-functions. We refer to [AC] and [Ni1] for details on this representation. For an explicit criterion, we need to introduce the Triangularization Theorem concretely. There are three cases to consider.

Case 1 : Let B be a Blaschke product and let $\Lambda := \{\lambda_n : n \geq 1\}$ be the sequence of zeros of B counted with their multiplicities. Write

$$\beta_1 := 1, \quad \beta_k := \prod_{n=1}^{k-1} \frac{\lambda_n - z}{1 - \overline{\lambda}_n z} \cdot \frac{|\lambda_n|}{\lambda_n} \qquad (k \geq 2),$$

and let

$$\delta_j := \frac{d_j}{1 - \overline{\lambda}_j z} \beta_j \qquad (j \geq 1),$$

where $d_j := (1 - |\lambda_j|^2)^{\frac{1}{2}}$. Let μ_B be the measure on $\mathbb{N}$ given by $\mu_B(\{n\}) := \frac{1}{2} d_n^2, (n \in \mathbb{N})$. Then the map $V_B : L^2(\mu_B) \to \mathcal{H}(B)$ defined by

$$V_B(c) := \frac{1}{\sqrt{2}} \sum_{n \geq 1} c(n) d_n \delta_n, \quad c \equiv \{c(n)\}_{n \geq 1}, \tag{38}$$

is unitary.

Case 2 : Let s be a singular inner function with continuous representing measure $\mu \equiv \mu_s$. Let μ_λ be the projection of μ onto the arc $\{\zeta : \zeta \in \mathbb{T},\ 0 < \arg\zeta \leq \arg\lambda\}$ and let

$$s_\lambda(\zeta) := \exp\Big(- \int_{\mathbb{T}} \frac{t + \zeta}{t - \zeta} d\mu_\lambda(t)\Big) \ \ (\zeta \in \mathbb{D}).$$

Then the map $V_s : L^2(\mu) \to \mathcal{H}(s)$ defined by

$$(V_s c)(\zeta) = \sqrt{2} \int_{\mathbb{T}} c(\lambda) s_\lambda(\zeta) \frac{\lambda d\mu(\lambda)}{\lambda - \zeta} \quad (\zeta \in \mathbb{D}) \tag{39}$$

is unitary.

Case 3 : Let Δ be a singular inner function with pure point representing measure $\mu \equiv \mu_\Delta$. We enumerate the set $\{t \in \mathbb{T} : \mu(\{t\}) > 0\}$ as a sequence $\{t_k\}_{k \in \mathbb{N}}$. Write $\mu_k := \mu(\{t_k\}),\ k \geq 1$. Further, let μ_Δ be a measure on $\mathbb{R}_+ = [0, \infty)$ such that $d\mu_\Delta(\lambda) = \mu_{[\lambda]+1} d\lambda$ and define a function Δ_λ on the unit disk $\mathbb{D}$ by the formula

$$\Delta_0 := 1, \quad \Delta_\lambda(\zeta) := \exp\left\{ - \sum_{k=1}^{[\lambda]} \mu_k \frac{t_k + \zeta}{t_k - \zeta} - (\lambda - [\lambda]) \mu_{[\lambda]+1} \frac{t_{[\lambda]+1} + \zeta}{t_{[\lambda]+1} - \zeta} \right\},$$

where $[\lambda]$ is the integer part of λ ($\lambda \in \mathbb{R}_+$). Then the map $V_\Delta : L^2(\mu_\Delta) \to \mathcal{H}(\Delta)$ defined by

$$(V_\Delta c)(\zeta) := \sqrt{2} \int_{\mathbb{R}_+} c(\lambda) \Delta_\lambda(\zeta) (1 - \overline{t}_{[\lambda]+1} \zeta)^{-1} d\mu_\Delta(\lambda) \ (\zeta \in \mathbb{D}) \tag{40}$$

is unitary.

Collecting together the above three cases we get:

Triangularization Theorem. ([Ni1, p.123]) Let θ be an inner function with the canonical factorization $\theta = B \cdot s \cdot \Delta$, where B is a Blaschke product, and s and Δ are singular inner functions with representing measures μ_s and μ_Δ respectively, with μ_s continuous and μ_Δ a pure point measure. Then the map $V: L^2(\mu_B) \times L^2(\mu_s) \times L^2(\mu_\Delta) \to \mathcal{H}(\theta) \equiv \mathcal{B}(B) \oplus B\mathcal{H}(s) \oplus Bs\mathcal{H}(\Delta)$ defined by

$$V := \begin{bmatrix} V_B & 0 & 0 \\ 0 & BV_s & 0 \\ 0 & 0 & BsV_\Delta \end{bmatrix} \tag{41}$$

is unitary, where $V_B, \mu_B, V_S, \mu_S, V_\Delta, \mu_\Delta$ are defined in (38), (39), (40) and $M := V^*\mathbf{S}_\theta V$ is a lower-triangular operator.

Now we note that every compression of the shift operator is completely non-unitary. Therefore M is an absolutely continuous contraction. Thus if $\Phi \in H^\infty_{M_n}$, then we can define $\Phi(M)$ as a H^∞-functional calculus (the Sz.-Nagy-Foiaş functional calculus). The following theorem was proved in [CHL3, Theorem 6.3].

Theorem 3.17. Let $\Phi \in H^\infty_{M_n}$ and let $\theta \in H^\infty$ be an inner function. If we write

$$M := V^*\mathbf{S}_\theta V \quad \text{and} \quad \mathcal{V} := V \otimes I_n, \tag{42}$$

where $V: L^2(\mu_B) \times L^2(\mu_s) \times L^2(\mu_\Delta) \to \mathcal{H}(\theta)$ is unitary as in (41), then

$$\mathcal{V}^*(T_\Phi)_\Theta \mathcal{V} = \Phi(M), \tag{43}$$

where $\Theta := I_\theta$ and $(T_\Phi)_\Theta := P_{\mathcal{H}(\Theta)} T_\Phi|_{\mathcal{H}(\Theta)}$

Remark 3.18. $\Phi(M)$ is called a *matrix representation* for $(T_\Phi)_\Theta$.

An application of the functional calculus. In [GHR], it was shown that (i) if $\Phi \in L^\infty_{M_n}$ is such that T_Φ is hyponormal, i.e., $[T^*_\Phi, T_\Phi] \geq 0$, then Φ is normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$ a.e. on $\mathbb{T}$ and that (ii) if $\Phi \in L^\infty_{M_n}$ is normal and

$$\mathcal{C}(\Phi) := \{K \in H^\infty_{M_n} : \ \Phi - K\Phi^* \in H^\infty_{M_n}\},$$

then

$$T_\Phi \text{ is hyponormal} \Longleftrightarrow \exists\, K \in \mathcal{C}(\Phi) \text{ with } ||K||_\infty \leq 1.$$

In [CHL2, Theorem 3.3], it was shown from (43) that if $\Phi := \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ is normal of the form

$$\Phi_+ = \theta_0\theta_1 A^* \quad \text{and} \quad \Phi_- = \theta_1 B^* \quad (A, B \in H^\infty_{M_n}) \tag{44}$$

and if $(T_A)_{\theta_0\theta_1} := P_{\mathcal{H}(\theta_0\theta_1)} T_A|_{\mathcal{H}(\theta_0\theta_1)}$ has dense range then

$$K(M) \text{ is contractive} \Longleftrightarrow T_\Phi \text{ is hyponormal}, \tag{45}$$

where $K \in \mathcal{C}(\Phi)$ and M is given by (42) with $\theta := \theta_0\theta_1$. Note that the form (44) is a necessary condition for T_Φ to be hyponormal when Φ and Φ^* are of bounded type.

We consider a revealing example that illustrates (45).

Example 3.19. Let θ be an inner function and consider the matrix-valued function

$$\Phi := \begin{bmatrix} \overline{z} & \overline{z\theta}+z\theta \\ \overline{z\theta}+z\theta & \overline{z} \end{bmatrix}.$$

We now use the equivalence (45) to determine the hyponormality of T_Φ. Under the above notation we have

$$\theta_0 = 1, \quad \theta_1 = z\theta, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \theta & 1 \\ 1 & \theta \end{bmatrix}.$$

If we put $K(z) := \left[\begin{smallmatrix} 1 & \theta \\ \theta & 1 \end{smallmatrix}\right]$, then a straightforward calculation shows that $K \in \mathcal{C}(\Phi)$. We can also see that that

$$(T_A)_{\theta_0\theta_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{is invertible.}$$

But since $K(M) = \left[\begin{smallmatrix} 1 & \theta(M) \\ \theta(M) & 1 \end{smallmatrix}\right]$, it follows that

$$I - K(M)^*K(M) = -\begin{bmatrix} \theta(M)^*\theta(M) & \theta(M)+\theta(M)^* \\ \theta(M)+\theta(M)^* & \theta(M)^*\theta(M) \end{bmatrix},$$

which is not positive (simply by looking at the upper-left entry). It therefore follows from (45) that T_Φ is not hyponormal. $\square$

We next extend the representation (43) to $\overline{H^\infty_{M_n}} + H^\infty_{M_n}$ (where $\overline{H^\infty_{M_n}}$ denotes the set of $n \times n$ matrix functions whose entries belong to $\overline{H^\infty} := \{g : \overline{g} \in H^\infty\}$). Let $Q \in \overline{H^\infty_{M_n}} + H^\infty_{M_n}$ be of the form $Q = Q_-^* + Q_+$. If $\Theta := I_\theta$ for an inner function θ, then we define

$$(T_Q)_\Theta := P_{\mathcal{H}(\Theta)} T_Q|_{\mathcal{H}(\Theta)}.$$

Then

$$(T_Q)_\Theta = (T_{Q_-^*})_\Theta + (T_{Q_+})_\Theta = (T_{Q_-})_\Theta^* + (T_{Q_+})_\Theta.$$

If $M := V^*\mathbf{S}_\theta V$, where $V : L \equiv L^2(\mu_B) \times L^2(\mu_s) \times L^2(\mu_\Delta) \to \mathcal{H}(\theta)$ is unitary as in (41), we also define $Q(M)$ by

$$Q(M) := (Q_-(M))^* + Q_+(M), \tag{46}$$

where $Q_\pm(M)$ is defined by the Sz.-Nagy-Foiaş functional calculus.

We then have ([CHL3, Lemma 6.8]):

Theorem 3.20. Let $Q \in \overline{H^\infty_{M_n}} + H^\infty_{M_n}$ and $\Theta := I_\theta$ for an inner function θ. Then $\mathcal{V}^*(T_Q)_\Theta\mathcal{V} = Q(M)$ and $Q(M)^* = Q^*(M)$, where $\mathcal{V}$ and M are given by (42).

Remark 3.21. We are tempted to guess that $Q(M)^*Q(M) = (Q^*Q)(M)$. But this is not the case. To see this, let $\theta = z^3$ and let $Q(z) := \left[\begin{smallmatrix} z^2 & 0 \\ 0 & z^2 \end{smallmatrix}\right]$. Then we have

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$Q(M) \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$Q(M)^*Q(M) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (Q^*Q)(M) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which gives $Q^*(M)Q(M) \neq (Q^*Q)(M)$.

4 Operator-valued function theory

In this section we consider operator-valued function theory related with Hankel, Toeplitz and model operators. We focus on meromorphic pseudo-continuation of bounded type, a canonical decomposition of strong L^2-functions, spectra of model operators, an operator-valued version of Potapov's matrix-valued factorization theorem, and hyponormality and subnormality of Toeplitz operators with operator-valued symbols.

4.1 Meromorphic pseudo-continuations of bounded type

We introduce the notion of "bounded type" for strong L^2-functions. Recall that a matrix-valued function of bounded type was defined by a matrix whose entries are of bounded type. But this definition is not appropriate for operator-valued functions, in particular strong L^2-functions, even though the terminology of matrix "entry" can be properly interpreted. Thus we need a new idea for a suitable definition of "bounded type" strong L^2-function, which is equivalent to the condition that each entry is of bounded type when the function is matrix-valued. Our motivation stems from the equivalence in (18) for the case of matrix-valued functions.

Definition 4.1. A function $\Phi \in L^2_s(\mathcal{B}(D,E))$ with values in $\mathcal{B}(D,E)$ is said to be of *bounded type* if $\ker H^*_\Phi = \Theta H^2_E$ for some two-sided inner function Θ with values in $\mathcal{B}(E)$.

It is known that if Δ is an inner function with values in $\mathcal{B}(D,E)$, then (cf. [CHL4, Corollary 2.25])

$$\breve{\Delta} \text{ is of bounded type} \iff [\Delta, \Delta_c] \text{ is two-sided inner.} \tag{47}$$

On the other hand, in general, if a strong L^2-function Φ is of bounded type then we cannot guarantee that each entry $\phi_{ij} \equiv \langle \Phi d_j,\ e_i \rangle$ is of bounded type, where $\{d_j\}$ and $\{e_i\}$ are orthonormal bases of D and E, respectively. We may ask:

Question 4.2. Under what conditions can we guarantee that each entry of Φ is of bounded type?

To examine this question, given a function

$$\Psi : \mathbb{D}^e \equiv \{z : 1 < |z| \leq \infty\} \to \mathcal{B}(D, E),$$

we define $\Psi_{\mathbb{D}} : \mathbb{D} \to \mathcal{B}(E, D)$ by $\Psi_{\mathbb{D}}(\zeta) := \Psi^*(1/\overline{\zeta})$ for $\zeta \in \mathbb{D}$. If $\Psi_{\mathbb{D}}$ is a strong H^2-function, inner, and two-sided inner with values in $\mathcal{B}(E, D)$, then we shall say that Ψ is a strong H^2-function, inner, and two-sided inner in $\mathbb{D}^e$ with values in $\mathcal{B}(D, E)$, respectively. A $\mathcal{B}(D, E)$-valued function Ψ is said to be *meromorphic of bounded type* in $\mathbb{D}^e$ if it can be represented by $\Psi = \frac{G}{\theta}$, where G is a strong H^2-function in $\mathbb{D}^e$, with values in $\mathcal{B}(D, E)$ and θ is a scalar inner function in $\mathbb{D}^e$ (cf. [Fu1], [Fu2]). A function $\Phi \in L^2_s(\mathcal{B}(D, E))$ is said to have a *meromorphic pseudo-continuation* $\hat{\Phi}$ *of bounded type* in $\mathbb{D}^e$ if $\hat{\Phi}$ is meromorphic of bounded type in $\mathbb{D}^e$ and Φ is the nontangential SOT limit of $\hat{\Phi}$, that is, for all $x \in D$,

$$\Phi(z)x = \hat{\Phi}(z)x := \lim_{rz \to z} \hat{\Phi}(rz)x \quad \text{for almost all } z \in \mathbb{T}.$$

Note that for almost all $z \in \mathbb{T}$,

$$\Phi(z)x = \lim_{rz \to z} \hat{\Phi}(rz)x = \lim_{rz \to z} \hat{\Phi}^*_{\mathbb{D}}(r^{-1}z)x = \hat{\Phi}^*_{\mathbb{D}}(z)x \quad (x \in D).$$

The following proposition was proved in [Fu1] under the more restrictive setting of $H^\infty(\mathcal{B}(D, E))$.

Proposition 4.3. ([CHL4, Lemma 2.28 and Corollary 2.29])
Let $\Phi \in L^\infty(\mathcal{B}(D, E)) \cup L^2_{\mathcal{B}(D,E)}$. Then the following are equivalent:

(a) Φ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$;

(b) $\theta H^2_E \subseteq \ker H_{\Phi^*}$ for some scalar inner function θ.

The following proposition gives an answer (for $L^2_{\mathcal{B}(D,E)}$) to Question 4.2 (cf. [CHL4, Proposition 2.30]).

Proposition 4.4. Let D and E be separable complex Hilbert spaces and let $\{d_j\}$ and $\{e_i\}$ be orthonormal bases of D and E, respectively. If $\Phi \in L^2_{\mathcal{B}(D,E)}$ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then $\breve{\phi}_{ij}(z) \equiv \langle \breve{\Phi}(z) d_j,\ e_i \rangle_E$ is of bounded type for each i, j.

Remark 4.5. For a function $\Phi \in L^2_s(\mathcal{B}(D, E))$, we can show (cf. [CHL4, Lemma 2.27]) that if Φ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then $\breve{\Phi}$ is of bounded type and that the converse is also true when Φ is matrix-valued. However, the converse is not true in general. To see this, let $\{\alpha_n\}$ be a sequence of distinct points in $\mathbb{D}$ such that $\sum_{n=1}^\infty (1 - |\alpha_n|) = \infty$ and put $\Delta := \text{diag}(b_{\alpha_n})$, where $b_{\alpha_n}(z) := \frac{z - \alpha_n}{1 - \overline{\alpha}_n z}$. Then Δ is two-sided inner, and hence $\breve{\Delta}$ is of bounded type. On the other hand, by Lemma 2.3, $\ker H_{\Delta^*} = \Delta H^2_{\ell^2}$. Thus if Δ had a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then by Proposition 4.3, we would have $\theta H^2_{\ell^2} \subseteq \Delta H^2_{\ell^2}$ for a scalar inner function θ, so that we should have $\theta(\alpha_n) = 0$ for each $n = 1, 2, \cdots$, and hence $\theta = 0$, a contradiction. Therefore, Δ cannot have a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$.

Remark 4.6. By contrast to the matrix-valued case, it may happen that an L^∞-function Φ is not of bounded type in the sense of Definition 4.1 even though each entry ϕ_{ij} of Φ is of bounded type. To see this, let $\{\alpha_j\}$ be a sequence of distinct points in $(0,1)$ satisfying $\sum_{j=1}^\infty (1-\alpha_j) < \infty$. For each $j \in \mathbb{Z}_+$, choose a sequence $\{\alpha_{ij}\}$ of distinct points on the circle $C_j := \{z \in \mathbb{C} : |z| = \alpha_j\}$. Let

$$B_{ij} := \frac{\overline{b}_{\alpha_{ij}}}{(i+j)!} \quad (i, j \in \mathbb{Z}_+),$$

where $b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}$, and let

$$\Phi := [B_{ij}] = \begin{bmatrix} \frac{\overline{b}_{\alpha_{11}}}{2!} & \frac{\overline{b}_{\alpha_{12}}}{3!} & \frac{\overline{b}_{\alpha_{13}}}{4!} & \cdots \\ \frac{\overline{b}_{\alpha_{21}}}{3!} & \frac{\overline{b}_{\alpha_{22}}}{4!} & \frac{\overline{b}_{\alpha_{23}}}{5!} & \cdots \\ \frac{\overline{b}_{\alpha_{31}}}{4!} & \frac{\overline{b}_{\alpha_{32}}}{5!} & \frac{\overline{b}_{\alpha_{33}}}{6!} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

Observe that

$$\sum_{i,j} |B_{ij}(z)|^2 = \sum_i \frac{i}{((1+i)!)^2} \le \sum_i \frac{1}{(1+i)^2} < \infty,$$

which implies that $\Phi \in L^\infty(\mathcal{B}(\ell^2))$. For a function $f \in H^2_{\ell^2}$, we write $f = (f_1, f_2, f_3, \cdots)^t$ ($f_n \in H^2$). Thus if $f = (f_1, f_2, f_3, \cdots)^t \in \ker H_\Phi$, then $\sum_j \frac{\overline{b}_{\alpha_{ij}}}{(i+j)!} f_j \in H^2$ for each $i \in \mathbb{Z}_+$, which forces that $f_j(\alpha_{ij}) = 0$ for each i, j. Thus $f_j = 0$ for each j (by the Identity Theorem). Therefore we can conclude that $\ker H^*_{\widetilde{\Phi}} = \{0\}$, so that $\widetilde{\Phi}$ is not of bounded type. But we note that every entry of $\widetilde{\Phi}$ is of bounded type.

We next consider an application to C_0-contractions.

The class C_{00} denotes the set of all contractions $T \in \mathcal{B}(\mathcal{H})$ such that $\lim_{n\to\infty} T^n x = 0$ and $\lim_{n\to\infty} T^{*n} x = 0$ for each $x \in \mathcal{H}$. It is known ([Ni1, p.43]) that if T is a C_0.-contraction with characteristic function Δ (i.e., $T \cong S_E^*|_{\mathcal{H}(\Delta)}$), then

$$T \in C_{00} \iff \Delta \text{ is } \textit{two-sided} \text{ inner.} \tag{48}$$

A contraction $T \in \mathcal{B}(\mathcal{H})$ is called a completely non-unitary (c.n.u.) if there exists no nontrivial reducing subspace on which T is unitary. The class C_0 is the set of all c.n.u. contractions T such that there exists a nonzero function $\varphi \in H^\infty$ annihilating T, i.e., $\varphi(T) = 0$, where $\varphi(T)$ is given by the Sz.-Nagy and Foiaş functional calculus. We can easily check that $C_0 \subseteq C_{00}$. Moreover, it is well known ([Ni1, p.73]) that if $T := P_{\mathcal{H}(\Delta)} S_E|_{\mathcal{H}(\Delta)} \in C_{00}$ and $\varphi \in H^\infty$, then

$$\varphi(T) = 0 \iff \exists\, G \in H^\infty(\mathcal{B}(E)) \text{ such that } G\Delta = \Delta G = \varphi I_E. \tag{49}$$

The theory of spectral multiplicity for operators of class C_0 has been well developed (see [Ni1, Appendix 1], [SFBK]). If $T \in C_0$, then there exists an inner function m_T such that $m_T(T) = 0$ and

$$\varphi \in H^\infty,\ \varphi(T) = 0 \implies \varphi/m_T \in H^\infty.$$

The function m_T is called the *minimal annihilator* of the operator T.

In view of (48), we may ask: What is a condition on the characteristic function Δ of T for a $C_{0\bullet}$-contraction T to belong to the class C_0. An answer to this question was given in [CHL4, Proposition 2.34].

Theorem 4.7. Let $T := S_E^*|_{\mathcal{H}(\Delta)}$ for an inner function Δ with values in $\mathcal{B}(D, E)$. Then $T \in C_0$ if and only if Δ is two-sided inner and has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Hence, in particular, if Δ is an inner matrix function then $T \in C_0$ if and only if $T \in C_{00}$.

4.2 A canonical decomposition of strong L^2-functions

There exists a matrix-valued L^2-function that does not admit a DSS factorization. To see this, assume that θ_1 and θ_2 are coprime inner functions. Consider

$$\Phi := \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & a \end{bmatrix} \equiv [\phi_1, \phi_2, \phi_3] \in H^\infty_{M_3},$$

where $a \in H^\infty$ is such that $\overline{a}$ is not of bounded type. Then a direct calculation shows that

$$\ker H_{\Phi^*} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & 0 \end{bmatrix} H^2_{\mathbb{C}^2} \equiv \Delta H^2_{\mathbb{C}^2}.$$

Since Δ is not two-sided inner, it follows from Lemma 2.1 that Φ does not admit a DSS factorization. For a decomposition of Φ, suppose that $\Phi = \Omega A^*$, where $\Omega, A \in H^2_{M_{3\times k}} (k = 1, 2)$, Ω is an inner function, and Ω and A are right coprime. We then have $\Phi^*\Omega = A \in H^2_{M_{3\times k}}$. But since $\overline{a}$ is not of bounded type, it follows that the third row vector of Ω is zero. Thus we must have $a = 0$, a contradiction. Therefore we could not get any decomposition of the form $\Phi = \Omega A^*$ with a $3 \times k$ inner matrix function Ω for each $k = 1, 2, 3$. To get to another idea, we note that $\ker \Delta^* = [0\ 0\ 1]^t H^2 \equiv \Delta_c H^2$. Then by a direct manipulation, we can get

$$\Phi = \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^* + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & a \end{bmatrix} \equiv \Delta A^* + \Delta_c C \tag{50}$$

where Δ and A are right coprime because $\widetilde{\Delta} H^2_{\mathbb{C}^3} \bigvee \widetilde{A} H^2_{\mathbb{C}^3} = H^2_{\mathbb{C}^2}$. The example (50) seems to signal that the decomposition of a matrix-valued H^2-functions Φ satisfying $\ker H^*_{\breve{\Phi}} = \Delta H^2_{\mathbb{C}^n}$ may be affected by the kernel of Δ^* and in turn, the complementary factor Δ_c of Δ. The following theorem gives a canonical decomposition of strong L^2-functions which realizes the idea inside the above example ([CHL4, Theorem 3.1]).

Theorem 4.8. (A canonical decomposition of strong L^2-functions) If Φ is a strong L^2-function with values in $\mathcal{B}(D,E)$, then Φ can be expressed in the form

$$\Phi = \Delta A^* + B, \tag{51}$$

where
(i) Δ is an inner function with values in $\mathcal{B}(E',E)$, $\widetilde{A} \in H^2_s(\mathcal{B}(D,E'))$, and $B \in L^2_s(\mathcal{B}(D,E))$;
(ii) Δ and A are right coprime;
(iii) $\Delta^* B = 0$; and
(iv) $\text{nc}\{\Phi_+\} \le \dim E'$.
In particular, if $\dim E' < \infty$ (for instance, if $\dim E < \infty$), then the expression (51) is unique (up to a unitary constant right factor).

If $\breve{\Phi}$ is of bounded type then Δ should be two-sided, and hence $B = 0$ in (51). Thus if $\breve{\Phi}$ is of bounded type the decomposition (51) reduces to the DSS factorization. From the viewpoint that the decomposition (51) is unique (except the case of $\dim E' = \infty$), we persist to say that the decomposition (51) is *canonical*. On the other hand, we can show (cf. [CHL4, Proof of Theorem 4.16]) that the inner function Δ in a canonical decomposition (51) of a strong L^2-function Φ can be obtained from the equation $\ker H^*_{\breve{\Phi}} = \Delta H^2_{E'}$ which is guaranteed by the Beurling-Lax-Halmos Theorem. In this case, the expression (51) will be called the *BLH-canonical decomposition* of Φ in the viewpoint that Δ comes from the Beurling-Lax-Halmos Theorem. However, if $\dim E' = \infty$ (even though $\dim D < \infty$), then it is possible to get another inner function Θ of a canonical decomposition (51) for the same function: in this case, $\ker H^*_{\breve{\Phi}} \ne \Theta H^2_{E''}$. Indeed, we can show that the canonical decomposition (51) is not unique in general. Indeed, if $\dim E' = \infty$ (even though $\dim D < \infty$), the canonical decomposition (51) may not be unique even if $\breve{\Phi}$ is of bounded type. To see this, let Φ be an inner function with values in $\mathcal{B}(\mathbb{C}^2, \ell^2)$ defined by

$$\Phi := \begin{bmatrix} \theta_1 & 0 \\ 0 & 0 \\ 0 & \theta_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix},$$

where θ_1 and θ_2 are scalar inner functions. Then

$$\ker H^*_{\breve{\Phi}} = \ker H_{\Phi^*} = \text{diag}(\theta_1, 1, \theta_2, 1, 1, 1, \cdots) H^2_{\ell^2} \equiv \Theta H^2_{\ell^2},$$

which implies that $\breve{\Phi}$ is of bounded type since Θ is two-sided inner (see Definition 4.1). Let

$$A := \Phi^* \Theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \quad \text{and} \quad B := 0.$$

Then $\widetilde{A}$ belongs to belongs to $H^2_s(\mathcal{B}(\mathbb{C}^2, \ell^2))$ and $\widetilde{\Theta} H^2_{\ell^2} \bigvee \widetilde{A} H^2_{\mathbb{C}^2} = H^2_{\ell^2}$, which implies that Θ and A are right coprime. Clearly, $\Theta^* B = 0$ and $\text{nc}\{\Phi_+\} \le \dim \ell^2 = \infty$. Therefore, $\Phi = \Theta A^*$ is the

BLH-canonical decomposition of Φ. On the other hand, to get another canonical decomposition of Φ, let

$$\Delta := \begin{bmatrix} \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \theta_2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then Δ is an inner function. If we define

$$A_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix} \quad \text{and} \quad B := 0\,,$$

then $\widetilde{A}_1$ belongs to $H^2_s(B(\mathbb{C}^2, \ell^2))$ such that Δ and A_1 are right coprime, $\Delta^* B = 0$ and $\text{nc}\{\Phi_+\} \leq \dim \ell^2 = \infty$. Therefore $\Phi = \Delta A_1^*$ is also a canonical decomposition of Φ. In this case, $\ker H^*_{\breve{\Phi}} \neq \Delta H^2_{\ell^2}$. Therefore, the canonical decomposition of Φ is not unique.

We are also interested in the following problem: *Given an inner function $\Delta \in \mathcal{B}(E', E)$, with $\dim E' < \infty$, describe the set F in H^2_E such that $\mathcal{H}(\Delta) = E^*_F$.* As a corollary of Theorem 4.8, we get an answer to the problem ([CHL4, Corollary 4.9]):

Corollary 4.9. Suppose Δ is an inner function with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$. If $\Phi = \Delta A^* + B$ is a canonical decomposition of Φ in $L^2_s(\mathcal{B}(D, E))$, we define a function F by $F(z) := \overline{z}\big(\Phi_+(z) - \hat{\Phi}(0)\big)$. We then have $E^*_{\{F\}} = \mathcal{H}(\Delta)$.

4.3 Spectra of model operators

If θ is a scalar inner function, then the spectrum, $\sigma(\theta)$, of θ is defined by the complement (in $\text{cl}\,\mathbb{D}$) of the set of all points $\lambda \in \text{cl}\,\mathbb{D}$, such that the function $\frac{1}{\theta}$ can be continued analytically into a (full) neighborhood of λ. We recall that if θ is a scalar inner function, then we may write

$$\theta(\zeta) = B(\zeta)\text{exp}\left(-\int_{\mathbb{T}} \frac{z+\zeta}{z-\zeta} d\mu(z)\right),$$

where B is a Blaschke product and μ is a singular measure on $\mathbb{T}$. It is known that the spectrum $\sigma(\theta)$ of θ is given as

$$\sigma(\theta) = \text{cl}\,\theta^{-1}(0) \bigcup \text{supp}\,\mu \qquad (cf.[\text{Ni1}, p.63]). \tag{52}$$

It is also known that if $\mathbf{S} := P_{\mathcal{H}(\Delta)} S_E|_{\mathcal{H}(\Delta)} \in C_0$, then (cf. [Ni1, p.72])

$$\sigma(\mathbf{S}) = \sigma(m_{\mathbf{S}}), \tag{53}$$

where $m_{\mathbf{S}}$ is the minimal annihilator of $\mathbf{S}$.

On the other hand, if Δ is a two-sided inner function with values in $\mathcal{B}(E)$, then the *spectrum*, denoted by $\sigma(\Delta)$, of Δ is defined as the complement (in $\text{cl}\,\mathbb{D}$) of the set of values $\lambda \in \text{cl}\,\mathbb{D}$ for which

$\Delta(\zeta)^{-1}$ exists in a certain relative neighborhood of λ and admits an analytic continuation to a (full) neighborhood of λ. By the operator-valued version of the Livšic-Moeller Theorem ([Ni1, p.75]), we know that if Δ is two-sided inner and $T := S_E^*|_{\mathcal{H}(\Delta)}$, then

$$\sigma(T) = \sigma(\widetilde{\Delta}). \tag{54}$$

Now, we may ask, what is the spectrum of the model operator $S_E^*|_{\mathcal{H}(\Delta)}$ (for an one-sided inner function Δ)? Here is a partial answer.

Proposition 4.10. Let $T := S_E^*|_{\mathcal{H}(\Delta)}$ for an inner function Δ with values in $\mathcal{B}(D,E)$. If $\breve{\Delta}$ is of bounded type then

$$\sigma(T) \subseteq \sigma(\widetilde{[\Delta,\Delta_c]}) \bigcup \overline{\sigma(S_E|_{\Delta_c H^2_{D'}})}, \tag{55}$$

where Δ_c is the complementary factor of Δ, with values in $\mathcal{B}(D',E)$. Moreover, if Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$, then

$$\sigma(\widetilde{[\Delta,\Delta_c]}) \subseteq \sigma(T). \tag{56}$$

Proof. Suppose first that $\breve{\Delta}$ is of bounded type. Then by (47), $[\Delta,\Delta_c]$ is two-sided inner. Observe that

$$[\Delta,\Delta_c]H^2_{D\oplus D'} = \Delta H^2_D \oplus \Delta_c H^2_{D'},$$

and hence

$$\mathcal{H}(\Delta) = \mathcal{H}([\Delta,\Delta_c]) \oplus \Delta_c H^2_{D'}.$$

Thus we may write

$$T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} \mathcal{H}([\Delta,\Delta_c]) \\ \Delta_c H^2_{D'} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}([\Delta,\Delta_c]) \\ \Delta_c H^2_{D'} \end{bmatrix}. \tag{57}$$

Note that $T_1 = S_E^*|_{\mathcal{H}([\Delta,\Delta_c])}$. Thus by (54), $\sigma(T_1) = \sigma(\widetilde{[\Delta,\Delta_c]})$. Also, since $T_2 = P_{\Delta_c H^2_{D'}} S_E^*|_{\Delta_c H^2_{D'}}$, it follows that $\sigma(T_2) = \overline{\sigma(T_2^*)} = \overline{\sigma(S_E|_{\Delta_c H^2_{D'}})}$. Since $\sigma(T) \subseteq \sigma(T_1) \cup \sigma(T_2)$, the assertion (55) follows at once.

Suppose next that Δ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then $\breve{\Delta}$ is of bounded type, so that again by (47), $[\Delta,\Delta_c]$ is two-sided inner. Observe, by (12),

$$\ker H_{\Delta^*} = [\Delta,\Delta_c]H^2_{D\oplus D'} = \ker H_{[\Delta,\Delta_c]^*}.$$

Thus by Proposition 4.3, $[\Delta,\Delta_c]$ has a meromorphic pseudo-continuation of bounded type in $\mathbb{D}^e$. Then by Theorem 4.7, $T_1 \equiv S_E^*|_{\mathcal{H}([\Delta,\Delta_c])}$ belongs to the class C_0. By the Model Theorem, T_1 is equivalent to $D \equiv P_{\mathcal{H}(\widetilde{[\Delta,\Delta_c]})} S_E|_{\mathcal{H}(\widetilde{[\Delta,\Delta_c]})}$ (cf. [Ni1, p.75]). Thus by (53),

$$\sigma(T_1) = \sigma(m_D) = \mathrm{cl}\, m_D^{-1}(0) \bigcup \mathrm{supp}\, \mu_D$$

(μ_D is a singular measure corresponding to m_D), which has no interior points. Thus $\sigma(T_1) \cap \sigma(T_2)$ has no interior points. Thus we have $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$ because in the Banach space setting, the passage from $\sigma\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ to $\sigma(A) \cup \sigma(B)$ consists of filling in certain holes in $\sigma\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$, occurring in $\sigma(A) \cap \sigma(B)$ (cf. [HLL]). Therefore, (55) holds with equality in place of inclusion, which gives (56). This complete the proof. $\square$

We would like to pose:

Problem 4.11. If $T := S_E^*|_{\mathcal{H}(\Delta)}$ for an inner function Δ with values in $\mathcal{B}(D,E)$ such that $\breve{\Delta}$ is of bounded type, describe the spectrum of T in terms of Δ.

4.4 An extension of Potapov's factorization theorem

In 1955, V.P. Potapov [Po] showed that every rational inner $n\times n$ matrix-valued function can be written as a finite Blaschke-Potapov product. We extend this result to the case of operator-valued functions.

As customarily done, we say that two inner functions $A, B \in H^\infty(B(E))$ are equal if they are equal up to a unitary constant right factor, i.e., there exists a unitary (constant) operator $V\in\mathcal{B}(E)$ such that $A = BV$. Note that if V is a unitary operator in $B(E)$, then for every $\Phi\in H^\infty(\mathcal{B}(E))$,

$$\Phi = V(V^*\Phi) = (\Phi V^*)V,$$

which implies that V is an inner divisor of Φ. If M is a nonzero closed subspace of a Hilbert space E, then a function of the form

$$b_\alpha P_M + (I_E - P_M) \quad (\text{where } b_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z})$$

is called a (*operator-valued*) *Blaschke-Potapov factor*, where P_M is the orthogonal projection of E onto M. A function D is called a (*operator-valued*) *finite Blaschke-Potapov product* if D is of the form

$$D = V\prod_{m=1}^{M}\Big(b_m P_m + (I-P_m)\Big),$$

where V is a unitary operator, b_m is a Blaschke factor, and P_m is a nonzero orthogonal projection in E for each $m = 1,\cdots,M$. In particular, a scalar-valued function D reduces to a finite Blaschke product $D = \nu\prod_{m=1}^M b_m$, where $\nu = e^{i\omega}$. It is known (cf. [Po]) that an $n\times n$ matrix function D is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product.

To proceed, we consider the follwoing question: What is a left inner divisor of zI_n? For this question, we may guess that each left inner divisor of zI_n is a Blaschke-Potapov factor. More specifically, we wonder if a left inner divisor of $\left[\begin{smallmatrix} z & 0\\ 0 & z\end{smallmatrix}\right]\equiv zI_2$ should be of the following form up to a unitary constant right factor (also up to unitary equivalence):

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 0\\ 0 & z\end{bmatrix},\quad \begin{bmatrix}z & 0\\ 0 & 1\end{bmatrix}\quad \text{or}\quad \begin{bmatrix}z & 0\\ 0 & z\end{bmatrix}.$$

For example, $A\equiv \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & -z\\ 1 & z\end{smallmatrix}\right]$ is a left inner divisor of $\left[\begin{smallmatrix} z & 0\\ 0 & z\end{smallmatrix}\right]\equiv zI_2$: indeed,

$$A\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}z & z\\ -1 & 1\end{bmatrix} = \begin{bmatrix}z & 0\\ 0 & z\end{bmatrix}.$$

In this case, if we take a unitary operator $V := \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1\\ -1 & 1\end{smallmatrix}\right]$, then

$$\begin{bmatrix}1 & 0\\ 0 & z\end{bmatrix} = V\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -z\\ 1 & z\end{bmatrix} = \left[V\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -z\\ 1 & z\end{bmatrix}\cdot V^*\right]\cdot V.$$

In fact, it was shown in [CHL1, Lemma 2.5] that

$$\text{every left inner divisor of } zI_n \in H^\infty_{M_n} \text{ is a Blaschke-Potapov factor.} \tag{58}$$

This fact is useful for the study of coprime-ness of functions (cf. [CHL2]). In [CHL3, p.23], the authors asked:

Question 4.12. Is the statement in (58) still true for operator-valued functions ?

We will call a function of the form zI_E the operator-valued coordinate function. This allows us to rephrase Question 4.12 as follows: Is every left inner divisor of the operator-valued coordinate function a Blaschke-Potapov factor ? For Question 4.12, we wish to study a more general case, and we first observe that if A is an inner divisor of $z^N I_E$, then there exists a function $\Omega \in H^\infty(\mathcal{B}(E))$ such that $A\Omega = \Omega A = z^N I_E$ a.e. on $\mathbb{T}$, so that $A^* z^N I_E = \Omega \in H^\infty(\mathcal{B}(E))$, which implies that A is a polynomial of degree at most N.

In [CHL5, Theorem 2.1], inner divisors of $z^N I_E$ were characterized:

Theorem 4.13. Let A be a polynomial of degree N. Then A is an inner divisor of $z^N I_E$ if and only if A is a finite Blaschke-Potapov product of the form

$$A(z) = V \prod_{m=1}^{N} \Big(zP_m + (I - P_m) \Big),$$

where P_m is the orthogonal projection from E onto $\bigoplus_{n=m}^{N} \operatorname{ran} \widehat{A}(n)^*$, and

$$V := \operatorname{diag}\big(\widehat{A}(0)|_{\operatorname{ran} \widehat{A}(0)^*}, \widehat{A}(1)|_{\operatorname{ran} \widehat{A}(1)^*}, \cdots, \widehat{A}(N)|_{\operatorname{ran} \widehat{A}(N)^*}\big).$$

The following corollary gives an affirmative answer to Question 4.12.

Corollary 4.14. If $\Delta \in H^\infty(\mathcal{B}(E))$ is a left inner divisor of zI_E, then Δ is a Blaschke-Potapov factor.

For $\Phi \in H^\infty(\mathcal{B}(E))$ and $\alpha \in \mathbb{D}$, write

$$\Phi_\alpha := \Phi \circ b_\alpha.$$

Then we can easily check the following:

(a) $\Phi_\alpha \in H^\infty(\mathcal{B}(E))$;

(b) If Δ is an inner function with values in $\mathcal{B}(E)$, then so is Δ_α.

Then the following corollary follows from Theorem 4.13 at once.

Corollary 4.15. Let $A \in H^\infty(\mathcal{B}(E))$. Then A is an inner divisor of $b_\alpha^N I_E$ if and only if A is a finite Blaschke-Potapov product of the form

$$A = V \prod_{m=1}^{N} \Big(b_\alpha P_m + (I - P_m) \Big),$$

where P_m and V are defined as in Theorem 4.13.

We now introduce the notion of operator-valued "rational" function. Recall that a matrix-valued function is rational if its entries are rational functions. But this definition is not appropriate for operator-valued functions, in particular H^∞-functions, even though the terminology of matrix "entry" may be properly interpreted. Thus, the new idea should capture and encapsulate a definition of operator-valued rational function which is equivalent to the condition that each entry is rational when the function is matrix-valued. In the sequel, we give a formal definition of operator-valued rational function.

To do so, we recall Definition 4.1 - the definition of bounded type: A function $\Phi \in L^2_s(B(D,E))$ is said to be of bounded type if $\ker H^*_\Phi = \Theta H^2_E$ for some two-sided inner function Θ with values in $\mathcal{B}(E)$.

We now introduce:

Definition 4.16. A function $\Phi \in H^\infty(\mathcal{B}(D,E))$ is said to be *rational* if

$$\theta H^2_E \subseteq \ker H_{\Phi^*} \tag{59}$$

for some finite Blaschke product θ.

Observe that if $\Phi \in H^\infty(\mathcal{B}(D,E))$, then

$$\Phi \text{ is rational} \Longrightarrow \breve{\Phi} \text{ is of bounded type.} \tag{60}$$

To see this, suppose Φ is rational. By definition and the Beurling-Lax-Halmos Theorem there exist a finite Blaschke product θ and an inner function $\Delta \in H^\infty(\mathcal{B}(E',E)))$ such that

$$\theta H^2_E \subseteq \ker H_{\Phi^*} = \Delta H^2_{E'},$$

which implies that Δ is a left inner divisor of θI_E. Thus Δ is two-sided inner, so that $\breve{\Phi}$ is of bounded type, which proves (60).

Also, if $\Phi \equiv (\phi_{ij}) \in H^\infty_{M_{m\times n}}$ is a rational function in the sense of Definition 4.16, then each entry ϕ_{ij} is rational. To see this suppose a matrix-valued function Φ satisfies the condition (59). Put $A := \theta\Phi^*$. Then $A \in H^\infty_{M_{n\times m}}$ and $\Phi = \theta A^*$. Thus ϕ_{ij} can be written as $\phi_{ij} = \theta\overline{a_{ij}}$ ($a_{ij} \in H^\infty$). Via Kronecker's Lemma [Ni1, p.183], we can see that

$$\phi_{ij} \text{ is rational} \Longleftrightarrow \phi_{ij} = \theta\overline{a_{ij}} \text{ with a finite Blaschke product } \theta,$$

which says that each ϕ_{ij} is rational.

In [CHL5, Corollary 3.7], Potapov's matrix-valued factorization theorem was extended to operator-valued functions.

Theorem 4.17. A two-sided inner function $\Phi \in H^\infty(\mathcal{B}(E))$ is rational if and only if it can be represented as a finite Blaschke-Potapov product.

We now present a key difference between matrix-valued functions and operator-valued functions.

If Φ and Ψ are not left coprime, then there exists a common left inner divisor Δ of both Φ and Ψ such that Δ is not a unitary operator. However, this is not the case for right coprime-ness. As we know, right coprime-ness is defined via left coprime-ness. As a result, there exist examples of two functions with no common right inner divisor and which are not right coprime. Example 4.18 below shows such an instance. To see this, we first recall that if $\theta \in H^\infty$ is an inner function, then θ can be written as $\theta = cbs$, where c is a constant of modulus 1, b is the Blaschke product formed from the zeros of θ in $\mathbb{D}$ and s is a singular inner function given by

$$s(\zeta) = \exp\left(-\int_{-\pi}^{\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta} d\mu(t)\right) \quad (\zeta \in \mathbb{D}), \tag{61}$$

where μ is a finite positive Borel measure on $\mathbb{T}$ which is singular with respect to Lebesgue measure. If s is the singular inner function given by (61) then it is well known that

$$\lim_{r\to 1_-} s(rz) = 0 \quad \text{for } \mu \text{ almost all } z \in \mathbb{T}. \tag{62}$$

We then have:

Example 4.18. Let f and g be defined as in Example 3.3. Put

$$\Delta := \begin{bmatrix} f \\ g \end{bmatrix} \quad (f, g \in H^\infty).$$

Then Δ is an inner function. Note that

$$\Delta(z) \neq 0 \quad \text{for all } z \in \mathbb{C}. \tag{63}$$

Put $\Phi := \widetilde{\Delta}$. Clearly, Δ and Φ are not right coprime. We claim that Δ and Φ do not have a common right inner divisor. Assume to the contrary that Φ has a right inner divisor Θ that is not a unitary matrix, i.e.,

$$\Phi = \Psi\Theta \quad (\Theta \text{ is inner and not a unitary matrix}).$$

Then $\Psi \in H^\infty_{M_{1\times 2}}$ and $\Theta \in H^\infty_{M_2}$, and hence Θ is two-sided inner. Thus it follows that Θ^* is of bounded type. Thus we can write

$$\Theta^* = \begin{bmatrix} \overline{\theta_1}b_1 & \overline{\theta_2}b_2 \\ \overline{\theta_3}b_3 & \overline{\theta_4}b_4 \end{bmatrix}, \quad \text{i.e.,} \quad \Theta = \begin{bmatrix} \theta_1\overline{b_1} & \theta_3\overline{b_3} \\ \theta_2\overline{b_2} & \theta_4\overline{b_4} \end{bmatrix},$$

where θ_i is inner and $b_i \in H^\infty$. Let θ be the least common multiple of $\{\theta_i : 1 \le i \le 4\}$. Then we can write

$$\Theta = \theta \begin{bmatrix} \overline{a_1} & \overline{a_3} \\ \overline{a_2} & \overline{a_4} \end{bmatrix} = \theta A^* \quad (\text{where } A \in H^\infty_{M_2}).$$

Thus $\Delta = \widetilde{\Theta}\widetilde{\Psi} = \widetilde{\theta}\breve{A}\widetilde{\Psi}$. If θ has a zero in $\mathbb{D}$ then so is Δ, which contradicts (63). If instead θ is a singular inner function of the form

$$\theta(\zeta) = \exp\left(-\int_{-\pi}^{\pi} \frac{e^{it}+\zeta}{e^{it}-\zeta} d\mu(t)\right),$$

then by (62), $\lim_{r\to 1_-} \Delta(rz) = 0$ for μ almost all $z \in \mathbb{T}$, which is a contradiction. □

4.5 Hyponormality and subnormality of Toeplitz operators

In the literature, many authors have considered the special cases of the following (scalar-valued or operator-valued) interpolation problem (cf. [Co1], [CHL2], [CHL3], [FF], [Ga], [Gu], [GHR], [HKL1], [HKL2], [HL1], [HL2], [NT], [Zh]).

Problem 4.19. For $\Phi \in L^\infty(\mathcal{B}(E))$, when does there exist a function $K \in H^\infty(\mathcal{B}(E))$ with $||K||_\infty \le 1$ satisfying

$$\Phi - K\Phi^* \in H^\infty(\mathcal{B}(E))\,? \tag{64}$$

For notational convenience, we write, for $\Phi \in L^\infty(\mathcal{B}(E))$,

$$\mathcal{C}(\Phi) := \Big\{K \in H^\infty(\mathcal{B}(E)) : \ \Phi - K\Phi^* \in H^\infty(\mathcal{B}(E))\Big\}.$$

We then have:

Theorem 4.20. ([CHL4, Theorem C.19]) Let $\Phi \equiv \breve{\Phi}_- + \Phi_+ \in L^\infty(\mathcal{B}(E))$. If $\mathcal{C}(\Phi)$ is nonempty then

$$\ker H^*_{\breve{\Phi}_+} \subseteq \ker H^*_{\Phi^*_-}\,. \tag{65}$$

In particular, $\text{nc}\{\Phi_+\} \le \text{nc}\{\widetilde{\Phi_-}\}$.

If Φ is a matrix-valued rational function then using Kronecker's Lemma we can show that the condition (65) is sufficient for $\mathcal{C}(\Phi) \ne \emptyset$ ([CHL2, Proposition 3.9]). Moreover, in this case, the solution $K \in \mathcal{C}(\Phi)$ is given by a polynomial via the classical Hermite-Fejér interpolation problem. However we were unable to determine whether the condition (65) is sufficient for the existence of a solution $K \in \mathcal{C}(\Phi)$ when $\Phi \in L^\infty_{M_n}$ is such that Φ and Φ^* are of bounded type.

Problem 4.21. Let $\Phi \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. If $\ker H_{\Phi^*_+} \subseteq \ker H_{\Phi^*_-}$, does there exist a solution $K \in H^\infty_{M_n}$ satisfying $\Phi - K\Phi^* \in H^\infty_{M_n}$?

For $\Phi \in L^\infty(\mathcal{B}(E))$, write

$$\mathcal{E}(\Phi) := \Big\{K \in H^\infty(\mathcal{B}(E)) : \ \Phi - K\Phi^* \in H^\infty(\mathcal{B}(E)) \ \text{ and } \ ||K||_\infty \le 1\Big\},$$

i.e., $\mathcal{E}(\Phi) = \{K \in \mathcal{C}(\Phi) : \ ||K||_\infty \le 1\}$. If $\dim E = 1$ and $\Phi \equiv \varphi$ is a scalar-valued function then an elegant theorem of C. Cowen (cf. [Co1], [NT], [CuL]) says that $\mathcal{E}(\varphi)$ is nonempty if and only if T_φ is hyponormal, i.e., the self-commutator $[T^*_\varphi, T_\varphi]$ is positive semi-definite. Cowen's Theorem is to recast the operator-theoretic problem of hyponormality into the problem of finding a solution of an interpolation problem. In [GHR], it was shown that the Cowen's theorem still holds for a Toeplitz operator T_Φ with a matrix-valued symbol $\Phi \in L^\infty_{M_n}$.

As we saw in Section 3.3, Halmos' Problem 5 has been partially answered in the affirmative by many authors. Despite considerable efforts, to date researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. For cases of matrix-valued symbols, the subnormality of Toeplitz operators was studied in [CHKL],[CHL1], [CHL2], [CHL3]. Also Theorem 3.15 shows that if the matrix-valued symbol Φ satisfies a general condition and T_Φ is subnormal then it is either normal or analytic. Also in [CHKL], it was conjectured that every subnormal Toeplitz operator with matrix-valued rational symbol is unitarily equivalent to a direct sum of a normal operator and a Toeplitz operator with analytic symbol. In fact, if an $n \times n$ matrix-valued function Φ is analytic then the normal extension of T_Φ is the multiplication operator M_Φ, so clearly T_Φ is subnormal. However, this is not the case for the operator-valued symbols. We have such an example (see Example 4.23 below). On the other hand, if Φ is matrix-valued and T_Φ is subnormal (even hyponormal), then Φ should be normal, i.e., $\Phi^*\Phi = \Phi\Phi^*$ a.e. on $\mathbb{T}$ (cf. [GHR]). However this may also fail for operator-valued symbols.

Example 4.22. Let $S := T_z$ on H^2 and $\Phi(z) = Sz^n \in H^\infty(\mathcal{B}(H^2))$ $(n \geq 0)$. Then

$$T_\Phi^* T_\Phi = T_{S^*S} = I_{H^2(\mathcal{B}(H^2))},$$

so that T_Φ is quasinormal and hence subnormal. However,

$$\Phi(z)\Phi^*(z) = SS^* \neq S^*S = \Phi^*(z)\Phi(z) \quad \text{for all } z \in \mathbb{T},$$

which implies that Φ is not normal. Here we don't need to expect that the multiplication operator $M_\Phi : L^2(\mathcal{B}(H^2)) \to L^2(\mathcal{B}(H^2))$ is a normal extension of T_Φ. Indeed, it is easy to show that M_Φ is not normal, and hence M_Φ can never be a normal extension of T_Φ. What is a normal extension of T_Φ ? Let $B := M_z$ on L^2 and $\Psi(z) := Bz^n \in H^\infty(\mathcal{B}(L^2))$. Then a straightforward calculation shows that the multiplication operator $M_\Psi : L^2(\mathcal{B}(L^2)) \to L^2(\mathcal{B}(L^2)))$ is a normal extension of T_Φ.

The following simple example shows that analytic Toeplitz operators with operator-valued symbols need not be even hyponormal.

Example 4.23. Let $\Phi(z) = S^* \in H^\infty(\mathcal{B}(H^2))$ and e_0 be the constant function $\mathbf{1} \in H^2(\mathbb{T})$. If $f(z) = e_0 z$, then

$$\langle (T_{\Phi^*}T_\Phi - T_\Phi T_{\Phi^*})f,\ f\rangle = \langle -e_0 z,\ e_0 z\rangle = -1 < 0,$$

which implies that T_Φ is not hyponormal and hence not subnormal even though Φ is analytic.

We would like to pose:

Question 4.24. Which analytic Toeplitz operators with operator-valued symbols are subnormal ?

For a sufficient condition, one may be tempted to conjecture that if $\Phi \in H^\infty(\mathcal{B}(H^2))$ and if $\Phi(z)$ is subnormal for almost all $z \in \mathbb{T}$, then T_Φ is subnormal. We have not been able to decide whether this is true.

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