CIRCLE COMPANIONS OF HARDY SPACES OF THE UNIT DISK

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Abstract. This paper gives a complete answer to the following problem: Find the circle companion of the Hardy space of the unit disk with values in the space of all bounded linear operators between two separable Hilbert spaces. Classically, the problem asks whether for each function h on the unit *disk*, there exists a "boundary function" bh on the unit *circle* such that the mapping $bh \mapsto h$ is an isometric isomorphism between Hardy spaces of the unit circle and the unit disk with values in some Banach space. For the case of bounded linear operator-valued functions, we construct a Hardy space of the unit circle such that its elements are SOT measurable, and their norms are integrable: indeed, this new space is isometrically isomorphic to the Hardy space of the unit disk via a "strong Poisson integral."

1. Introduction

We solve an old and outstanding problem in the theory of Hardy spaces. For $1 \le p \le \infty$ and X a Banach space, consider the Hardy space $H^p(\mathbb{D}, X)$ of X–valued functions defined on the unit disk $\mathbb{D}$. For each $h \in H^p(\mathbb{D}, X)$, we try to associate a function bh, which captures the boundary values of h. Our goal is to identify a Banach space $\mathcal{C}$ of X–valued functions defined on the unit circle $\mathbb{T}$ which represent, in a natural and canonical way, the boundary values of functions in $H^p(\mathbb{D}, X)$. When the mapping $h \mapsto bh$ is an isometric isomorphism from $H^p(\mathbb{D}, X)$ onto $\mathcal{C}$, we say that $\mathcal{C}$ is the "circle companion" of $H^p(\mathbb{D}, X)$.

In this paper, we find the circle companion of the Hardy space of the unit disk with values in $\mathcal{B}(D, E)$, the space of all bounded linear operators between two separable Hilbert spaces D and E. That is, we focus on the cases where the above-mentioned Banach space X is $\mathcal{B}(D, E)$.

A study on the boundary values of functions in Banach-space-valued Hardy spaces $H^p(\mathbb{D}, X)$ of the unit disk was initiated in 1976 by A.V. Bukhvalov [Bu]. Since then, many researchers have studied the spaces of boundary values of functions in $H^p(\mathbb{D}, X)$ (see the bibliographical references at the end of this paper). In particular, in 1982 A.V. Bukhvalov and A.A. Danilevich [BD] showed that if a Banach space X has the analytic Radon-Nikodým property (ARNP) (or equivalently, every function in $H^1(\mathbb{D}, X)$ has radial limits a.e. on $\mathbb{T}$; cf. [Bl], [BD], [DE], [Do], [Ed]), then the space of boundary values of

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functions in $H^p(\mathbb{D}, X)$ is $H^p(\mathbb{T}, X)$; more precisely, the mapping $h \mapsto bh$ is an isometric isomorphism from $H^p(\mathbb{D}, X)$ onto $H^p(\mathbb{T}, X)$ and moreover, $P[bh] = h$, where $P[\cdot]$ denotes the Poisson integral, or equivalently, the mapping $f \mapsto P[f]$ is an isometric isomorphism from $H^p(\mathbb{T}, X)$ *onto* $H^p(\mathbb{D}, X)$. However, this is no longer true for spaces of operator-valued functions. Indeed, if $X = \mathcal{B}(D, E)$, then X need not satisfy the ARNP in general, so that we cannot guarantee that the mapping $f \mapsto P[f]$ is an isometric isomorphism from $H^p(\mathbb{T}, X)$ onto $H^p(\mathbb{D}, X)$. In fact, for each $1 \le p \le \infty$, there exists a function $h \in H^p(\mathbb{D}, \mathcal{B}(\ell^2))$ such that $h \ne P[f]$ for any $f \in H^p(\mathbb{T}, \mathcal{B}(\ell^2))$ (see Example 2.2). Thus, the following problem remained unsolved until now:

$$\text{Find the circle companion of } H^p(\mathbb{D}, \mathcal{B}(D, E)) \text{ for } 1 \le p \le \infty. \tag{1}$$

Although not necessarily explicitly stated as an open problem, the problem (1) appears in Nikolski's book [Ni, p. 62, lines 14-15], where it is mentioned implicitly. In this paper, we solve problem (1). Our solution aims to shed additional insights into the study of boundary values, and how the Poisson transform serves as a bridge between those boundary values and the initial Hardy space function. Towards our solution, we introduce a new space $L^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ $(1 \le p \le \infty)$ defined by the space of all (equivalence classes of) SOT measurable functions $f : \mathbb{T} \to \mathcal{B}(X, Y)$ such that $N(f) \in L^p(\mathbb{T})$ (where $N(f)(z) := ||f(z)||_{\mathcal{B}(X,Y)}$); we identify f and g when $f(z) = g(z)$ for almost all $z \in \mathbb{T}$. In this case, let

$$||f||_{L^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))} := ||N(f)||_{L^p(\mathbb{T})}.$$

Also for $1 \le p \le \infty$, let $H^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ be defined by the space of functions in $L^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ such that $f(\cdot)x \in H^p(\mathbb{T}, Y)$ for every $x \in X$. On the other hand, we define the "strong Poisson integral" $P_s[f]$ of f in $H^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ by

$$P_s[f](\zeta)x := P[f(\cdot)x](\zeta) \quad (x \in X, \ \zeta \in \mathbb{D}).$$

The aim of this paper is to prove that for $1 \le p \le \infty$, the mapping $f \mapsto P_s[f]$ is an isometric isomorphism from $H^p_{sot}(\mathbb{T}, \mathcal{B}(D, E))$ onto $H^p(\mathbb{D}, \mathcal{B}(D, E))$. In [Ni, p. 53, Theorem 3.11.10] it is shown that the mapping $f \mapsto P_s[f]$ provides an isometric isomorphism from $H^\infty_{WOT}(\mathbb{T}, \mathcal{B}(D, E))$ onto $H^\infty(\mathbb{D}, \mathcal{B}(D, E))$ when D and E are separable Hilbert spaces - in fact, we can show that $H^\infty_{WOT}(\mathbb{T}, \mathcal{B}(D, E)) = H^\infty_{sot}(\mathbb{T}, \mathcal{B}(D, E))$ in our language. This provides a sound rationale for denoting this new space as $H^p_{sot}(\mathbb{T}, \mathcal{B}(D, E))$, in a manner fully consistent with the well-known result. In fact, we can get a more general version of the Banach space setting. The following is the main result of this paper.

Theorem 1.1. Let X be a separable Banach space and Y be a Banach space satisfying the analytic Radon-Nikodým property. Then, for $1 \le p \le \infty$, the mapping $f \mapsto P_s[f]$ is an isometric isomorphism from $H^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ onto $H^p(\mathbb{D}, \mathcal{B}(X, Y))$.

The following corollary is immediate from Theorem 1.1.

Corollary 1.2. Let D and E be separable Hilbert spaces. Then, for $1 \le p \le \infty$, the mapping $f \mapsto P_s[f]$ is an isometric isomorphism from $H^p_{sot}(\mathbb{T}, \mathcal{B}(D, E))$ onto $H^p(\mathbb{D}, \mathcal{B}(D, E))$. As a result, $H^p_{sot}(\mathbb{T}, \mathcal{B}(D, E))$ is the circle companion of $H^p(\mathbb{D}, \mathcal{B}(D, E))$.

In Section 2, we give a few essential facts that will be needed to prove Theorem 1.1. Section 3 is devoted to a proof of Theorem 1.1. In the Appendix, we consider relevant results for strong H^p-spaces.

2. Preliminaries

We review here the preliminary background needed to prove the main theorem, using [HNVW] and [Ni] as general references. Let m be the normalized Lebesgue measure on $\mathbb{T}$. For a Banach space X, a function $f : \mathbb{T} \to X$ is said to be essentially separably valued if there exists a Lebesgue measurable set $\mathbb{T}' \subseteq \mathbb{T}$ such that the range $f(\mathbb{T}')$ is separable and $m(\mathbb{T} \setminus \mathbb{T}') = 0$.

We begin with:

Pettis Measurability Theorem ([HNVW]). Let X be a Banach space and X^* denote the dual space of X. For a function $f : \mathbb{T} \to X$, the following are equivalent:

(a) f is strongly measurable (i.e., there exists a sequence of simple functions f_n such that $f(z) = \lim_{n\to\infty} f_n(z)$ for almost all $z \in \mathbb{T}$);
(b) f is essentially separably valued and weakly measurable (i.e., the mapping $z \mapsto \langle f(z), x^*\rangle$ is Lebesgue measurable for every $x^* \in X^*$).

Observation. By the Pettis Measurability Theorem, the almost everywhere limit of a sequence of strongly measurable functions is also strongly measurable.

Given a function $f : \mathbb{T} \to X$, let

$$N(f)(z) := \|f(z)\|_X \,.$$

For $1 \le p \le \infty$, let $L^p(\mathbb{T}, X)$ be the space of all (equivalence classes of) strongly measurable functions $f : \mathbb{T} \to X$ such that $N(f) \in L^p(\mathbb{T})$. Endowed with the norm

$$||f||_{L^p(\mathbb{T},X)} := ||N(f)||_{L^p(\mathbb{T})},$$

the space $L^p(\mathbb{T}, X)$ is a Banach space. For $f \in L^1(\mathbb{T}, X)$, the n-th Fourier coefficient of f, denoted by $\widehat{f}(n)$, is defined by

$$\widehat{f}(n) := \int_{\mathbb{T}} \overline{z}^n f(z)\, dm(z) \quad \text{for each } n \in \mathbb{Z},$$

where the integral is understood in the sense of the Bochner integral. Also, $H^p(\mathbb{T}, X)$ is defined as the space of functions $f \in L^p(\mathbb{T}, X)$ with $\widehat{f}(n) = 0$ for $n < 0$.

Hereafter, let X and Y be Banach spaces and $\mathcal{B}(X, Y)$ denote the space of all bounded linear operators from X to Y, and abbreviate $\mathcal{B}(X, X)$ as $\mathcal{B}(X)$. We write $\mathrm{Hol}(\mathbb{D}, X)$ for the set of all X-valued functions holomorphic in $\mathbb{D}$.

Equivalent conditions of holomorphic functions ([Ni]). If $h : \mathbb{D} \to \mathcal{B}(X, Y)$, then the following are equivalent.

(a) $h \in \mathrm{Hol}(\mathbb{D}, \mathcal{B}(X, Y))$;
(b) $h(\cdot)x \in \mathrm{Hol}(\mathbb{D}, Y)$ for all $x \in X$;
(c) $\langle h(\cdot)x, y^*\rangle \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ for all $x \in X$ and $y^* \in Y^*$.

Let us associate to any function $h : \mathbb{D} \to X$, a family of functions h_r on $\mathbb{T}$, defined by

$$h_r(z) := h(rz) \quad (z \in \mathbb{T},\ 0 \le r < 1).$$

For $1 \le p \le \infty$, let $H^p(\mathbb{D}, X)$ be the space of all functions $h \in \mathrm{Hol}(\mathbb{D}, X)$ satisfying

$$||h||_{H^p(\mathbb{D},X)} := \sup\bigl\{||N(h_r)||_{L^p(\mathbb{T})} : r < 1\bigr\} < \infty.$$

Then $H^p(\mathbb{D}, X)$ is a Banach space (cf. [Do]). If $h \in \mathrm{Hol}\,(\mathbb{D}, X)$, then we may write

$$h(\zeta) = \sum_{n=0}^{\infty} x_n \zeta^n \quad (\zeta \in \mathbb{D},\ x_n \in X).$$

Hence for each $0 \le r < 1$,

$$h_r(z) = \sum_{n=0}^{\infty} x_n r^n z^n \quad (z \in \mathbb{T}),$$

which implies that h_r is essentially separably valued. For each $x^* \in X^*$,

$$\langle h_r(z), x^* \rangle = \sum_{n=0}^{\infty} \bigl\langle x_n r^n,\, x^* \bigr\rangle z^n \quad (z \in \mathbb{T}),$$

which implies that h_r is weakly measurable. Thus, by the Pettis Measurability Theorem, h_r is strongly measurable. Therefore we have that

$$||h||_{H^p(\mathbb{D},X)} = \sup_{0 \le r < 1} ||h_r||_{L^p(\mathbb{T},X)}.$$

For $f \in L^1(\mathbb{T}, X)$, let $P[f]$ denote the Poisson integral of f defined by

$$P[f](\zeta) := \int_{\mathbb{T}} P_\zeta(z) f(z) dm(z) \quad (\zeta \in \mathbb{D}), \tag{2}$$

where $P_\zeta(z)$ is the Poisson kernel.

The following are basic properties of Poisson integrals.

Lemma 2.1. [Ni, Lemma 3.11.6.] If $f \in L^p(\mathbb{T}, X)$ $(1 \le p \le \infty)$, then

(a) $||(P[f])_r||_{L^p(\mathbb{T},X)} \le ||f||_{L^p(\mathbb{T},X)}$ for all $0 \le r < 1$;
(b) If $p < \infty$, then $\lim_{r \to 1} ||(P[f])_r - f||_{L^p(\mathbb{T},X)} = 0$;
(c) $\lim_{r \to 1} ||(P[f])_r(z) - f(z)||_X = 0$ for almost all $z \in \mathbb{T}$.

On the other hand, the function $P : H^p(\mathbb{T}, X) \to H^p(\mathbb{D}, X)$ given by (2), is an isometry for all $1 \le p \le \infty$ (cf. [Bl]). As we noticed in the introduction, if X has the ARNP and $1 \le p \le \infty$, then the function $P : H^p(\mathbb{T}, X) \to H^p(\mathbb{D}, X)$ given by (2) is an isometric isomorphism (cf. [BD]). However, the function $P : H^p(\mathbb{T}, \mathcal{B}(D, E)) \to H^p(\mathbb{D}, \mathcal{B}(D, E))$ given by (2) is not onto in general, as we see in the following example.

Example 2.2. Let $h : \mathbb{D} \to \mathcal{B}(\ell^2)$ be defined by $(h(\zeta)x)(n) := \zeta^n x(n)$ for each $x \in \ell^2$. Then $h \in \operatorname{Hol}(\mathbb{D}, \mathcal{B}(\ell^2))$ and $||h||_{H^\infty(\mathbb{D},\mathcal{B}(\ell^2))} = 1$, so that $h \in H^p(\mathbb{D}, \mathcal{B}(\ell^2))$ for all $1 \le p \le \infty$. Suppose that there exists $p \in [1, \infty]$ such that $P : H^p(\mathbb{T}, \mathcal{B}(\ell^2)) \to H^p(\mathbb{D}, \mathcal{B}(\ell^2))$ is onto. Then there exists a function $f \in H^p(\mathbb{T}, \mathcal{B}(\ell^2))$ such that $P[f] = h$. For each $z \in \mathbb{T}$, define a "strong boundary function" $bh : \mathbb{T} \to \mathcal{B}(\ell^2)$ by

$$(bh)(z)x := \lim_{r\to 1} h_r(z)x = (z^n x(n)) \quad (x \equiv (x(n)) \in \ell^2). \tag{3}$$

Then it follows from Lemma 2.1(c) that for all $x \in \ell^2$,

$$(bh)(z)x = \lim_{r\to 1} (P[f])_r(z)x = f(z)x$$

for almost all $z \in \mathbb{T}$, which implies $f = bh$. Let $z_1 \neq z_2$ in $\mathbb{T}$. For $k = 1, 2$, write $z_k = e^{i\theta_k}$ $(0 \le \theta_k < 2\pi)$. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{\pi}{2} < n_0|\theta_2 - \theta_1| \le \pi \pmod{2\pi}$. Let $\{e_n : n = 1, 2, \cdots\}$ be the canonical orthonormal basis for ℓ^2. Then it follows from (3) that

$$\left\|\big(f(z_1) - f(z_2)\big)e_{n_0}\right\|_{\ell^2} = |z_1^{n_0} - z_2^{n_0}| = |1 - (z_2\overline{z_1})^{n_0}| > \sqrt{2},$$

which implies that f is not essentially separably valued. Thus, by the Pettis Measurability Theorem, f is not strongly measurable, a contradiction. Therefore, $P : H^p(\mathbb{T}, \mathcal{B}(\ell^2)) \to H^p(\mathbb{D}, \mathcal{B}(\ell^2))$ is not onto for any $p \in [1, \infty]$. □

3. Proof of the main result

A function $f : \mathbb{T} \to \mathcal{B}(X, Y)$ is called *SOT measurable* if the mapping $z \mapsto f(z)x$ is strongly measurable for every $x \in X$.

We introduce a new normed space.

Definition 3.1. For $1 \le p \le \infty$, define $L^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ by the space of all (equivalence classes of) SOT measurable functions $f : \mathbb{T} \to \mathcal{B}(X, Y)$ such that $N(f) \in L^p(\mathbb{T})$; we identify f and g when $f(z) = g(z)$ for almost all $z \in \mathbb{T}$. In this case, define

$$||f||_{L^p_{sot}(\mathbb{T},\mathcal{B}(X,Y))} := ||N(f)||_{L^p(\mathbb{T})}.$$

We can easily check that $L^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ is a normed space and $L^q_{sot}(\mathbb{T}, \mathcal{B}(X, Y)) \subseteq L^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ if $1 \le p \le q \le \infty$. Further, the space $L^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ is a Banach space.

Lemma 3.2. For $1 \le p \le \infty$, $L^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ is a Banach space.

Proof. The proof follows from a slight variation of the standard proof (cf. [Ru]) for the completeness of scalar-valued L^p-spaces, except for SOT-measurability. To be completely rigorous, we sketch a proof of the validity of SOT-measurability.

Suppose (f_n) is a Cauchy sequence in $L^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$. Then we can choose a subsequence (f_{n_i}) such that

$$||f_{n_{i+1}} - f_{n_i}||_{L^p_{sot}(\mathbb{T},\mathcal{B}(X,Y))} < 2^{-i} \text{ for all } i = 1, 2, 3, \cdots .$$

If we put $g := \sum_{i=1}^{\infty}(f_{n_{i+1}} - f_{n_i})$, then it is easy to show that $g(z) \in \mathcal{B}(X,Y)$ for almost all $z \in \mathbb{T}$ and in turn,

$$f(z) := f_{n_1}(z) + g(z)$$

converges for almost all $z \in \mathbb{T}$. Therefore for each $x \in X$, $f(z)x = \lim_{i\to\infty} f_{n_i}(z)x$ for almost all $z \in \mathbb{T}$. Since f_{n_i} is SOT measurable, the mapping $z \mapsto f_{n_i}(z)x$ is strongly measurable, so that the mapping $z \mapsto f(z)x$ is also strongly measurable. Therefore f is SOT measurable. □

Remark 3.3. In the definition of $L^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$, we implicitly suppose $N(f)$ is (Lebesgue) measurable. In fact, we don't guarantee that if f is SOT measurable then $N(f)$ is measurable in general. To see this, let $\ell^2(\mathbb{T})$ be the set of all functions $x : \mathbb{T} \to \mathbb{C}$ such that $x(z) = 0$ for all but a countable number of z's and $\sum_{z\in\mathbb{T}} |x(z)|^2 < \infty$. For x and y in $\ell^2(\mathbb{T})$ define

$$\langle x, y\rangle := \sum_{z\in\mathbb{T}} x(z)\overline{y(z)}.$$

Then $\ell^2(\mathbb{T})$ is a (non-separable) Hilbert space. Let F be a nonmeasurable set in $\mathbb{T}$. For $z \in \mathbb{T}$, let $f : \mathbb{T} \to \mathcal{B}(\ell^2(\mathbb{T}))$ be defined by

$$(f(z)x)(s) := \begin{cases} x(z), & \text{if } s = z \in F \\ 0, & \text{if } z \notin F \text{ or } s \neq z. \end{cases}$$

Then for each $x \in \ell^2(\mathbb{T})$, we have that $f(z)x = 0$ for almost all $z \in \mathbb{T}$, and hence f is SOT measurable.

We now claim that

$$N(f) = \mathbf{1}_F \quad (\mathbf{1}_F \text{ denotes the indicator function of the set } F), \tag{4}$$

which implies that $N(f)$ is not measurable because F is a nonmeasurable set. To see this, for each $z \in \mathbb{T}$, let

$$x_z(s) := \begin{cases} 1, & \text{if } s = z \\ 0, & \text{if } s \neq z. \end{cases}$$

Then, $x_z \in \ell^2(\mathbb{T})$ and $||x_z|| = 1$. If $z \in F$, then $(f(z)x_z)(s) = x_z(s)$, so that $||f(z)x_z||_{\ell^2(\mathbb{T})} = 1$. But since $f(z)$ is a contraction, it follows that $N(f)(z) = 1$ for all $z \in F$. If instead $z \notin F$, then $f(z) = 0$, so that $N(f)(z) = 0$. This proves (4). □

We note that in the above remark, $\ell^2(\mathbb{T})$ is not separable. However, we can show that the SOT-measurability of f implies the measurability of $N(f)$ if X is a separable Banach space.

Lemma 3.4. Let X be a separable Banach space. If $f : \mathbb{T} \to \mathcal{B}(X,Y)$ is SOT measurable then $N(f)$ is measurable.

Proof. Suppose that $f : \mathbb{T} \to \mathcal{B}(X,Y)$ is SOT measurable. Then for all $x \in X$, the mapping $z \mapsto f(z)x$ is strongly measurable, and hence the mapping $z \mapsto ||f(z)x||$ is measurable. Thus the mapping $z \mapsto \frac{||f(z)x||}{||x||}$ is measurable for all nonzero $x \in X$. Choose a countable dense subset X_0 of X. Then we can easily see that

$$N(f)(z) = \sup\left\{ \frac{||f(z)x||}{||x||} : 0 \neq x \in X_0 \right\}.$$

Thus the mapping $z \mapsto N(f)(z)$ is measurable. $\square$

We now introduce a space which fits our purpose:

Definition 3.5. For $1 \leq p \leq \infty$, let $H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$ be the space of all (equivalence classes of) functions $f \in L^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$ such that $f(\cdot)x \in H^p(\mathbb{T}, Y)$ for every $x \in X$.

Observe that for $1 \leq p \leq \infty$, $H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$ is a closed subspace of $L^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$, so that by Lemma 3.2, $H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$ is a Banach space.

Example 3.6. In general, $H^p(\mathbb{T}, \mathcal{B}(X,Y)) \neq H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$ for all $1 \leq p \leq \infty$. To see this, let $H^2 \equiv H^2(\mathbb{T})$ and define the function $f : \mathbb{T} \to \mathcal{B}(H^2)$ by

$$f(z)x(s) := x(zs).$$

Since the set of all polynomials on $\mathbb{T}$ is dense in H^2, it follows that the mapping $z \mapsto f(z)x$ is (uniformly) continuous for each $x \in H^2$. Thus, by the Pettis Measurability Theorem, f is SOT measurable. Since $N(f)(z) = 1$ for all $z \in \mathbb{T}$, it follows that $f \in L^\infty_{sot}(\mathbb{T}, \mathcal{B}(H^2))$ with $||f||_{L^\infty_{sot}(\mathbb{T}, \mathcal{B}(H^2))} = 1$. Moreover for each $x \in H^2$ and $n \in \mathbb{Z}$,

$$\big(\widehat{f}(n)x\big)(s) = \int_{\mathbb{T}} \overline{z}^n f(z)x(s)dm(z) = \big\langle x(zs),\ z^n \big\rangle_{H^2} = \widehat{x}(n)s^n,$$

which implies that $f \in H^\infty_{sot}(\mathbb{T}, \mathcal{B}(H^2)) \subseteq H^p_{sot}(\mathbb{T}, \mathcal{B}(H^2))$ for all $1 \leq p \leq \infty$. However we have that $f \notin H^p(\mathbb{T}, \mathcal{B}(H^2))$. To see this we use the same argument as Example 2.2. Let $z_1 \neq z_2$ in $\mathbb{T}$. Write $z_k = e^{i\theta_k}$ $(0 \leq \theta_k < 2\pi)$. Then there exists $n_0 \in \mathbb{N}$ such that $\frac{\pi}{2} < n_0|\theta_2 - \theta_1| \leq \pi \pmod{2\pi}$. We thus have

$$\begin{aligned} ||(f(z_1) - f(z_2))s^{n_0}||^2_{H^2} &= \int_{\mathbb{T}} |(z_1 s)^{n_0} - (z_2 s)^{n_0}|^2 dm(s) \\ &= \int_{\mathbb{T}} |1 - (z_2\overline{z_1})^{n_0}|^2 dm(s) > 2, \end{aligned}$$

which implies that f is not essentially separably valued. Thus, by the Pettis Measurability Theorem, f is not strongly measurable, so that, $f \notin H^p(\mathbb{T}, \mathcal{B}(H^2))$. $\square$

Definition 3.7. For $f \in H^1_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$ and $x \in X$, let $P_s[f](\cdot)x : \mathbb{D} \to Y$ be defined by

$$P_s[f](\zeta)x := P[f(\cdot)x](\zeta) \quad (\zeta \in \mathbb{D}),$$

where $P[\cdot]$ denotes the Poisson integral. In this case, $P_s[f]$ is called the *strong Poisson integral* of f.

Lemma 3.8. For $1 \leq p \leq \infty$, the mapping $f \mapsto P_s[f]$ is a contraction from $H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$ to $H^p(\mathbb{D}, \mathcal{B}(X,Y))$.

Proof. Let $f \in H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$ $(1 \le p \le \infty)$ and $\zeta = re^{i\theta} \in \mathbb{D}$. Clearly, $P_s[f](\zeta)$ is linear on X. For each $x \in X$,

$$\begin{aligned} ||P_s[f](\zeta)x|| &= \Big|\Big| \int_{\mathbb{T}} P_\zeta(z) f(z) x dm(z) \Big|\Big| \\ &\le \frac{1+r}{1-r} \cdot ||f||_{L^1_{sot}(\mathbb{T}, \mathcal{B}(X,Y))} ||x||, \end{aligned}$$

which implies that $P_s[f](\zeta) \in \mathcal{B}(X,Y)$. Since $P_s[f](\cdot)x \in H^1(\mathbb{D}, Y)$ for every $x \in X$, it follows $P_s[f] \in \mathrm{Hol}\,(\mathbb{D}, \mathcal{B}(X,Y))$. We now claim that

$$P_s[f] \in H^p(\mathbb{D},\, \mathcal{B}(X,Y)) \text{ and } ||P_s[f]||_{H^p(\mathbb{D},\, \mathcal{B}(X,Y))} \le ||f||_{L^p_{sot}(\mathbb{T},\, \mathcal{B}(X,Y))}.$$

For each $\zeta \in \mathbb{D}$ and a unit vector $x \in X$,

$$||P_s[f](\zeta)x|| \le \int_{\mathbb{T}} P_\zeta(z) ||f(z)|| dm(z) = P[N(f)](\zeta).$$

Thus $||P_s[f](\zeta)|| \le P[N(f)](\zeta)$ for all $\zeta \in \mathbb{D}$ and hence, by Lemma 2.1(a), we have

$$||(P_s[f])_r||_{L^p(\mathbb{T},\, \mathcal{B}(X,Y))} \le \big|\big|(P[N(f)])_r\big|\big|_{L^p(\mathbb{T})} \le ||f||_{L^p_{sot}(\mathbb{T},\, \mathcal{B}(X,Y))},$$

which implies that $P_s[f] \in H^p(\mathbb{D}, \mathcal{B}(X,Y))$ and

$$||P_s[f]||_{H^p(\mathbb{D},\, \mathcal{B}(X,Y))} \le ||f||_{L^p_{sot}(\mathbb{T},\, \mathcal{B}(X,Y))}.$$

This completes the proof. □

We are ready to prove our main theorem. Before doing it, we would like to underline a reason why our proof is little intricate. Let $h \in H^1(\mathbb{D}, \mathcal{B}(X,Y))$ and assume that Y has the ARNP. Since $h(\cdot)x \in H^1(\mathbb{D}, Y)$ for each $x \in X$, there exists the following radial strong limit bh a.e. on $\mathbb{T}$: i.e., for each $x \in X$,

$$bh(z)x := \lim_{r \to 1} h_r(z)x \quad (z \in \mathbb{T}).$$

Write

$$E_x := \{z \in \mathbb{T} : bh(z)x \text{ does not exist}\} \quad \text{and} \quad E := \bigcup_{x \in X} E_x.$$

Then $m(E_x) = 0$ for each $x \in X$, but we don't guarantee $m(E) = 0$. Thus the function bh may not be defined almost everywhere on $\mathbb{T}$. Therefore bh is not appropriate for a boundary function of h. The crucial point of our proof is how to construct a "boundary function" defined almost everywhere on $\mathbb{T}$ for a function in $H^p(\mathbb{D}, \mathcal{B}(X,Y))$.

We will now prove Theorem 1.1, which we restate for the reader's convenience:

Theorem 1.1. *Let X be a separable Banach space and Y be a Banach space satisfying the analytic Radon-Nikodým property. Then, for $1 \le p \le \infty$, the mapping $f \mapsto P_s[f]$ is an isometric isomorphism from $H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$ onto $H^p(\mathbb{D}, \mathcal{B}(X,Y))$.*

Proof. Let X be a separable Banach space and Y be a Banach space satisfying the analytic Radon-Nikodým property. Let $h \in H^1(\mathbb{D}, \mathcal{B}(X,Y))$. Our first task is to define a "boundary function" $b_s h$ a.e. on $\mathbb{T}$ for h. To do so, let $X_0 = \{x_n \in X : n = 1, 2, \cdots\}$ be a countable dense subset of X. Then for each $n = 1, 2, \cdots$, there exists a measurable set E_n with $m(E_n) = 0$ such that $bh(z)x_n = \lim_{r\to 1} h_r(z)x_n$ exists for all $z \in \mathbb{T} \setminus E_n$. Then $bh(\cdot)x_n \in H^1(\mathbb{T}, Y)$ for each $n = 1, 2, \cdots$. Put $E_0 := \cup_{n\geq 1} E_n$. Then $m(E_0) = 0$. For $z \in \mathbb{T} \setminus E_0$, let

$$q(z) := \sup \left\{ \frac{||bh(z)x||}{||x||} : 0 \neq x \in X_0 \right\}. \tag{5}$$

Observe that for all $z \in \mathbb{T} \setminus E_0$ and each $x \in X_0$,

$$||bh(z)x|| = \lim_{r\to 1} ||h_r(z)x|| \leq \liminf_{r\to 1} ||h_r(z)|| \cdot ||x||. \tag{6}$$

Let $u(z) := \liminf_{r\to 1} N(h_r)(z)$. Since $h \in H^1(\mathbb{D}, \mathcal{B}(X,Y))$, $N(h_r)$ is in $L^1(\mathbb{T})$ for each $0 \leq r < 1$, so that u is measurable. Also by (5) and (6), we have

$$0 \leq q(z) \leq u(z) \text{ for all } z \in \mathbb{T} \setminus E_0. \tag{7}$$

On the other hand, by Fatou's lemma, we have

$$\int_{\mathbb{T}} u(z)dm(z) \leq \lim\ \mathrm{inf}_{r\to 1} \int_{\mathbb{T}} N(h_r)(z)dm(z) \leq ||h||_{H^1(\mathbb{D}, \mathcal{B}(X,Y))} < \infty,$$

which implies that $u \in L^1(\mathbb{T})$. Thus there exists a subset E_u of $\mathbb{T}$ with $m(E_u) = 0$ such that $u(z) < \infty$ for all $z \in \mathbb{T} \setminus E_u$. Hence, by (7), $q(z) \leq u(z) < \infty$ for all $z \in \mathbb{T} \setminus (E_0 \cup E_u)$. Therefore $bh(z)$ can be extended to a bounded linear operator $b_s h(z)$ on X for almost all $z \in \mathbb{T}$: for each $z \in \mathbb{T} \setminus (E_0 \cup E_u)$ and $x \in X$, define

$$b_s h(z)x := \lim_{n\to\infty} bh(z)x_n, \tag{8}$$

where (x_n) is a sequence in X_0 such that $x_n \to x$. We note that (8) is independent of the particular choice of the dense subset X_0 of X and a sequence (x_n) in X_0 : indeed let Y_0 is another countable dense subset of X and (y_n) is a sequence in Y_0 such that $y_n \to x$. By the same argument above, we see that for almost all $z \in \mathbb{T}$,

$$q'(z) := \sup \left\{ \frac{||bh(z)x||}{||x||} : 0 \neq x \in X_0 \cup Y_0 \right\} < \infty.$$

Thus

$$||bh(z)x_n - bh(z)y_n|| \leq q'(z)||x_n - y_n|| \to 0 \quad \text{as } n \to \infty,$$

which implies that the function $b_s h(z)$ is well-defined on X for almost all $z \in \mathbb{T}$. (We call $b_s h$ the *strong boundary function* of h.)

Now let $1 \leq p \leq \infty$ and suppose $h \in H^p(\mathbb{D}, \mathcal{B}(X,Y))$. Then $b_s h(z) \in \mathcal{B}(X,Y)$ for almost all $z \in \mathbb{T}$ and it is easy to show that $b_s h$ is SOT measurable and hence, by Lemma 3.4, $N(b_s h)$ is measurable because X is separable. We claim that

$$b_s h \in H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y)). \tag{9}$$

To see this, we first observe that, by (7), $N(b_s h)(z) = q(z) \le \liminf_{r\to 1} N(h_r)(z)$ for almost all $z \in \mathbb{T}$. Thus for $1 \le p < \infty$, it follows from Fatou's lemma that

$$\begin{aligned} \int_{\mathbb{T}} N(b_s h)(z)^p dm(z) &\le \text{lim inf}_{r\to 1} \int_{\mathbb{T}} N(h_r)(z)^p dm(z) \\ &\le ||h||^p_{H^p(\mathbb{D},\, \mathcal{B}(X,Y))} < \infty. \end{aligned} \tag{10}$$

Let $x \in X$ be arbitrary and (x_n) be a sequence in X_0 such that $x_n \to x$. Then it follows from (10) that

$$\begin{aligned} ||b_s h(\cdot)x - bh(\cdot)x_n||_{L^p(\mathbb{T},Y)} &= \left(\int_{\mathbb{T}} ||b_s h(z)(x - x_n)||^p dm(z) \right)^{\frac{1}{p}} \\ &\le ||h||_{H^p(\mathbb{D},\, \mathcal{B}(X,Y))} ||x - x_n|| \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

But since $H^p(\mathbb{T}, Y)$ is a closed subspace of $L^p(\mathbb{T}, Y)$ and $bh(\cdot)x_n \in H^p(\mathbb{T}, Y)$, we have $b_s h(\cdot)x \in H^p(\mathbb{T}, Y)$, which together with (10) implies that $b_s h \in H^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ and

$$||b_s h||_{H^p_{sot}(\mathbb{T},\mathcal{B}(X,Y))} \le ||h||_{H^p(\mathbb{D},\, \mathcal{B}(X,Y))}.$$

If instead $p = \infty$, then $h \in H^1(\mathbb{D}, \mathcal{B}(X, Y))$, so that $b_s h \in H^1_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$. Also, it follows from (6) that

$$||b_s h||_{L^\infty_{sot}(\mathbb{T},\, \mathcal{B}(X,Y))} \le ||h||_{H^\infty(\mathbb{D},\, \mathcal{B}(X,Y))}. \tag{11}$$

Thus $b_s h \in H^\infty_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$. This proves (9).

We next claim that

$$P_s[b_s h] = h. \tag{12}$$

Let $x \in X$ be arbitrary. Then for each $\zeta = re^{i\theta} \in \mathbb{D}$,

$$\begin{aligned} ||P_s[b_s h](\zeta)x|| &\le \frac{1+r}{1-r} \int_{\mathbb{T}} ||b_s h(z)x|| dm(z) \\ &\le \frac{1+r}{1-r} \cdot ||b_s h||_{L^1_{sot}(\mathbb{T},\ \mathcal{B}(X,Y))} \cdot ||x||. \end{aligned} \tag{13}$$

Choose a sequence (x_n) in X_0 such that $x_n \to x$. Then for each $\zeta \in \mathbb{D}$,

$$h(\zeta)x = \lim_{n\to\infty} h(\zeta)x_n = \lim_{n\to\infty} P_s[b_s h](\zeta)x_n = P_s[b_s h](\zeta)x,$$

where the last equality follows from (13). This proves (12). Thus the mapping $f \mapsto P_s[f]$ is a surjection from $H^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ to $H^p(\mathbb{D}, \mathcal{B}(X, Y))$. Therefore, by Lemma 3.8, (10) and (11), the mapping $f \mapsto P_s[f]$ is an isometry from $H^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ onto $H^p(\mathbb{D}, \mathcal{B}(X, Y))$. This completes the proof. $\square$

Theorem 1.1 may fail if the separability condition on X is dropped. For $z \in \mathbb{T}$ and $x \in \ell^2(\mathbb{T})$, let $f : \mathbb{T} \to \mathcal{B}(\ell^2(\mathbb{T}))$ be defined by

$$(f(z)x)(s) := \begin{cases} x(z), & \text{if } s = z \\ 0, & \text{if } s \ne z. \end{cases}$$

Then by the argument in Remark 3.3, we have $N(f)=1$, and hence $f\in H^p_{sot}(\mathbb{T},\mathcal{B}(\ell^2(\mathbb{T})))$ with $||f||_{H^p_{sot}(\mathbb{T},\mathcal{B}(\ell^2(\mathbb{T})))}=1$ for all $1\le p\le\infty$. Since $(f(z)x)(s)$ is zero for all $z\neq s$, it follows that for each $x\in\ell^2(\mathbb{T})$, $\zeta\in\mathbb{D}$ and $s\in\mathbb{T}$,

$$\big(P_s[f](\zeta)x\big)(s)=\int_{\mathbb{T}}P_\zeta(z)(f(z)x)(s)dm(z)=0,$$

which implies that $P_s[f]=0$ in $H^p(\mathbb{D},\mathcal{B}(\ell^2(\mathbb{T})))$. Therefore, the mapping $f\mapsto P_s[f]$ is *not* an isometry.

We conclude with consideration on adjoints of functions in $H^p_{sot}(\mathbb{T},\mathcal{B}(X,Y))$.

For a function $f:\mathbb{T}\to\mathcal{B}(X,Y)$, define the "adjoint" $f^*:\mathbb{T}\to\mathcal{B}(Y^*,X^*)$ of f by

$$f^*(z):=f(z)^*\quad(z\in\mathbb{T}).$$

We may ask the following question: for $1\le p\le\infty$, does it follow that

$$f\in H^p_{sot}(\mathbb{T},\mathcal{B}(X,Y))\implies f^*\in H^p_{sot}(\mathbb{T},\mathcal{B}(Y^*,X^*))\,?$$

In the sequel, we give an affirmative answer to this question if X is reflexive. To begin with we review some definitions.

A function $f:\mathbb{T}\to X$ is called *weakly integrable* if $\langle f,x^*\rangle\in L^1(\mathbb{T})$ for every $x^*\in X^*$. If f is weakly integrable then the function $T_f:X^*\to L^1(\mathbb{T})$, defined by $T_fx^*:=\langle f,x^*\rangle$, is a bounded linear operator. A weakly integrable function $f:\mathbb{T}\to X$ is called *Pettis integrable* if the adjoint T_f^* of the operator T_f maps $L^\infty(\mathbb{T})$ into X. It is well known that

$$f\text{ is Bochner integrable}\Longrightarrow f\text{ is Pettis integrable}\Longrightarrow f\text{ is weakly integrable.}$$

Also it is known (cf. [HNVW, Proposition 1.2.36.]) that for a weakly integrable function $f:\mathbb{T}\to X$, the following are equivalent:

(a) f is Pettis integrable;
(b) for each measurable set B in $\mathbb{T}$, there exists an element $x_B\in X$ such that for every $x^*\in X^*$ we have $\langle x_B,x^*\rangle=\int_B\langle f(z),x^*\rangle dm(z)$.

In this case, we shall write

$$x_B=:(p)\text{–}\int_B f(z)dm(z),$$

and call it the *Pettis integral* of f over B.

We then have:

Lemma 3.9. Let X be a reflexive Banach space, and let $f\in L^1_{sot}(\mathbb{T},\mathcal{B}(X,Y))$. Then for each $y^*\in Y^*$ and $\zeta\in\mathbb{D}$, we have

$$P_s[f]^*(\zeta)y^*=(p)\text{–}\int_{\mathbb{T}}P_\zeta(z)f^*(z)y^*dm(z),$$

where $P_s[f]^*(\zeta):=P_s[f](\zeta)^*$.

Proof. Let X be reflexive and $f\in L^1_{sot}(\mathbb{T},\mathcal{B}(X,Y))$. Then for all $x\in X$ and $y^*\in Y^*$, the mapping $z\mapsto\big\langle x,f^*(z)y^*\big\rangle=\big\langle f(z)x,y^*\big\rangle$ is measurable. But since X is reflexive, the mapping $z \mapsto f^*(z)y^*$ is weakly measurable. Thus the mapping $z \mapsto P_\zeta(z)f^*(z)y^*$ is weakly measurable for each $\zeta = re^{i\theta} \in \mathbb{D}$. For each $x \in X$,

$$\begin{aligned}\int_{\mathbb{T}} \big|\langle x, P_\zeta(z)f^*(z)y^*\rangle\big| dm(z) &= \int_{\mathbb{T}} P_\zeta(z)\big|\langle f(z)x, y^*\rangle\big| dm(z)\\ &\le \frac{1+r}{1-r}\cdot ||y^*||\cdot||x||\cdot||f||_{L^1_{sot}(\mathbb{T},\mathcal{B}(X,Y))}\\ &< \infty,\end{aligned}$$

which implies that $P_\zeta(\cdot)f^*(\cdot)y^*$ is weakly integrable and hence Pettis integrable. Thus for all $x \in X$ and $\zeta \in \mathbb{D}$,

$$\begin{aligned}\big\langle P_s[f](\zeta)x, y^*\big\rangle &= \int_{\mathbb{T}} \big\langle x, P_\zeta(z)f^*(z)y^*\big\rangle dm(z)\\ &= \Big\langle x, (p)\text{-}\!\!\int_{\mathbb{T}} P_\zeta(z)f^*(z)y^* dm(z)\Big\rangle,\end{aligned}$$

which gives the result. □

We now have:

Theorem 3.10. Let X be a reflexive Banach space and $1 \le p \le \infty$. If $f \in H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$, then $f^* \in H^p_{sot}(\mathbb{T}, \mathcal{B}(Y^*, X^*))$. Moreover, $P_s[f^*] = P_s[f]^*$.

Proof. Let X be reflexive, $1 \le p \le \infty$, and $f \in H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$. Since $\big\langle x, P_s[f]^*(\zeta)y^*\big\rangle = \big\langle P_s[f](\zeta)x, y^*\big\rangle$ for all $x \in X$ and $y^* \in Y^*$, it follows from Lemma 3.8 that $P_s[f]^* \in \mathrm{Hol}\,(\mathbb{D}, \mathcal{B}(Y^*, X^*))$. For all $y^* \in Y^*$ and $\zeta \in \mathbb{D}$,

$$\begin{aligned}||P_s[f]^*(\zeta)y^*|| &= \sup_{||x||=1}\big|\big\langle x, P_s[f]^*(\zeta)y^*\big\rangle\big|\\ &\le \int_{\mathbb{T}} P_\zeta(z)||f(z)||dm(z)\cdot||y^*||\\ &= P[N(f)](\zeta)\cdot||y^*||,\end{aligned}$$

which implies that $||P_s[f]^*(\zeta)|| \le P[N(f)](\zeta)$. It thus follows from Lemma 2.1(a) that

$$\int_{\mathbb{T}} ||\big(P_s[f]^*\big)_r(z)||dm(z) \le \int_{\mathbb{T}} (P[N(f)])_r(z)dm(z) \le ||f||_{L^1_{sot}(\mathbb{T},\mathcal{B}(X,Y))}.$$

This proves that $P_s[f]^* \in H^1(\mathbb{D}, \mathcal{B}(Y^*, X^*))$. On the other hand, for all $x \in X$ and $y^* \in Y^*$, we have that for almost all $z \in \mathbb{T}$,

$$\begin{aligned}\lim_{r\to1}\big\langle x, P_s[f]^*(rz)y^*\big\rangle &= \lim_{r\to1}\langle P_s[f](rz)x, y^*\rangle\\ &= \lim_{r\to1}\big(P[\langle f(\cdot)x, y^*\rangle]\big)_r(z)\\ &= \langle x, f^*(z)y^*\rangle,\end{aligned}\tag{14}$$

where the last equality follows from the fact that $\langle f(\cdot)x, y^*\rangle \in L^p(\mathbb{T})$. Since X^* has the ARNP and $P_s[f]^*(\cdot)y^* \in H^1(\mathbb{D}, X^*)$, it follows that

$$bP_s[f]^*(z)y^* := \lim_{r\to1} P_s[f]^*(rz)y^*$$

exists for almost all $z \in \mathbb{T}$. Since X is reflexive, by the Hahn-Banach Theorem and (14), $f^*(\cdot)y^* = bP_s[f]^*(\cdot)y^* \in H^1(\mathbb{T}, X^*)$. In particular, f^* is SOT measurable, and hence $f^* \in H^p_{sot}(\mathbb{T}, \mathcal{B}(Y^*, X^*))$. On the other hand, since $f \in H^1_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$, it follows from Lemma 3.9 that for each $y^* \in Y^*$ and $\zeta \in \mathbb{D}$,

$$P_s[f]^*(\zeta)y^* = (p)\text{–}\int_{\mathbb{T}} P_\zeta(z) f^*(z) y^* dm(z) = P_s[f^*](\zeta)y^*,$$

which implies $P_s[f^*] = P_s[f]^*$. This completes the proof. □

Theorem 3.10 may fail if the reflexive condition on X is dropped. To see this, let $f : \mathbb{T} \to \mathcal{B}(\ell^1)$ be defined by

$$(f(z)x)(n) := z^n x(n) \quad (x \equiv (x(n)) \in \ell^1).$$

Then it is not difficult to show that $f \in H^\infty_{sot}(\mathbb{T}, \mathcal{B}(\ell^1))$ and f^* is not SOT measurable (cf. Example 2.2), so that $f^* \notin H^\infty_{sot}(\mathbb{T}, \mathcal{B}(\ell^\infty))$. Note that ℓ^1 is not reflexive.

4. Appendix: Strong H^p-spaces

We devote this section to a general discussion of the circle companions of strong H^p-spaces.

For $1 \le p \le \infty$, let $H^p_s(\mathbb{D}, \mathcal{B}(X, Y))$ be the space of all functions h in $\mathrm{Hol}(\mathbb{D}, \mathcal{B}(X, Y))$ such that $h(\cdot)x \in H^p(\mathbb{D}, Y)$ for every $x \in X$: $H^p_s(\mathbb{D}, \mathcal{B}(X, Y))$ is called a *strong H^p-space* (cf. [Ni]). If $h \in H^p_s(\mathbb{D}, \mathcal{B}(X, Y))$, then we can easily show that the mapping $x \mapsto h(\cdot)x$ is a closed linear transformation from X into $H^p(\mathbb{D}, Y)$, so that by the Closed Graph Theorem, it is bounded. Let

$$||h||_{H^p_s(\mathbb{D}, \mathcal{B}(X,Y))} := \sup\big\{||h(\cdot)x||_{H^p(\mathbb{D},Y)} : x \in X \text{ with } ||x|| \le 1\big\}.$$

Then $H^p_s(\mathbb{D}, \mathcal{B}(X, Y))$ is a normed space and

$$H^p(\mathbb{D}, \mathcal{B}(X, Y)) \subseteq H^p_s(\mathbb{D}, \mathcal{B}(X, Y)) \quad (1 \le p \le \infty). \tag{15}$$

Also we can easily check that $H^\infty_s(\mathbb{D}, \mathcal{B}(X, Y)) = H^\infty(\mathbb{D}, \mathcal{B}(X, Y))$. However, if $1 \le p < \infty$ then the inclusion in (15) may be proper.

Example 4.1. Let $1 \le p < \infty$. For $\zeta \in \mathbb{D}$, define $h(\zeta) : H^p(\mathbb{T}) \to \mathbb{C}$ by

$$h(\zeta)f := P[f](\zeta) \quad (f \in H^p(\mathbb{T})).$$

Then for each $\zeta = re^{i\theta} \in \mathbb{D}$,

$$\begin{aligned} ||h(\zeta)||_{\mathcal{B}(H^p(\mathbb{T}),\mathbb{C})} &= \sup\big\{||P[f](re^{i\theta})|| : ||f||_{H^p(\mathbb{T})} = 1\big\} \\ &\le \frac{1+r}{1-r} \cdot \sup\big\{||f||_{H^1(\mathbb{T})} : ||f||_{H^p(\mathbb{T})} = 1\big\} \\ &\le \frac{1+r}{1-r}, \end{aligned}$$

which implies that $h(\zeta)$ is a bounded linear operator. Thus it is easy to show that $h \in H_s^p(\mathbb{D}, \mathcal{B}(H^p(\mathbb{T}), \mathbb{C}))$ and $||h||_{H_s^p(\mathbb{D}, \mathcal{B}(H^p(\mathbb{T}),\mathbb{C}))} = 1$. However, $h \notin H^p(\mathbb{D}, \mathcal{B}(H^p(\mathbb{T}), \mathbb{C}))$: indeed, for each $z \in \mathbb{T}$, let

$$f_z(s) := e^{\frac{s+z}{s-z}} \quad (s \in \mathbb{T}).$$

Then f_z is inner, so that $||f_z||_{H^p(\mathbb{T})} = 1$. Thus

$$||h_r(z)|| \geq |h_r(z) f_z| = e^{\frac{r+1}{r-1}},$$

so that

$$\sup_{0 \leq r < 1} \left(\int_{\mathbb{T}} ||h_r(z)||^p dm(z) \right)^{\frac{1}{p}} \geq \sup_{0 \leq r < 1} e^{\frac{r+1}{r-1}} = \infty,$$

which implies that $h \notin H^p(\mathbb{D}, \mathcal{B}(H^p(\mathbb{T}), \mathbb{C}))$. □

Let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the set of all linear transformations between normed spaces $\mathcal{X}$ and $\mathcal{Y}$. For a subset F of a Banach space X, let $\mathrm{sp}(F)$ denote the linear span of F. For $1 \leq p \leq \infty$, let $L_s^p(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$ be the space of all (equivalence classes of) functions $f : \mathbb{T} \to \mathcal{L}(\mathrm{sp}(F), Y)$ satisfying

(i) $f(\cdot)x \in L^p(\mathbb{T}, Y)$ for all $x \in \mathrm{sp}(F)$; as usual, we identify f and g when $f(\cdot)x = g(\cdot)x$ in $L^p(\mathbb{T}, Y)$ for all $x \in \mathrm{sp}(F)$;

(ii) $||f||_{L_s^p(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F),Y))} := \sup\big\{||f(\cdot)x||_{L^p(\mathbb{T},Y)} : x \in \mathrm{sp}(F) \text{ with } ||x|| \leq 1\big\} < \infty$.

Then $L_s^p(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$ is a normed space and

$$L_s^q(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y)) \subseteq L_s^p(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y)) \quad \text{if } 1 \leq p \leq q \leq \infty.$$

We now define $H_s^p(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$ as the space of all (equivalence classes of) functions $f \in L_s^p(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$ such that $f(\cdot)x \in H^p(\mathbb{T}, Y)$ for all $x \in \mathrm{sp}(F)$. We note that Definition 3.7 is still well-defined for functions in $H_s^1(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$; i.e., for $f \in H_s^1(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$ and $x \in \mathrm{sp}(F)$,

$$P_s[f](\zeta)x := P[f(\cdot)x](\zeta) \quad (\zeta \in \mathbb{D}).$$

We then have:

Lemma 4.2. Let X, Y be Banach spaces and $F \subseteq X$. Suppose $f \in H_s^1(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$. If (x_n) is a Cauchy sequence in $\mathrm{sp}(F)$, then the sequence $(P_s[f](\cdot)x_n)$ converges uniformly on every compact subset of $\mathbb{D}$.

Proof. Suppose $f \in H_s^1(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$ and K is a compact subset of $\mathbb{D}$. Then $r \equiv \max\big\{|\zeta| : \zeta \in K\big\} < 1$. Let (x_n) be a Cauchy sequence in $\mathrm{sp}(F)$ and $\epsilon > 0$ be arbitrary. For all $\zeta = re^{i\theta} \in K$, there exists $N > 0$ such that if $m > n > N$, then

$$\begin{aligned} ||P_s[f](\zeta)x_n - P_s[f](\zeta)x_m|| &= \left|\left| \int_{\mathbb{T}} P_\zeta(z) f(z)(x_n - x_m) dm(z) \right|\right| \\ &\leq \frac{1+r}{1-r} \cdot \int_{\mathbb{T}} ||f(z)(x_n - x_m)|| dm(z) \\ &\leq \frac{1+r}{1-r} \cdot ||f||_{L_s^1(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F),Y))} ||x_n - x_m|| < \frac{\epsilon}{2}. \end{aligned} \tag{16}$$

Thus $(P_s[f](\cdot)x_n)$ converges pointwise to a function $h : \mathbb{D} \to Y$. Fixing $n > N$ and letting $m \to \infty$, (16) leads to

$$||P_s[f](\zeta)x_n - h(\zeta)|| = \lim_{m\to\infty} ||P_s[f](\zeta)x_n - P_s[f](\zeta)x_m|| < \epsilon \quad \text{for all } \zeta \in K,$$

which implies $(P_s[f](\cdot)x_n)$ converges uniformly on K. □

Now if X is separable Banach space, we may define $P_s[f](\zeta)$ on X for all $\zeta \in \mathbb{D}$ by virtue of Lemma 4.2. This is a reason why we introduce $\mathrm{sp}(F)$. Indeed, let X, Y be Banach spaces and assume that X is separable and F is a dense subset of X. Then by Lemma 4.2, given a function $f \in H_s^1(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$, we may define an extension $\overline{P}_s[f](\zeta)$ of $P_s[f](\zeta)$ to X for each $\zeta \in \mathbb{D}$: in other words, if $x \in X$, then there exists a sequence (x_n) in $\mathrm{sp}(F)$ such that $x_n \to x$, so that by Lemma 4.2, $(P_s[f](\zeta)x_n)$ is a convergent sequence for each $\zeta \in \mathbb{D}$ and hence, we can define, for each $x \in X$,

$$\overline{P}_s[f](\zeta)x := \lim_{n\to\infty} P_s[f](\zeta)x_n \quad (\zeta \in \mathbb{D}). \tag{17}$$

We note that the limit in (17) is independent of the particular choice of (x_n) because if (y_n) is another sequence in $\mathrm{sp}(F)$ such that $y_n \to x$, then by the same argument as in (16) we have, for all $\zeta \in \mathbb{D}$,

$$||P_s[f](\zeta)x_n - P_s[f](\zeta)y_n|| \to 0 \quad \text{as } n \to \infty,$$

which implies that the function $\overline{P}_s[f](\zeta)$ is well-defined on X. For simplicity, and since doing so will not lead to confusion, we will keep denoting by $P_s[f]$ the extension $\overline{P}_s[f]$ defined by (17).

We then have:

Theorem 4.3. Let X, Y be Banach spaces and F be a dense subset of X. Then the mapping $f \mapsto P_s[f]$ is an isometry from $H_s^p(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$ to $H_s^p(\mathbb{D}, \mathcal{B}(X, Y))$ for each $1 \le p \le \infty$.

Proof. Let $f \in H_s^p(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))$ $(1 \le p \le \infty)$ and $\zeta = re^{i\theta} \in \mathbb{D}$. Clearly, $P_s[f](\zeta)$ is linear on X. If $x \in X$, then there exists a sequence (x_n) in $\mathrm{sp}(F)$ such that $x_n \to x$. Thus we have

$$\begin{aligned} ||P_s[f](\zeta)x|| &= \lim_{n\to\infty} \left|\left| \int_{\mathbb{T}} P_\zeta(z) f(z) x_n dm(z) \right|\right| \\ &\le \frac{1+r}{1-r} \cdot ||f||_{L_s^1(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))} ||x||, \end{aligned}$$

which implies that $P_s[f](\zeta) \in \mathcal{B}(X, Y)$. For each $y^* \in Y^*$, it follows from Lemma 4.2 that $\langle P_s[f](\zeta)x_n,\ y^*\rangle$ converges uniformly to $\langle P_s[f](\zeta)x,\ y^*\rangle$ on every compact subset of $\mathbb{D}$. Thus $P_s[f] \in \mathrm{Hol}\,(\mathbb{D}, \mathcal{B}(X, Y))$. Now we claim that

$$P_s[f] \in H_s^p(\mathbb{D}, \mathcal{B}(X, Y)) \text{ and } ||P_s[f]||_{H_s^p(\mathbb{D}, \mathcal{B}(X,Y))} \le ||f||_{L_s^p(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y))}. \tag{18}$$

We split the proof into two cases.

Case 1 ($1 \le p < \infty$): Let x be an arbitrary unit vector in X and (x_n) be a sequence in $\text{sp}(F)$ such that $x_n \to x$. Since $P[f(\cdot)x_n] \in H^p(\mathbb{D}, Y)$, the mapping $z \mapsto ||(P_s[f])_r(z)x_n||^p$ is measurable for all $n \in \mathbb{N}$. Thus it follows from Fatou's lemma and Lemma 2.1(a) that

$$\begin{aligned} ||(P_s[f])_r(\cdot)x||_{L^p(\mathbb{T},Y)} &= \left(\int_{\mathbb{T}} \lim_{n\to\infty} ||(P_s[f])_r(z)x_n||^p dm(z)\right)^{\frac{1}{p}} \\ &\le \liminf_{n\to\infty} ||(P_s[f])_r(\cdot)x_n||_{L^p(\mathbb{T},Y)} \\ &\le \liminf_{n\to\infty} ||f(\cdot)x_n||_{L^p(\mathbb{T},Y)} \\ &\le ||f||_{L^p_s(\mathbb{T},\,\mathcal{L}(\text{sp}(F),Y))}, \end{aligned}$$

which proves (18).

Case 2 ($p = \infty$): Assume to the contrary that

$$||P_s[f]||_{H^\infty_s(\mathbb{D},\,\mathcal{B}(X,Y))} > ||f||_{L^\infty_s(\mathbb{T},\,\mathcal{L}(\text{sp}(F),Y))}.$$

Then there exists a unit vector x_0 in X and $\zeta_0 \in \mathbb{D}$ such that $||P_s[f](\zeta_0)x_0|| > ||f||_{L^\infty_s(\mathbb{T},\,\mathcal{L}(\text{sp}(F),Y))}$. Choose a sequence (x_n) in $\text{sp}(F)$ such that $x_n \to x_0$. Then for sufficiently large N,

$$||P_s[f](\zeta_0)x_N|| > ||f||_{L^\infty_s(\mathbb{T},\,\mathcal{L}(\text{sp}(F),Y))} \ge ||f(\cdot)x_N||_{L^\infty(\mathbb{T},Y)},$$

which is a contradiction by Lemma 2.1(a). This proves (18) with $p = \infty$.

Now for all $x \in \text{sp}(F)$, it follows that $||f(\cdot)x||_{L^p(\mathbb{T},Y)} = ||P_s[f](\cdot)x||_{H^p(\mathbb{D},Y)}$ and hence $||f||_{L^p_s(\mathbb{T},\,\mathcal{L}(\text{sp}(F),Y))} \le ||P_s[f]||_{H^p_s(\mathbb{D},\mathcal{B}(X,Y))}$. Therefore, by (18), the mapping $f \mapsto P_s[f]$ is an isometry from $H^p_s(\mathbb{T}, \mathcal{L}(\text{sp}(F), Y))$ to $H^p_s(\mathbb{D}, \mathcal{B}(X, Y))$. This completes the proof. □

Corollary 4.4. Let X, Y be Banach spaces, and assume that X is separable and Y has the ARNP. For a countable dense subset F of X and $1 \le p \le \infty$, the mapping $f \mapsto P_s[f]$ is an isometric isomorphism from $H^p_s(\mathbb{T}, \mathcal{L}(\text{sp}(F), Y))$ onto $H^p_s(\mathbb{D}, \mathcal{B}(X, Y))$.

Proof. A similar argument to (12) shows that the mapping $f \mapsto P_s[f]$ is a surjection from $H^p_s(\mathbb{T}, \mathcal{L}(\text{sp}(F), Y))$ into $H^p_s(\mathbb{D}, \mathcal{B}(X, Y))$. Thus the result follows at once from Theorem 4.3. □

For $1 \le p \le \infty$, let $L^p_s(\mathbb{T}, \mathcal{B}(X, Y))$ be the space of all (equivalence classes of) SOT measurable functions $f : \mathbb{T} \to \mathcal{B}(X, Y)$ such that $f(\cdot)x \in L^p(\mathbb{T}, Y)$ for all $x \in X$; we identify f and g when $f(\cdot)x = g(\cdot)x$ in $L^p(\mathbb{T}, Y)$ for all $x \in X$. If $f \in L^p_s(\mathbb{T}, \mathcal{B}(X, Y))$, then it follows from the Closed Graph Theorem that

$$||f||_{L^p_s(\mathbb{T},\mathcal{B}(X,Y))} := \sup\big\{||f(\cdot)x||_{L^p(\mathbb{T},Y)} : x \in X \text{ with } ||x|| \le 1\big\} < \infty.$$

Then $L^p_s(\mathbb{T}, \mathcal{B}(X, Y))$ is a normed space (cf. [CHL1], [HNVW], [Pe]). In general, the space $L^p_s(\mathbb{T}, \mathcal{B}(X, Y))$ is not complete (cf. [HNVW, p.64]). Also we define $H^p_s(\mathbb{T}, \mathcal{B}(X, Y))$ by the space of all (equivalence classes of) functions $f \in L^p_s(\mathbb{T}, \mathcal{B}(X, Y))$ such that $f(\cdot)x \in H^p(\mathbb{T}, Y)$ for every $x \in X$.

Example 4.5. In view of Lemma 3.4, we may ask whether or not $L^p_s(\mathbb{T}, \mathcal{B}(X, Y)) = L^p_{sot}(\mathbb{T}, \mathcal{B}(X, Y))$ if X and Y are separable Banach spaces. The answer, however, is negative. To see this, we use the notation

$$(\mathbf{1}_F \otimes x)(z) := \mathbf{1}_F(z)x \quad \text{for } x \in X,\ F \subseteq \mathbb{T}.$$

Write $H^2 \equiv H^2(\mathbb{T})$ and define a function $f : \mathbb{T} \to \mathcal{B}(H^2)$ by

$$(f(z)x)(s) := \widehat{x}(0) + \sum_{n=1}^{\infty} \big(\mathbf{1}_{F_n} \otimes \sqrt{2n}\,\widehat{x}(n)\big)(z)s^n \quad (x \in H^2), \tag{19}$$

where $F_n := \{e^{i\theta} : (2 - \frac{1}{n})\pi \le \theta < 2\pi\}$ for $n = 1, 2, \cdots$. Let

$$E_0 := \{e^{i\theta} : 0 < \theta < \pi\} \quad \text{and} \quad E_n := F_n \setminus F_{n+1} \ \text{ for } n = 1, 2, \cdots .$$

Then for each $1 \neq z \in \mathbb{T}$, there exists $N \ge 0$ such that $z \in E_N$. Thus, by (19), we have that for $N \ge 1$,

$$(f(z)x)(s) = \widehat{x}(0) + \sum_{n=1}^{N} \sqrt{2n}\,\widehat{x}(n)s^n \quad (z \in E_N), \tag{20}$$

which implies $f(z) \in \mathcal{B}(H^2)$. For $x \in H^2$, it follows from (20) that

$$\begin{aligned}
||f(\cdot)x||^2_{L^2(\mathbb{T},\,H^2)} &= \sum_{N=0}^{\infty} \int_{E_N} ||f(z)x||^2 dm(z) \\
&= \frac{1}{2}\big|\widehat{x}(0)\big|^2 + \sum_{N=1}^{\infty} \left(\big|\widehat{x}(0)\big|^2 + \sum_{n=1}^{N} 2n\big|\widehat{x}(n)\big|^2 \right) m(E_N) \\
&= \big|\widehat{x}(0)\big|^2 + \sum_{n=1}^{\infty} 2n\big|\widehat{x}(n)\big|^2 m(F_n) \\
&= ||x||^2_{H^2},
\end{aligned}$$

which proves that $f \in L^2_s(\mathbb{T}, \mathcal{B}(H^2))$. However, we have $f \notin L^2_{sot}(\mathbb{T}, \mathcal{B}(H^2))$: indeed if $z \in E_n$ $(n \ge 1)$ then it follows from (20) that $||f(z)s^n||_{H^2} = \sqrt{2n}$. Thus

$$\int_{\mathbb{T}} ||f(z)||^2 dm(z) = \sum_{n=0}^{\infty} \int_{E_n} ||f(z)||^2 dm(z) \ge \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty,$$

which implies that $f \notin L^2_{sot}(\mathbb{T}, \mathcal{B}(H^2))$. □

The following diagram summarizes the preceding arguments.

Let X be a separable Banach space, Y be a Banach space satisfying the ARNP and F be a countable dense subset of X. Then for all $1 \le p \le \infty$,

$$\begin{array}{ccccccc}
H^p(\mathbb{T}, \mathcal{B}(X,Y)) & \subsetneqq & H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y)) & \subsetneqq & H^p_s(\mathbb{T}, \mathcal{B}(X,Y)) & \subsetneqq & H^p_s(\mathbb{T}, \mathcal{L}(\mathrm{sp}(F), Y)) \\
 & & \cong \big\downarrow P_s & & & & \cong \big\downarrow P_s \\
 & & H^p(\mathbb{D}, \mathcal{B}(X,Y)) & & & & H^p_s(\mathbb{D}, \mathcal{B}(X,Y)).
\end{array} \tag{21}$$

We note that all the inclusions on the first line of (21) are strict. Indeed, in Example 3.6, we saw $H^p(\mathbb{T}, \mathcal{B}(X,Y)) \neq H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y))$ in general. We give some examples such that the other two inclusions are strict.

Example 4.6. (a) In general, $H^p_{sot}(\mathbb{T}, \mathcal{B}(X,Y)) \neq H^p_s(\mathbb{T}, \mathcal{B}(X,Y))$. Indeed, for a separable Hilbert space E, there exists a function $g \in H^2_s(\mathbb{T}, \mathcal{B}(E))$ such that $g \notin H^2_{sot}(\mathbb{T}, \mathcal{B}(E))$. To see this, we introduce some notations. For $f \in L^2_s(\mathbb{T}, \mathcal{B}(E))$, we denote by f_- and f_+ the functions

$$f_-(z)x := \big(P_-(f(\cdot)x)\big)(\overline{z}) \quad (z \in \mathbb{T},\ x \in E);$$
$$f_+(z)x := \big(P_+(f(\cdot)x)\big)(z) \quad (z \in \mathbb{T},\ x \in E),$$

where P_+ and P_- are the orthogonal projections from $L^2(\mathbb{T}, E)$ onto $H^2(\mathbb{T}, E)$ and $L^2(\mathbb{T}, E) \ominus H^2(\mathbb{T}, E)$, respectively (cf. [Pe]). Then, $f_-, f_+ \in H^2_s(\mathbb{T}, \mathcal{B}(E))$ and we may write

$$f(z) = f_+(z) + f_-(\overline{z}) \quad (z \in \mathbb{T}).$$

Let f be the function given in (19). Assume that $f_+ \in H^2_{sot}(\mathbb{T}, \mathcal{B}(H^2))$ and $f_- \in H^2_{sot}(\mathbb{T}, \mathcal{B}(H^2))$. Observe that for $z \in \mathbb{T}$,

$$||f(z)||^2 \leq ||f_-(\overline{z})||^2 + ||f_+(z)||^2 + 2||f_-(\overline{z})||||f_+(z)||.$$

It thus follows from Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{T}} ||f(z)||^2 dm(z) &\leq \int_{\mathbb{T}} ||f_-(z)||^2 dm(z) + \int_{\mathbb{T}} ||f_+(z)||^2 dm(z) \\ &\quad + 2\Big(\int_{\mathbb{T}} ||f_-(z)||^2 dm(z)\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{T}} ||f_+(z)||^2 dm(z)\Big)^{\frac{1}{2}} \\ &< \infty,\end{aligned}$$

which is a contradiction. We thus have $f_+ \notin H^2_{sot}(\mathbb{T}, \mathcal{B}(H^2))$ or $f_- \notin H^2_{sot}(\mathbb{T}, \mathcal{B}(H^2))$. Note that H^2 is a separable Hilbert space.

(b) In general, $H^p_s(\mathbb{T}, \mathcal{B}(X,Y)) \neq H^p_s(\mathbb{T}, \mathcal{L}(\text{sp}(F), Y))$. To see this, define $f : \mathbb{T} \to \mathcal{L}(\ell^2, \mathbb{C})$ by

$$f(z)x := \sum_{n=1}^{\infty} x(n) z^n \qquad (x \equiv (x(n)) \in \ell^2).$$

Then $f(z)$ is not bounded for all $z \in \mathbb{T}$ because for any $z_0 \in \mathbb{T}$, if we let

$$x_0(n) := \frac{\overline{z}_0^n}{n} \quad (n = 1, 2, \cdots),$$

then $f(z_0)x_0 = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. Thus, $f \notin H^2_s(\mathbb{T}, \mathcal{B}(\ell^2, \mathbb{C}))$. On the other hand, let

$$F := \Big\{\sum_{n \in \Omega} \alpha_n e_n : \alpha_n \in \mathbb{Q} \text{ and } \Omega \text{ is a finite subset of } \mathbb{N}\Big\},$$

where $\mathbb{Q}$ is a countable dense subset of $\mathbb{C}$ and $\{e_n : n = 1, 2, \cdots\}$ is the canonical orthonormal basis for ℓ^2. Then F is a countable dense subset of ℓ^2 and we can easily see that $f \in H^2_s(\mathbb{T}, \mathcal{L}(\text{sp}(F), \mathbb{C}))$.

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