Higher-order de Branges–Rovnyak and sub-Bergman spaces

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Abstract

The sub-Bergman spaces are de Branges–Rovnyak subspaces of Bergman space A^2 defined by the contraction T_b or T_b^* for an analytic symbol b. The fact that both T_b and T_b^* are 2-hypercontractions on A^2 leads to the introduction of a new type of sub-Bergman spaces, which will be called higher-order sub-Bergman spaces. We show these new spaces are different and yet connected in a nice way with the sub-Bergman spaces. The close relationship of these new spaces to the original de Branges–Rovnyak subspaces of the Hardy spaces are also explored. A similar study is conducted on weighted Bergman spaces A^2_α where both T_b and T_b^* are $[\alpha + 2]$-hypercontractions.

Contents

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1 Introduction

Let H^2 be the Hardy space on the unit disk $\mathbb{D}$. By the Beurling theorem, an invariant subspace of the multiplication by z (the shift operator S) is of the form θH^2, where θ is an inner function. The model space $K_\theta := H^2 \ominus \theta H^2$ is an invariant subspace of S^* that plays an important role in Sz.-Nagy and Foiaş model theory of Hilbert space contractions. The de Branges–Rovnyak model spaces are submanifolds (not necessarily closed) of H^2 which are invariant under S^*. Thus they are also called sub-Hardy spaces [39]. A beautiful extension of Beurling invariant subspace theorem due to de Branges and Rovnyak characterizes Hilbert spaces contained contractively inside H^2 which are invariant under S. The de Branges–Rovnyak model spaces and their analogue inside vector-valued Hardy spaces are indeed fundamental in model theory of Hilbert space contractions [7]. See also a recent paper [34] where de Branges–Rovnyak model spaces are also useful in modeling expansive operators and m-isometries. Since the introduction of de Branges–Rovnyak spaces more than a half century ago, they have been useful in operator theory, function theory, and applications in engineering, [8], [11], [16], [22], [23], [31], [33], [42], in particular, see the recent two volumes [24] devoted to these spaces. Even these two volumes covered primarily the de Branges–Rovnyak spaces inside the scalar-valued Hardy space. See a recent paper [4], where de Branges–Rovnyak type spaces of vector-valued analytic functions are discussed.

The pioneering work of Zhu [43], [44] studied the analogues of de Branges–Rovnyak spaces inside the Bergman space A^2 called sub-Bergman spaces. Several fundamental results for sub-Bergman spaces are given and interesting, but their structures are much less understood. One fundamental reason is that the invariant subspaces of S (hence the invariant subspaces of S^*) are much more complicated on A^2 [5]. Since the work of Zhu [43], [44], there have been several papers answering Zhu's questions and extending his results [1], [12], [13], [41]. In particular, Chu [12] gives an affirmative answer to the question of whether polynomials are dense in the sub-Bergman space. It turns out that unlike sub-Hardy spaces, sub-Bergman spaces cannot be finite dimensional. As is well-known, K_θ on H^2 is finite dimensional if and only if θ is a finite Blaschke product. These finite dimensional model spaces K_θ are important in approximation theory and in understanding more general sub-Hardy spaces. They are often building blocks in various mathematical constructions [24]. Furthermore, the truncated Toeplitz operators [40] acting on them are closely related to Toeplitz and Hankel type structured matrices that have wide applications in physical science and engineering [32], [35], [37].

Our motivation is to find "sub-Bergman" spaces which are finite dimensional. This leads us to introduce in Section 2 the higher-order de Branges–Rovnyak space $G_k(H, A)$ for a k-hypercontraction A^* on a complex Hilbert space H. The class of k-hypercontractions was introduced by Agler [2], where a functional model of k-hypercontractions on subspaces of weighted Bergman spaces is established. Since then, k-hypercontractions have been studied intensively, see for example [19], [20], [36], and see also [28], where an analogue of Agler's model for k-hypercontractions on Banach spaces is obtained. The main result of Section 2 is Theorem 2.14, which shows that $G_k(H, A)$ can be viewed as an iterated de Branges–Rovnyak space.

In Section 3, we study the higher-order de Branges–Rovnyak spaces $G_k(H, A)$, where H is the weighted Bergman space A^2_α ($\alpha > -1$) of the unit disk $\mathbb{D}$ and A is the Toeplitz operator T_b on A^2_α with b in the unit ball of H^∞. We denote such $G_k(A^2_\alpha, T_b)$ by $G_{k,\alpha}(b)$ and call them the *higher-order sub-Bergman spaces*. We recall some basic results of reproducing kernel Hilbert spaces (RKHS) [3], [6], [17]. Using the powerful techniques of RKHS, we show that $G_{k,\alpha}(b)$ is defined for $1 \le k \le [\alpha + 2]$ and $G_{k,\alpha}(\bar{b}) := G_k(A^2_\alpha, T^*_b)$ is well-defined for all $k \ge 1$. We then identify reproducing kernels of $G_{k,\alpha}(b)$ and $G_{k,\alpha}(\bar{b})$ in Theorem 3.15 and Proposition 3.20, respectively, and establish the invariance property of $G_{k,\alpha}(\bar{b})$ and $G_{k,\alpha}(b)$ in Proposition 3.13 and Proposition 3.18, respectively.

In Section 4, we study $G_{k,\alpha}(b)$, where b is a finite Blaschke product. The sub-Bergman space

$\mathcal{H}(A^2_\alpha, T_b)$ in this case is identified in [44] for A^2 and in [41] for A^2_α, see also [1], [13] for different proofs and refinements. Our abstract operator theoretic approach not only extends previous results but also gives more transparent proofs than function theoretic and computational proofs. We identify $G_{k,\alpha}(b)$ as norm equivalent to certain weighted Bergman spaces in Theorem 4.10. In particular, when α is a nonnegative integer, $G_{\alpha+2,\alpha}(b)$ is shown to be finite dimensional, which answers our motivating question of when "sub-Bergman" spaces are finite dimensional. We then find the dimension of $G_{\alpha+2,\alpha}(b)$ and write down an orthonormal basis of $G_{\alpha+2,\alpha}(b)$ in some generic cases.

In Section 5, we identify $G_{k,\alpha}(\bar{b})$, where b is a finite Blaschke product. It turns out they are norm equivalent to weighted Bergman spaces and Dirichlet type spaces [9]. Again our proofs benefit from the abstract Theorem 4.7.

In Section 6, by modifying the method of Chu [12], we prove that polynomials are dense in $G_{k,\alpha}(\bar{b})$ for all $k \geq 1$. Furthermore, we demonstrate that polynomials are dense in $G_{1,\alpha}(b)$. These theorems extend the result of Chu [12] that polynomials are dense in both $G_{k,\alpha}(\bar{b})$ and $G_{k,\alpha}(b)$ for $k = 1$ and $\alpha = 0$. In view of these results, we conjecture that polynomials are dense in $G_{k,\alpha}(b)$ for $1 \leq k < [\alpha + 2]$.

In summary, we introduce the novel concept of higher-order de Branges–Rovnyak spaces by combining the studies of de Branges–Rovnyak spaces and hypercontractions, both topics have been studied intensively in the last several decades. We develop some basic properties of these higher-order de Branges–Rovnyak spaces. In particular, we show they can be viewed as iterated de Branges–Rovnyak spaces. We apply an abstract operator theoretic approach to the study of higher-order sub-Bergman spaces. We compute the reproducing kernels of higher-order sub-Bergman spaces and use them effectively to answer a number of questions. We identify these higher-order sub-Bergman spaces when the associated symbols are finite Blaschke products. We demonstrate that some natural function spaces are contained in higher-order sub-Bergman spaces for general associated symbols. We find finite dimensional higher-order sub-Bergman spaces and produce explicit orthonormal bases for these spaces. Our approach also leads to transparent and unified proofs for several fundamental results on sub-Bergman spaces where the original proofs were function theoretic and highly technical. In comparison with the extensive theory of de Branges–Rovnyak spaces and sub-Hardy spaces, it is clear that there are many questions about higher-order de Branges–Rovnyak spaces and sub-Bergman spaces for further exploration.

2 Hypercontractions and higher-order de Branges–Rovnyak spaces

2.1 Hypercontractions

The origin of de Branges–Rovnyak spaces is in the geometric definition of complementary subspaces [17]. Later D. Sarason [39] formulated de Branges–Rovnyak spaces as operator or defect range spaces of a contraction in a complex Hilbert space, in particular de Branges–Rovnyak spaces associated with analytic Toeplitz operator T_b and conjugate analytic Toeplitz operator T_b^* on the Hardy space (called sub-Hardy spaces) were analyzed in depth.

Let H and K be complex Hilbert spaces and $\mathcal{B}(K, H)$ be the set of bounded linear operators from K into H. We abbreviate $\mathcal{B}(H, H)$ to $\mathcal{B}(H)$. Let $A \in \mathcal{B}(H)$ be a contraction. We define $\mathcal{M}(H, A)$ (briefly, $\mathcal{M}(A)$) as the operator range of A with the Hilbert space structure that makes A a coisometry, i.e., the inner product is defined by

$$\langle Ah_1, Ah_2\rangle_{\mathcal{M}(A)} = \langle h_1, h_2\rangle_H,$$

where $h_1 \in (\ker A)^\perp$ and $h_2 \in H$. Let $D_{A^*} = (I - AA^*)^{1/2}$. Then the de Branges–Rovnyak space (or the complementary space) $\mathcal{H}(H, A)$ (briefly, $\mathcal{H}(A)$) is the operator range of D_{A^*} with the following inner product:

$$\mathcal{H}(A) = \mathcal{M}(D_{A^*}), \qquad \langle (I - AA^*)h_1, h_2 \rangle_{\mathcal{H}(A)} = \langle h_1, h_2 \rangle_H, \tag{1}$$

where $h_1 \in H$ and $h_2 \in \mathcal{H}(A)$. Similarly, we have $\mathcal{M}(A^*)$ and $\mathcal{H}(A^*)$. From (1), we can see

$$\|(I - AA^*)h_1\|^2_{\mathcal{H}(A)} = \|h_1\|^2_H - \|A^* h_1\|^2_H \qquad (h_1 \in H). \tag{2}$$

Definition 2.1. A Hilbert space $K \subseteq H$ is said to be *contractively contained in* H if the inclusion map $\iota : K \to H$ is a contraction from K into H, which is denoted by $K \hookrightarrow H$. More generally, if $A \in \mathcal{B}(K, H)$, then $AK \hookrightarrow H$ means A is a contraction from K into H.

Observe that when A is a contraction on H,

$$\mathcal{M}(A),\ \mathcal{M}(A^*),\ \mathcal{H}(A),\ \mathcal{H}(A^*) \hookrightarrow H.$$

We note that if $A \in \mathcal{B}(H)$ is hyponormal, i.e., $AA^* \leq A^*A$, and in turn, $I - A^*A \leq I - AA^*$, then by the Douglas range inclusion lemma [18], we can see that

$$\mathcal{M}(A) \subseteq \mathcal{M}(A^*) \quad \text{and} \quad \mathcal{H}(A^*) \subseteq \mathcal{H}(A). \tag{3}$$

It is also easy to see that if $\iota : K \hookrightarrow H$ is the inclusion map then $K = \mathcal{M}(\iota\iota^*)$.

For $A \in \mathcal{B}(H)$ and $m \geq 0$, let

$$\beta_m(A) := \sum_{i=0}^{m} (-1)^i \binom{m}{i} A^{*i} A^i.$$

Then for $h \in H$,

$$\langle \beta_m(A)h, h \rangle = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \left\| A^i h \right\|^2. \tag{4}$$

The following recursive formula is useful:

$$\beta_m(A) = \beta_{m-1}(A) - A^* \beta_{m-1}(A) A. \tag{5}$$

Definition 2.2. An operator $A \in \mathcal{B}(H)$ is called an *m-hypercontraction* if

$$\beta_k(A) \geq 0 \quad \text{for all } 1 \leq k \leq m.$$

For example, an operator $A \in \mathcal{B}(H)$ is a 2-hypercontraction if

$$I - A^*A \geq 0 \quad \text{and} \quad I - 2A^*A + A^{*2}A^2 \geq 0,$$

equivalently if for $h \in H$,

$$\|Ah\|^2 \leq \|h\|^2 \quad \text{and} \quad \|Ah\|^2 - \left\|A^2 h\right\|^2 \leq \|h\|^2 - \|Ah\|^2.$$

The notion of m-hypercontractions was introduced in [2] and has been studied in literature [10], [19], [20], [36], etc. See also a recent extension of this notion to Banach space operators [26], [27], [28].

If $A \in \mathcal{B}(H)$ is an m-hypercontraction, set

$$\Delta_k(A) := \beta_k(A)^{1/2}, \quad 1 \leq k \leq m.$$

Note that $\Delta_1(A) = D_A$. Then (5) and (4) become

$$\Delta_k^2(A) = \Delta_{k-1}^2(A) - A^*\Delta_{k-1}^2(A)A, \tag{6}$$

$$\|\Delta_k(A)h\|^2 = \|\Delta_{k-1}(A)h\|^2 - \|\Delta_{k-1}(A)Ah\|^2 = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left\|A^ih\right\|^2. \tag{7}$$

While it is obvious that A is a contraction if and only if A^* is a contraction, this does not hold for an m-hypercontraction. To see this we recall that for $f \in L^\infty$, the Toeplitz operator T_f on H^2 with symbol f is defined by

$$T_f h = P(fh), \quad h \in H^2,$$

where P denotes the orthogonal projection from L^2 onto H^2.

Example 2.3. *Let $A = cT_z$ on H^2, where $|c| \leq 1$. Then A is a 2-hypercontraction, but A^* is a 2-hypercontraction if and only if $|c| \leq 1/\sqrt{2}$.*

Proof. Since $T_z^*T_z = I$ and $T_zT_z^* = I - 1 \otimes 1$,

$$\beta_2(A) = I - 2|c|^2T_z^*T_z + |c|^4T_z^{*2}T_z^2 = (1-|c|^2)^2I \geq 0$$

and

$$\begin{aligned}\beta_2(A^*) &= I - 2|c|^2T_zT_z^* + |c|^4T_z^2T_z^{*2}\\ &= (1-|c|^2)^2I + (2|c|^2 - |c|^4)1\otimes 1 - |c|^4 z\otimes z.\end{aligned}$$

Therefore $\beta_2(A^*) \geq 0$ if and only if $(1-|c|^2)^2 \geq |c|^4$. The result follows from this inequality. □

Recall that for $h_1 \in H$ and $h_2 \in \mathcal{M}(C)$,

$$\langle CC^*h_1, h_2\rangle_{\mathcal{M}(C)} = \langle h_1, h_2\rangle_H. \tag{8}$$

The following lemma is useful in the sequel. We here note that by the closed graph theorem and standard arguments in $\mathcal{M}(A)$ spaces, the inclusion $B\mathcal{M}(C) \subset \mathcal{M}(D)$ implies that B is a bounded operator from $\mathcal{M}(C)$ into $\mathcal{M}(D)$.

Lemma 2.4. *Let $C \in \mathcal{B}(H_1, H_2)$, $B \in \mathcal{B}(H_2, H_3)$, and $D \in \mathcal{B}(H_4, H_3)$. Assume B maps $\mathcal{M}(C)$ into $\mathcal{M}(D)$. Let $B_{\mathcal{M}} = B|\mathcal{M}(C) \in \mathcal{B}(\mathcal{M}(C), \mathcal{M}(D))$. Then*

$$B_{\mathcal{M}}^*DD^* = CC^*B^*.$$

Proof. For $h \in H_3$, $h_2 \in \mathcal{M}(C)$,

$$\begin{aligned}\langle B_{\mathcal{M}}^*DD^*h, h_2\rangle_{\mathcal{M}(C)} &= \langle DD^*h, B_{\mathcal{M}}h_2\rangle_{\mathcal{M}(D)}\\ &= \langle DD^*h, Bh_2\rangle_{\mathcal{M}(D)} = \langle h, Bh_2\rangle_{H_3}\\ &= \langle B^*h, h_2\rangle_{H_2} = \langle CC^*B^*h, h_2\rangle_{\mathcal{M}(C)},\end{aligned}$$

where the first equality follows from the definition of $B_{\mathcal{M}}^*$, and the third and the last equalities follow from (8). Therefore, $B_{\mathcal{M}}^*DD^*h = CC^*B^*h$ for all $h \in H_3$. □

The above simple lemma is useful for characterizing certain operators between two de Branges–Rovnyak spaces as seen in Lemma 2.5 and Corollary 2.6 below.

Next by using Lemma 2.4, we characterize when the adjoint of an operator on a de Branges–Rovnyak space is a k-hypercontraction.

Lemma 2.5. *Let $B, C \in \mathcal{B}(H)$. Assume B maps $\mathcal{M}(C)$ into $\mathcal{M}(C)$. Let $B_{\mathcal{M}} = B|\mathcal{M}(C) \in \mathcal{B}(\mathcal{M}(C))$. Then $B_{\mathcal{M}}^*$ is a k-hypercontraction if and only if*

$$\beta_m(B^*, C) := \sum_{i=0}^{m} (-1)^i \binom{m}{i} B^i CC^* B^{*i} \geq 0 \text{ on } H \text{ for all } 1 \leq m \leq k.$$

Proof. By definition, $B_{\mathcal{M}}^*$ is a k-hypercontraction if and only if for $1 \leq m \leq k$, $\langle \beta_m(B_{\mathcal{M}}^*) h_1, h_1 \rangle_{\mathcal{M}(C)} \geq 0$ for all $h_1 \in \mathcal{M}(C)$. Since CC^*H is dense in $\mathcal{M}(C)$, this happens if and only if for all $h \in H$,

$$\langle \beta_m(B_{\mathcal{M}}^*) CC^* h, CC^* h \rangle_{\mathcal{M}(C)} \geq 0 :$$

indeed,

$$\begin{aligned} \langle \beta_m(B_{\mathcal{M}}^*) CC^* h, CC^* h \rangle_{\mathcal{M}(C)} &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} \left\langle B_{\mathcal{M}}^{*i} CC^* h, B_{\mathcal{M}}^{*i} CC^* h \right\rangle_{\mathcal{M}(C)} \\ &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} \left\langle CC^* B^{*i} h, CC^* B^{*i} h \right\rangle_{\mathcal{M}(C)} \\ &= \sum_{i=0}^{m} (-1)^i \binom{m}{i} \left\langle CC^* B^{*i} h, B^{*i} h \right\rangle_H = \langle \beta_m(B^*, C) h, h \rangle_H \geq 0, \end{aligned}$$

where the second equality follows from Lemma 2.4 and the third equality follows from (8). □

It is curious to ask when $B_{\mathcal{M}}$ is a k-hypercontraction. Similarly, we can give characterizations of when $B_{\mathcal{M}}^*$ belongs to other classes of operators such as $B_{\mathcal{M}}^*$ being a k-isometry, i.e., $\beta_k(B_{\mathcal{M}}^*) = 0$. The case of when $B_{\mathcal{M}}^*$ is an isometry is useful for us. Furthermore, this result can be stated for an operator between different spaces, and it can also be viewed as a refinement of Corollary 16.11 in [24].

Corollary 2.6. *Let $C \in \mathcal{B}(H_1, H_2)$, $B \in \mathcal{B}(H_2, H_3)$ and $D \in \mathcal{B}(H_4, H_3)$. Then $DD^* = BCC^*B^*$ on H_3 if and only if $B_{\mathcal{M}}^*$ is an isometry, where $B_{\mathcal{M}} = B|\mathcal{M}(C) \in \mathcal{B}(\mathcal{M}(C), \mathcal{M}(D))$. Furthermore, $B_{\mathcal{M}}$ is a unitary (or $B_{\mathcal{M}}^*$ is an onto isometry) if and only if* $\ker(B) \cap \mathcal{M}(C) = \{0\}$.

Proof. Assume $DD^* = BCC^*B^*$. Then by Corollary 16.10(ii) in [24], B maps $\mathcal{M}(C)$ into $\mathcal{M}(D)$ and $B_{\mathcal{M}} = B|\mathcal{M}(C) \in \mathcal{B}(\mathcal{M}(C), \mathcal{M}(D))$. Note that $B_{\mathcal{M}}^*$ is an isometry if and only if $\langle B_{\mathcal{M}}^* h_1, B_{\mathcal{M}}^* h_1 \rangle_{\mathcal{M}(C)} = \langle h_1, h_1 \rangle_{\mathcal{M}(D)}$ for all $h_1 \in \mathcal{M}(D)$. Since DD^*H_3 is dense in $\mathcal{M}(D)$, this happens if and only if for all $h \in H_3$,

$$\begin{aligned} 0 &= \langle B_{\mathcal{M}}^* DD^* h, B_{\mathcal{M}}^* DD^* h \rangle_{\mathcal{M}(C)} - \langle DD^* h, DD^* h \rangle_{\mathcal{M}(D)} \\ &= \langle CC^* B^* h, CC^* B^* h \rangle_{\mathcal{M}(C)} - \langle DD^* h, DD^* h \rangle_{\mathcal{M}(D)} \\ &= \langle CC^* B^* h, B^* h \rangle_{H_2} - \langle DD^* h, h \rangle_{H_3} = \langle (BCC^*B^* - DD^*) h, h \rangle_{H_3}, \end{aligned}$$

where the second equality follows from Lemma 2.4 and the third equality follows from (8). Since $B_{\mathcal{M}}^*$ is an isometry, it is clear that $B_{\mathcal{M}}$ is a unitary if and only if $\ker(B) \cap \mathcal{M}(C) = \{0\}$. □

2.2 Higher-order de Branges–Rovnyak spaces

We now introduce the new spaces to be studied in this paper.

Definition 2.7. If $A^* \in \mathcal{B}(H)$ is an m-hypercontraction, we define

$$G_k(A) \equiv G_k(H, A) := \mathcal{M}(H, \Delta_k(A^*)), \quad 1 \leq k \leq m.$$

Recall that $\Delta_k(A^*) = \beta_k(A^*)^{1/2}$. Thus we have that for $h_1 \in H$ and $h_2 \in G_k(A)$,

$$\langle \beta_k(A^*) h_1, h_2 \rangle_{G_k(A)} = \langle h_1, h_2 \rangle_H. \tag{9}$$

Note that $G_1(A) = \mathcal{H}(A)$ is the de Branges–Rovnyak space. For notational convenience, set $G_0(A) := H$.

The following lemma describes a dense set of $G_k(A)$ and how one can compute the norms of the vectors in this dense set.

Lemma 2.8. *If $A^* \in \mathcal{B}(H)$ is a k-hypercontraction, then $\beta_k(A^*)H$ is dense in $G_k(A)$. Furthermore, for $h \in H$,*

$$\|\beta_k(A^*)h\|^2_{G_k(A)} = \|\Delta_k(A^*)h\|^2_H = \sum_{i=0}^{k}(-1)^i \binom{k}{i} \left\|A^{*i}h\right\|^2_H . \tag{10}$$

Proof. The first assertion follows from Lemma 16.15 in [24]. For $h \in H$, by (9),

$$\|\beta_k(A^*)h\|^2_{G_k(A)} = \langle \beta_k(A^*)h, \beta_k(A^*)h\rangle_{G_k(A)} = \langle h, \beta_k(A^*)h\rangle_H = \|\Delta_k(A^*)h\|^2_H.$$

The result follows from (7). □

Lemma 2.9. *Let $A^* \in \mathcal{B}(H)$ be an m-hypercontraction. Then*

$$G_m(A) \hookrightarrow G_{m-1}(A) \hookrightarrow \cdots \hookrightarrow G_2(A) \hookrightarrow \mathcal{H}(A) = G_1(A) \hookrightarrow H = G_0(A).$$

Proof. For $1 \leq k \leq m$, by definition, $G_k(A) = \mathcal{M}(\Delta_k(A^*))$ and $G_{k-1}(A) = \mathcal{M}(\Delta_{k-1}(A^*))$. By (6),

$$\Delta_k^2(A^*) = \Delta_{k-1}^2(A^*) - A\Delta_{k-1}^2(A^*)A^* \leq \Delta_{k-1}^2(A^*),$$

which proves $G_k(A) \hookrightarrow G_{k-1}(A)$. □

Problem 2.10. *When $G_k(A)$ is closed in $G_{k-1}(A)$? In particular, when $G_2(A)$ is closed in $\mathcal{H}(A)$?*

The following observation shows that the notion of m-hypercontractions has a natural connection with the contraction on de Branges–Rovnyak spaces.

Lemma 2.11. *Let $A^* \in \mathcal{B}(H)$ be a k-hypercontraction. Then A is a contraction on $G_k(A)$ if and only if A^* is a $(k+1)$-hypercontraction on H. As a consequence, if $A^* \in \mathcal{B}(H)$ is a k-hypercontraction, then $AG_{k-1}(A) \hookrightarrow G_{k-1}(A)$.*

Proof. By Corollary 16.10(ii) in [24] with $\mathcal{M}(\Delta_k(A^*)) = G_k(A)$, the operator $A : G_k(A) \to G_k(A)$ is a contraction if and only if

$$A\Delta_k(A^*)\Delta_k(A^*)A^* \leq \Delta_k(A^*)\Delta_k(A^*),$$

equivalently if and only if

$$A\beta_k(A^*)A^* \leq \beta_k(A^*).$$

By definition and (5), this is the same as A^* being a $(k+1)$-hypercontraction. □

On Lemma 2.11, we say something more:

Corollary 2.12. *Let $A^* \in \mathcal{B}(H)$ be a k-hypercontraction. Then A maps $G_k(A)$ into $G_k(A)$ and $A^*_{G_k}$ is an m-hypercontraction, where $A_{G_k} = A|G_k(A) \in \mathcal{B}(G_k(A))$, if and only if A^* is a $(k+m)$-hypercontraction on H. As a consequence, if $A^* \in \mathcal{B}(H)$ is an n-hypercontraction for some $n \geq 2$, then $A^*_{G_l}$ is an $(n-l)$-hypercontraction, where $A_{G_l} = A|G_l(A) \in \mathcal{B}(G_l(A))$ for all $1 \leq l < n$.*

Proof. Since $G_k(A) = \mathcal{M}(\beta_k(A^*)^{1/2})$, by Lemma 2.5 with $C = \beta_k(A^*)^{1/2}$ and $B = A$, $A^*_{G_k}$ is an m-hypercontraction on $G_k(A)$ if and only if for all $1 \le j \le m$,

$$\begin{aligned}\beta_j(A^*, \beta_k(A^*)^{1/2}) &= \sum_{i=0}^{j}(-1)^i \binom{j}{i} A^i \beta_k(A^*)^{1/2}\beta_k(A^*)^{1/2}A^{*i} \\ &= \sum_{i=0}^{j}(-1)^i \binom{j}{i} A^i \beta_k(A^*)A^{*i} = \beta_{k+j}(A^*) \ge 0,\end{aligned}$$

where the third equality can be proved by induction using (5). By definition, this happens if and only if A^* is a $(k+m)$-hypercontraction on H, where the backward implication uses Lemma 2.11. □

When $m = 1$, $A^*_{G_k}$ is an m-hypercontraction if and only if A_{G_k} is a contraction, so the above corollary generalizes Lemma 2.11 from $m = 1$ to $m \ge 1$.

If A^* is a 2-hypercontraction on H, then $A\mathcal{H}(A) \hookrightarrow \mathcal{H}(A)$. Thus we can define a new de Branges–Rovnyak space by using the contraction A viewed as an operator on $\mathcal{H}(A)$. In the definition of this new space, we need to use the adjoint $(A|\mathcal{H}(A))^*$ which is different from A^* since A^* even does not necessarily map $\mathcal{H}(A)$ into $\mathcal{H}(A)$. We add explicitly the Hilbert space where A acts for the de Branges–Rovnyak space $\mathcal{H}(A)$ to elucidate the situation. Namely, let

$$\mathcal{H}(H, A) = \mathcal{M}(D_{A^*}) \text{ which as a set is } (I - AA^*)^{1/2}H,$$

where A is viewed as a contraction on H. We first need a simple observation about $(A|\mathcal{H}(A))^*$. In fact we study the general case $(B|G_k(A))^*$.

Corollary 2.13. *Let $A^* \in \mathcal{B}(H)$ be a k-hypercontraction and $B \in \mathcal{B}(H)$ be a contraction. Assume $BG_k(A) \hookrightarrow G_k(A)$. Let $B_1 = B|G_k(A)$. Then $B_1^*G_k(A) \hookrightarrow G_k(A)$ and*

$$B_1^*\beta_k(A^*) = \beta_k(A^*)B^*.$$

Proof. This follows at once from Lemma 2.4 by applying with $C = D = \Delta_k(A^*)$. □

With the above notation we have the following interpretation of $G_k(A)$.

Theorem 2.14. *Let A^* be a k-hypercontraction on H. Then*

$$G_k(A) = \mathcal{H}(G_{k-1}(A), A). \tag{11}$$

Proof. Let $A_1 = A|G_{k-1}(A)$. By Lemma 2.11, A_1 is a contraction on $G_{k-1}(A)$. We first note that as a set,

$$\mathcal{H}(G_{k-1}(A), A) = (I - A_1A_1^*)^{1/2}G_{k-1}(A) = (I - A_1A_1^*)^{1/2}\Delta_{k-1}(A^*)H.$$

By Corollary 2.13, for $h \in H$,

$$\begin{aligned}(I - A_1A_1^*)\beta_{k-1}(A^*)h &= \beta_{k-1}(A^*)h - A_1A_1^*\beta_{k-1}(A^*)h \\ &= \beta_{k-1}(A^*)h - A\beta_{k-1}(A^*)A^*h = \beta_k(A^*)h,\end{aligned} \tag{12}$$

where the last equality follows from (5). Thus,

$$(I - A_1A_1^*)\beta_{k-1}(A^*)H = \beta_k(A^*)H. \tag{13}$$

Furthermore,

$$\begin{aligned}
&\|(I - A_1A_1^*)\beta_{k-1}(A^*)h\|^2_{\mathcal{H}(G_{k-1}(A),A)} \\
&= \|\beta_{k-1}(A^*)h\|^2_{G_{k-1}(A)} - \|A_1^*\beta_{k-1}(A^*)h\|^2_{G_{k-1}(A)} \\
&= \|\beta_{k-1}(A^*)h\|^2_{G_{k-1}(A)} - \|\beta_{k-1}(A^*)A^*h\|^2_{G_{k-1}(A)} \qquad (14)\\
&= \|\Delta_{k-1}(A^*)h\|^2_H - \|\Delta_{k-1}(A^*)A^*h\|^2_H \\
&= \|\Delta_k(A^*)h\|^2_H = \|\beta_k(A^*)h\|^2_{G_k(A)},
\end{aligned}$$

where the first equality follows from (2), the second equality follows from Corollary 2.13, and the third equality follows from (10) with k being $k-1$. The fourth equality follows from (7). The last equality follows from again (10). By Lemma 2.8, $\beta_{k-1}(A^*)H$ is dense in $G_{k-1}(A)$ and $(I - A_1A_1^*)G_{k-1}(A)$ is dense in $\mathcal{H}(G_{k-1}(A), A)$. Thus $(I-A_1A_1^*)\beta_{k-1}(A^*)H$ is dense in $\mathcal{H}(G_{k-1}(A), A)$. By (13), $\beta_k(A^*)H$ is dense in $\mathcal{H}(G_{k-1}(A), A)$. By Lemma 2.8 again, $\beta_k(A^*)H$ is dense in $G_k(A)$. Therefore $\mathcal{H}(G_{k-1}(A), A)$ and $G_k(A)$ contain a common dense set $\beta_k(A^*)H$, where the two norms are equal. So $G_k(A) = \mathcal{H}(G_{k-1}(A), A)$. □

Corollary 2.15. *Let A^* be a 3-hypercontraction on H. Then*

$$G_3(A) = \mathcal{H}(G_2(A), A) = \mathcal{H}(\mathcal{H}(\mathcal{H}(H, A), A), A).$$

In the above sense, we call $G_k(A)$ an *order-k* (or *iterated*) *de Branges–Rovnyak space.*

On the other hand, for $1 \le l < k$, in order for $G_{k-l}(G_l(H, A), A)$ to be defined, A as an operator in $\mathcal{B}(G_l(H, A))$ has to be such that $(A|G_l(A))^*$ is a $(k-l)$-hypercontraction which indeed is the case by Corollary 2.12. Then we have the following corollary of Theorem 2.14.

Corollary 2.16. *Let $A^* \in \mathcal{B}(H)$ be a k-hypercontraction. Then*

$$G_k(H, A) = G_{k-l}(G_l(H, A), A), \quad 1 \le l < k.$$

We now give an answer to Problem 2.10. By Theorem 16.21 in [24] and Theorem 2.14, $G_2(A)$ is a closed subspace of $\mathcal{H}(A)$ if and only if $A|\mathcal{H}(A)$ is a partial isometry. However it is not clear what this actually means for A viewed as an operator on H. The following result gives some orthogonality condition.

Proposition 2.17. *Let A^* be a k-hypercontraction on H. Then $G_k(A)$ is a closed subspace of $G_{k-1}(A)$ if and only if*

$$\langle \beta_k(A^*)h_1, A\beta_{k-1}(A^*)h_2 \rangle_{G_{k-1}(A)} = 0$$

for all $h_1, h_2 \in H$. In particular, $G_2(A)$ is a closed subspace of $\mathcal{H}(A)$ if and only if for all $h_1, h_2 \in H$,

$$\big\langle (I - 2AA^* + A^2A^{*2})h_1, A(I - AA^*)h_2 \big\rangle_{\mathcal{H}(A)} = 0.$$

Proof. Recall that an operator B on a complex Hilbert space is a partial isometry if and only if $B = BB^*B$ if and only if $B^*(I - BB^*) = 0$. Let $A_1 = A|G_{k-1}(A)$. By Theorem 16.21 in [24] and Theorem 2.14, $G_k(A)$ is a closed subspace of $G_{k-1}(A)$ if and only if for $k_1, k_2 \in \beta_{k-1}(A^*)H$ (by Lemma 2.8, $\beta_{k-1}(A^*)H$ is dense in $G_{k-1}(A)$),

$$\langle (I - A_1A_1^*)k_1, Ak_2 \rangle_{G_{k-1}(A)} = \langle A_1^*(I - A_1A_1^*)k_1, k_2 \rangle_{G_{k-1}(A)} = 0.$$

Write $k_i = \beta_{k-1}(A^*)h_i$, where $h_i \in H$. Then

$$\begin{aligned}&\langle \beta_k(A^*)h_1, A\beta_{k-1}(A^*)h_2\rangle_{G_{k-1}(A)}\\&= \langle (I - A_1A_1^*)\beta_{k-1}(A^*)h_1, A\beta_{k-1}(A^*)h_2\rangle_{G_{k-1}(A)} = 0,\end{aligned}$$

where the first equality follows from (12). This completes the proof. □

If $f \in L^\infty$ with $\|f\|_\infty \le 1$, write $\mathcal{M}(f)$ for $\mathcal{M}(T_f)$ and $\mathcal{H}(f)$ for $\mathcal{H}(T_f)$. In the sequel, let $(H^\infty)_1$ denote the unit ball of H^∞.

Proposition 2.18. *Let $b \in (H^\infty)_1$. Then T_b on H^2 is an m-hypercontraction for any $m \ge 1$.*

Proof. This follows at once from the observation:

$$\beta_m(T_b) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} T_b^{*i} T_b^i = \sum_{i=0}^{m} (-1)^i \binom{m}{i} T_{\overline{b^i} b^i} = T_{(1-|b|^2)^m} \ge 0.$$

□

By Proposition 2.18, if $b \in (H^\infty)_1$, then T_b on H^2 is an m-hypercontraction for all $m \ge 1$, so that we may define

$$G_k(\bar{b}) := G_k(H^2, T_{\bar{b}}) = \mathcal{M}(\Delta_k(T_b)) \quad \text{for each } k \ge 1.$$

In particular, $G_1(\bar{b}) = \mathcal{H}(\bar{b})$.

On the other hand, the nested sequence $(G_k(A))_{k=0}^m$ in Lemma 2.9 often results in a sequence of dense subsets of H^2.

Theorem 2.19. *If $b \in (H^\infty)_1$ is nonextreme, i.e., $\log(1 - |b|^2)$ is integrable on the unit circle $\partial\mathbb{D}$, then $G_k(\bar{b})$ is dense in H^2 for all $k \ge 1$.*

Proof. If b is nonextreme then we may choose an outer function a such that $|a|^2 = 1 - |b|^2$ on the unit circle where $a(0) > 0$ (cf. [39, (IV-1)]). Then for all $k \ge 1$,

$$\beta_k(T_b) = T_{(1-|b|^2)^k} = T_{|a|^{2k}} = T_{a^k}^* T_{a^k}.$$

It is easy to see that $\ker(T_{a^k}^* T_{a^k}) = \ker(T_{a^k}) = \{0\}$. Thus, $\beta_k(T_b)H^2$ is dense in H^2. By definition, $G_k(\bar{b}) \supseteq \beta_k(T_b)H^2$. So $G_k(\bar{b})$ is dense in H^2. □

3 Higher-order sub-Bergman spaces

3.1 Preliminaries on reproducing kernel Hilbert spaces

Let $K(z, w)$ be a reproducing kernel of holomorphic Hilbert space $H(K)$ on the unit disk $\mathbb{D}$. Let c be a multiplier of $H(K)$, i.e., $H(K)$ is invariant under M_c, where M_c is the multiplication operator on $H(K)$ with symbol c. Then we can prove that for each $w \in \mathbb{D}$,

$$M_c^* K(z, w) = \overline{c(w)} K(z, w). \tag{15}$$

Definition 3.1. For two complex Hilbert spaces H_1 and H_2 with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively, we write $H_1 \approx H_2$ if $H_1 = H_2$ as sets and there exist two positive constants γ and δ such that

$$\gamma\|h\|_1 \leq \|h\|_2 \leq \delta\|h\|_1.$$

We use $K_1(z,w) \succeq 0$ to indicate $K_1(z,w)$ is a positive semi-definite kernel, equivalently $K_1(z,w)$ is a reproducing kernel.

The following lemma contains some basic properties of reproducing kernels which hold in more general context [3], [6], [38]. We shall use this lemma sometimes without explicitly mentioning it.

Lemma 3.2. *Let $K_i(z,w)$ be three reproducing kernel functions on $\mathbb{D} \times \mathbb{D}$, that is, $K_i(z,w) \succeq 0$ for $i = 1,2,3$. The following statements hold.*

(i) *For $c_1 \geq 0$ and $c_2 \geq 0$, $c_1 K_1(z,w) + c_2 K_2(z,w) \succeq 0$.*

(ii) *$K_1(z,w)K_2(z,w) \succeq 0$. Thus if $K_1(z,w) \succeq K_2(z,w)$, then*

$$K_1(z,w)K_3(z,w) \succeq K_2(z,w)K_3(z,w).$$

(iii) *$H(K_1) \subseteq H(K_2)$ if and only if there exists a positive constant γ such that*

$$\gamma K_1(z,w) \preceq K_2(z,w).$$

Thus $H(K_1) \approx H(K_2)$ if and only if there exist two positive constants γ and δ such that

$$\gamma K_1(z,w) \preceq K_2(z,w) \preceq \delta K_1(z,w).$$

In this case, we write $K_1(z,w) \approx K_2(z,w)$.

(iv) *Let $K(z,w)$ be a reproducing kernel of holomorphic Hilbert space $H(K)$ on the unit disk $\mathbb{D}$. Then*

$$K(z,w) = \sum_{i\in I} e_i(z)\overline{e_i(w)} \quad (z,w \in \mathbb{D}),$$

where $\{e_i(z)\}_{i\in I}$ is any orthonormal basis of $H(K)$. In particular, $f \in H(K)$ with $\|f\|_{H(K)} \leq 1$ if and only if

$$f(z)\overline{f(w)} \preceq K(z,w).$$

The following result can be viewed as a converse of Lemma 3.2(iv).

Lemma 3.3. *Let $K(z,w)$ be a reproducing kernel of holomorphic Hilbert space $H(K)$ on the unit disk $\mathbb{D}$. Assume that*

$$K(z,w) = \sum_{j\in J} f_j(z)\overline{f_j(w)} \quad (z,w \in \mathbb{D}),$$

where $\{f_j : j \in J\}$ are finitely linearly independent vectors of $H(K)$, in the sense that for any finite set $F \subseteq J$, $\{f_j : j \in F\}$ are linearly independent. Then $\{f_j : j \in J\}$ forms an orthonormal basis for $H(K)$.

Proof. Define H_0 by

$$H_0 = \left\{ \sum_{j\in F} a_j f_j : F \subseteq J, \text{ a finite index set} \right\}$$

with inner product

$$\left\langle \sum_{j\in F} a_j f_j, \sum_{j\in F} b_j f_j \right\rangle_{H_0} = \sum_{j\in F} a_j \overline{b_j} \quad (a_j, b_j \in \mathbb{C}).$$

Then H_0 is an inner product space. This inner product is well-defined by the finitely linear independence assumption of $\{f_j : j \in J\}$. Let H_1 be the completion of H_0. Then H_1 is a Hilbert space and $\{f_j : j \in J\}$ is an orthonormal set in H_1. If $f \perp f_j$ for all $j \in J$, then $f \perp H_0$. Hence $f = 0$. It follows that $\{f_j : j \in J\}$ is an orthonormal basis for H_1. Thus, formally,

$$H_1 = \left\{ f(z) = \sum_{j\in J} c_j f_j(z) : \|f(z)\|^2_{H_1} = \sum_{i\in J} |c_j|^2 < \infty \right\}.$$

We note that $f(z)$ as above is a well-defined function since for $z \in \mathbb{D}$,

$$\left(\sum_{j\in J} |c_j f_i(z)| \right)^2 \leq \left(\sum_{i\in J} |c_j|^2 \right) \left(\sum_{j\in J} |f_j(z)|^2 \right) = \left(\sum_{i\in J} |c_j|^2 \right) K(z,z) < \infty.$$

On the other hand, if $w \in \mathbb{D}$, then

$$\sum_{j\in J} \left| \overline{f_j(w)} \right|^2 = K(w,w) < \infty,$$

which implies that $K(z,w) \in H_1$. Let $f \in H_1$. Then $f(z) = \sum_{j\in J} \langle f, f_j \rangle_{H_1} f_j(z)$, so that

$$\langle f(z), K(z,w) \rangle_{H_1} = \left\langle f(z), \sum_{j\in J} f_j(z) \overline{f_j(w)} \right\rangle_{H_1} = \sum_{j\in J} \langle f, f_j \rangle_{H_1} f_j(w) = f(w).$$

Therefore H_1 is a reproducing kernel Hilbert space on $\mathbb{D}$ with kernel K. By the one-to-one correspondence between a reproducing kernel and its associated Hilbert space, we conclude that $H(K) = H_1$ as Hilbert spaces. In particular, $\{f_j : j \in J\}$ is an orthonormal basis for $H(K)$. □

Next we give a connection between the hypercontractive multipliers and the reproducing kernels. Similar versions of the following result appear in literature frequently [3], where their proofs are also pretty standard.

Lemma 3.4. *Let A be an operator on $H(K)$ such that*

$$A = \sum_{i,j=1}^{N} a_{ij} M_{b_i} M^*_{c_j},$$

where $a_{ij} \in \mathbb{C}$ and $b_i(z), c_j(z)$ are multipliers of $H(K)$. Then $A \geq 0$ if and only if

$$\left(\sum_{i,j=1}^{N} a_{ij} b_i(z) \overline{c_j(w)} \right) K(z,w) \succeq 0.$$

That is, the function on the left side of the above equation is a positive semi-definite kernel.

Proof. Let $h(z) = \sum_{p=1}^{l} \delta_p K(z, w_p)$, where $\delta_p \in \mathbb{C}$ and $\{w_1, \cdots w_l\} \subseteq \mathbb{D}$ is a set of l distinct points. Then

$$\begin{aligned}
\langle Ah, h\rangle_{H(K)} &= \left\langle A\sum_{p=1}^{l} \delta_p K(z, w_p), \sum_{q=1}^{l} \delta_q K(z, w_q)\right\rangle \\
&= \sum_{p,q=1}^{l} \delta_p\overline{\delta_q} \sum_{i,j=1}^{N} \left\langle a_{ij} M_{b_i} M_{c_j}^* K(z, w_p), K(z, w_q)\right\rangle \\
&= \sum_{p,q=1}^{l} \delta_p\overline{\delta_q} \sum_{i,j=1}^{N} \left\langle a_{ij} M_{c_j}^* K(z, w_p), M_{b_i}^* K(z, w_q)\right\rangle \\
&= \sum_{p,q=1}^{l} \delta_p\overline{\delta_q} \sum_{i,j=1}^{N} \left\langle a_{ij} \overline{c_j(w_p)} K(z, w_p), \overline{b_i(w_q)} K(z, w_q)\right\rangle \quad \text{(by (15))} \\
&= \sum_{p,q=1}^{l} \delta_p\overline{\delta_q} \sum_{i,j=1}^{N} a_{ij} \overline{c_j(w_p)} b_i(w_q) K(w_q, w_p).
\end{aligned}$$

The result follows from the density of kernel functions in $H(K)$ and the definitions. □

Corollary 3.5. *The analytic function c is a multiplier of $H(K)$ such that $\|M_c\| \le \gamma$ if and only if*

$$c(z)\overline{c(w)}K(z, w) \preceq \gamma^2 K(z, w)$$

Furthermore, if $1/c$ is also a multiplier such that $\|M_{1/c}\| \le \delta$, then

$$c(z)\overline{c(w)}K(z, w) \succeq K(z, w)/\delta^2. \tag{16}$$

Proof. The assumption $\|M_c\| \le \gamma$ is the same as $\gamma^2 - M_c M_c^* \ge 0$. By Lemma 3.3, this happens if and only if

$$\left(\gamma^2 - c(z)\overline{c(w)}\right) K(z, w) \succeq 0.$$

The assumption $\|M_{1/c}\| \le \delta$ implies that

$$\frac{1}{c(z)} \frac{1}{\overline{c(w)}} K(z, w) \preceq \delta^2 K(z, w).$$

Multiplying both sides of the above inequality by the reproducing kernel $c(z)\overline{c(w)}$, we see that (16) holds. □

Corollary 3.6. *Let $b(z)$ be a multiplier of $H(K)$. Then M_b^* on $H(K)$ is a k-hypercontraction if and only if*

$$(1 - b(z)\overline{b(w)})^m K(z, w) \succeq 0 \quad \textit{for } 1 \le m \le k.$$

Proof. Note that,

$$\beta_m(M_b^*) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} M_b^i M_b^{*i} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} M_{b^i} M_{b^i}^*.$$

By Lemma 3.4, $\beta_m(M_b^*) \ge 0$ if and only if

$$\left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} b(z)^i \overline{b(w)^i}\right) K(z, w) = (1 - b(z)\overline{b(w)})^m K(z, w) \succeq 0.$$

□

For $\alpha > -2$, let A^2_α be the Hilbert space with reproducing kernel

$$K_\alpha(z,w) = \frac{1}{(1-z\overline{w})^{\alpha+2}} = \sum_{i\geq 0} c_{i,\alpha} z^i \overline{w}^i,$$

where

$$c_{i,\alpha} = \binom{i+\alpha+1}{i} = \frac{\Gamma(i+\alpha+2)}{i!\,\Gamma(\alpha+2)}. \tag{17}$$

Thus, $\left\{\sqrt{c_{i,\alpha}} z^i : i \geq 0\right\}$ is an orthonormal basis of A^2_α.

When $\alpha > -1$, let

$$dA_\alpha(z) := (\alpha+1)(1-|z|^2)^\alpha \, dA(z),$$

where $dA(z)$ is the normalized area measure on $\mathbb{D}$. Let $L^2(\mathbb{D}, dA_\alpha(z))$ be the L^2 space on $\mathbb{D}$ with measure $dA_\alpha(z)$. Equivalently, the weighted Bergman space A^2_α is the closed subspace of $L^2(\mathbb{D}, dA_\alpha(z))$ consisting of holomorphic functions in $\mathbb{D}$. The Bergman projection $P_\alpha : L^2(\mathbb{D}, dA_\alpha(z)) \to A^2_\alpha$ is given by

$$P_\alpha[g](z) = \int_{\mathbb{D}} \frac{g(u)}{(1-z\overline{u})^{\alpha+2}} \, dA_\alpha(u), \quad g \in L^2(\mathbb{D}, dA_\alpha(z)). \tag{18}$$

The Bergman space with $\alpha = 0$ is denoted by A^2, and when $\alpha = -1$, we get the Hardy space H^2.

For $f \in L^\infty \equiv L^\infty(\mathbb{D})$, the Toeplitz operator T_f on A^2_α with symbol f is defined by

$$T_f h = P_\alpha(fh), \quad h \in A^2_\alpha.$$

The space of multipliers of A^2_α with $\alpha \geq -1$ is H^∞. The space of multipliers of A^2_α with $-2 < \alpha < -1$ is a proper subset of H^∞ containing functions holomorphic in a neighborhood of the closed unit disk $\overline{\mathbb{D}}$.

3.2 Reproducing kernels of higher-order sub-Bergman spaces

For $b \in (H^\infty)_1$, define

$$G_{k,\alpha}(b) := G_k(T_b) = \mathcal{M}(\Delta_k(T_b^*)),$$

where T_b is defined on A^2_α and $1 \leq k \leq [\alpha+2]$ (this is justified in Corollary 3.8 below), where $[\alpha+2]$ denotes the integer part of $\alpha+2$. The $G_{k,\alpha}(b)$ will be called an *order-k sub-Bergman space.* In particular, for $\alpha > -1$, let

$$\mathcal{A}_\alpha(b) := \mathcal{H}(A^2_\alpha, T_b) = G_{1,\alpha}(b) \quad \text{and} \quad \mathcal{A}(b) \equiv \mathcal{A}_0(b).$$

Theorem 3.7. *Let $H(K)$ be a holomorphic Hilbert space with reproducing kernel $K(z,w)$ and assume $c(z) = z$ is a multiplier of $H(K)$. If M_c^* on $H(K)$ is a k-hypercontraction, then for any $b \in (H^\infty)_1$, M_b^* on $H(K)$ is a k-hypercontraction.*

Proof. By Corollary 3.6, if M_c^* on $H(K)$ is a k-hypercontraction, then

$$(1-z\overline{w})^m K(z,w) \succeq 0 \quad \text{for } 1 \leq m \leq k.$$

For $b \in (H^\infty)_1$, M_b^* on $H(K)$ is a k-hypercontraction if and only if

$$(1-b(z)\overline{b(w)})^m K(z,w) = \left(\frac{1-b(z)\overline{b(w)}}{1-z\overline{w}}\right)^m (1-z\overline{w})^m K(z,w) \succeq 0 \tag{19}$$

for $1 \le m \le k$. By the result on H^2 [39], we know

$$\frac{1 - b(z)\overline{b(w)}}{1 - z\overline{w}} \succeq 0.$$

Therefore, (19) holds for $1 \le m \le k$. □

Corollary 3.8. *Let $b \in (H^\infty)_1$. Then T_b^* on A^2_α is an $[\alpha + 2]$-hypercontraction.*

Proof. Note that T_z^* is an $[\alpha + 2]$-hypercontraction on A^2_α since

$$(1 - z\overline{w})^m K_\alpha(z, w) = \frac{1}{(1 - z\overline{w})^{\alpha+2-m}} \succeq 0$$

for $1 \le m \le [\alpha + 2]$. The result follows from Theorem 3.7. □

The proof of the above theorem is relatively straightforward using reproducing kernels. However, to determine if an operator is a 2-hypercontraction is difficult in general. For example, we cannot completely resolve the following question.

Problem 3.9. *For which $a, b \in \mathbb{C}$, the Toeplitz operator $T_{az+b\overline{z}}$ on H^2 is a 2-hypercontraction?*

The following result from [21] is useful. In fact, Theorem 1.2 in [21] states that if A is k-hyponormal, then A is $2k$-contractive.

Lemma 3.10. [21] *Let $A \in \mathcal{B}(H)$ be a contraction. If A is hyponormal, then A is a 2-hypercontraction.*

Proof. We include a short proof. Since A is hyponormal,

$$A^* (A^* A - AA^*) A \ge 0 \quad \text{or} \quad A^{*2} A^2 \ge A^* A A^* A.$$

Hence

$$\begin{aligned} \beta_2(A) = I - 2A^* A + A^{*2} A^2 &\ge I - 2A^* A + A^* A A^* A \\ &= (I - A^* A)^2 \ge 0. \end{aligned}$$

The proof is complete. □

The study of hyponormal Toeplitz operators on H^2 and other spaces are extensive [14], [15], [29]. We need to introduce Hankel operators on A^2_α $(\alpha > -1)$. If $f \in L^\infty$, then the Hankel operator $H_f : A^2_\alpha \to L^2(\mathbb{D}, dA_\alpha(z)) \ominus A^2_\alpha$ with symbol f is defined by

$$H_f h = [I - P_\alpha] (fh), \quad h \in A^2_\alpha,$$

and the dual Toeplitz operator S_f on $L^2(\mathbb{D}, dA_\alpha(z)) \ominus A^2_\alpha$ is defined by

$$S_f u = [I - P_\alpha] (fu), \quad u \in L^2(\mathbb{D}, dA_\alpha(z)) \ominus A^2_\alpha.$$

Then the following well-known relations between Toeplitz, Hankel and dual Toeplitz operators hold: for $f, g \in L^\infty$,

$$\begin{aligned} T_{fg} - T_f T_g &= H^*_{\overline{f}} H_g; \\ H_{fg} &= H_f T_g + S_f H_g. \end{aligned}$$

In particular, if $f \in H^\infty$, then $H_{fg} = S_f H_g$ for $g \in L^\infty$. Then we have the following sufficient condition for T_f to be a 2-hypercontraction.

Corollary 3.11. *Let $f \in L^\infty$ be such that $\|f\|_\infty \le 1$. If there exists $k \in (H^\infty)_1$ such that $f - k\overline{f} \in H^\infty$, then T_f on A^2_α $(\alpha > -1)$ is a 2-hypercontraction.*

Proof. Let $k \in (H^\infty)_1$ and $g \in H^\infty$ be such that

$$f = k\overline{f} + g.$$

By Lemma 3.10, we just need to show that T_f is hyponormal. Note that

$$\begin{aligned} T_f^* T_f - T_f T_f^* &= T_{\overline{f}} T_f - T_f T_{\overline{f}} = T_{\overline{f} f} - H_f^* H_f - T_{\overline{f} f} + H_{\overline{f}}^* H_{\overline{f}} \\ &= H_{\overline{f}}^* H_{\overline{f}} - H_f^* H_f = H_{\overline{f}}^* H_{\overline{f}} - H_{k\overline{f}}^* H_{k\overline{f}} \\ &= H_{\overline{f}}^* (I - S_k^* S_k) H_{\overline{f}} \ge 0 \quad (\text{since } \|S_k\| = \|k\|_\infty \le 1), \end{aligned}$$

which implies that T_f is hyponormal and hence T_f is a 2-hypercontraction. □

By Corollary 3.11, we get a partial answer to Problem 3.9: if $|a| + |b| \le 1$ and $|b| \le |a|$, then $T_{az+b\overline{z}}$ is a 2-hypercontraction. But the converse is not true in general. C. Cowen [14] showed that T_f on H^2 is hyponormal if and only if there exists $k \in (H^\infty)_1$ such that $f - k\overline{f} \in H^\infty$, but this result does not extend to T_f on A^2_α [30]. Some Toeplitz operators in Example 2.3 are 2-hypercontractions, but they are not hyponormal. The Toeplitz operator T_f with an analytic symbol f admits a simple answer.

Proposition 3.12. *Let $b \in (H^\infty)_1$. Then T_b on A^2_α $(\alpha \ge -1)$ is an m-hypercontraction for any $m \ge 1$.*

Proof. Same as the proof of Proposition 2.18. □

For $b \in (H^\infty)_1$, the above observation allows us to define for any $k \ge 1$,

$$G_{k,\alpha}(\overline{b}) := G_k(T_{\overline{b}}) = \mathcal{M}(\Delta_k(T_b)),$$

where T_b is defined on A^2_α. In particular, for $\alpha \ge -1$, let

$$\mathcal{A}_\alpha(\overline{b}) := \mathcal{H}(A^2_\alpha, T_{\overline{b}}) = G_{1,\alpha}(\overline{b}) \quad \text{and} \quad \mathcal{A}(\overline{b}) \equiv \mathcal{A}_0(\overline{b}).$$

Proposition 3.13. *Let $\varphi \in H^\infty$ and $b \in (H^\infty)_1$. Then for all $m \ge 1$, $G_{m,\alpha}(\overline{b})$ is invariant under $T_{\overline{\varphi}}$ and the norm of the operator $T_{\overline{\varphi}} : G_{m,\alpha}(\overline{b}) \to G_{m,\alpha}(\overline{b})$ does not exceed $\|\varphi\|_\infty$.*

Proof. Assume $\varphi \in (H^\infty)_1$. We need to show $T_{\overline{\varphi}} G_{m,\alpha}(\overline{b}) \hookrightarrow G_{m,\alpha}(\overline{b})$. This happens if and only if

$$\begin{aligned} \beta_m(T_b) - T_{\overline{\varphi}} \beta_m(T_b) T_\varphi &= T_{(1-|b|^2)^m} - T_{\overline{\varphi}} T_{(1-|b|^2)^m} T_\varphi \\ &= T_{(1-|b|^2)^m (1-|\varphi|^2)} \ge 0, \end{aligned}$$

where the second equality uses the fact that $T_\psi T_\varphi = T_{\psi\varphi}$ if either $\overline{\psi}$ or φ is in H^∞ and the last inequality holds since $(1 - |b|^2)^m (1 - |\varphi|^2)$ is a positive function on $\mathbb{D}$. □

Problem 3.14. *Let $f \in L^\infty$ be such that $\|f\|_\infty \le 1$. When is T_f on the Bergman space A^2 a 2-hypercontraction?*

We have not found $f \in L^\infty$ with $\|f\|_\infty \le 1$ such that T_f on A^2 is not a 2-hypercontraction.

Next we note that as an order-k de Branges–Rovnyak space, $G_{k,\alpha}(b)$ is a reproducing kernel space on $\mathbb{D}$ with the following kernel.

Theorem 3.15. *Let $b \in (H^\infty)_1$. For $1 \le k \le [\alpha+2]$, the reproducing kernel of $G_{k,\alpha}(b)$, denoted by $F^b_{k,\alpha}(z,w)$, is*

$$F^b_{k,\alpha}(z,w) = \frac{(1-b(z)\overline{b(w)})^k}{(1-z\overline{w})^{\alpha+2}}.$$

Proof. By the idea in (I-3) in [39] or Theorem 16.13 in [24] and (9), the reproducing kernel of $G_{k,\alpha}(b)$ is

$$\begin{aligned} F^b_{k,\alpha}(z,w) = \beta_k(T_b^*)K_\alpha(z,w) &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}T_{b^i}T^*_{b^i}K_\alpha(z,w) \\ &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}b(z)^i\overline{b(w)}^iK_\alpha(z,w) = \frac{(1-b(z)\overline{b(w)})^k}{(1-z\overline{w})^{\alpha+2}}. \end{aligned}$$

The proof is complete. □

It is a relatively simple fact for two contractions A_1, A_2 on H that $\mathcal{H}(A_1A_2) \hookleftarrow \mathcal{H}(A_1)$, see [24, (16.23)] for a precise relation between $\mathcal{H}(A_1)$ and $\mathcal{H}(A_1A_2)$. But it is not true that $G_2(A_1A_2) \hookleftarrow G_2(A_1)$ for two 2-hypercontractions A_1^*, A_2^* on H. Even $G_2(A_1A_2)$ could be undefined since the product of two 2-hypercontractions is not necessarily a 2-hypercontraction. However, for higher order sub-Bergman spaces, we do have the following result.

Corollary 3.16. *Let $b_1, b_2 \in (H^\infty)_1$. Then on A^2_α,*

(a) *$G_{k,\alpha}(b_1) \hookrightarrow G_{k,\alpha}(b_1b_2)$ for $1 \le k \le [\alpha+2]$;*

(b) *$G_{m,\alpha}(\overline{b_1}) \hookrightarrow G_{m,\alpha}(\overline{b_1b_2})$ for all $m \ge 1$.*

Proof. (a) By Theorem 3.15 and Lemma 3.2, for $1 \le k \le [\alpha+2]$, the relation $G_{k,\alpha}(b_1) \hookrightarrow G_{k,\alpha}(b_1b_2)$ follows from

$$\begin{aligned} &\frac{(1-b_1(z)b_2(z)\overline{b_1(w)b_2(w)})^k}{(1-z\overline{w})^{\alpha+2}} - \frac{(1-b_1(z)\overline{b_1(w)})^k}{(1-z\overline{w})^{\alpha+2}} \\ &= \frac{b_1(z)\big(1-b_2(z)\overline{b_2(w)}\big)\overline{b_1(w)}\left\{\sum_{i=0}^{k-1}s(z,w)^it(z,w)^{k-1-i}\right\}}{(1-z\overline{w})^{\alpha+2}} \succeq 0, \end{aligned}$$

where $s(z,w) = (1-b_1(z)b_2(z)\overline{b_1(w)b_2(w)})$, $t(z,w) = (1-b_1(z)\overline{b_1(w)})$, and

$$\begin{aligned} &\frac{b_1(z)\big(1-b_2(z)\overline{b_2(w)}\big)\overline{b_1(w)}s(z,w)^it(z,w)^{k-1-i}}{(1-z\overline{w})^{\alpha+2}} \\ &= \frac{b_1(z)\big(1-b_2(z)\overline{b_2(w)}\big)\overline{b_1(w)}}{(1-z\overline{w})}\frac{s(z,w)^i}{(1-z\overline{w})^i}\frac{t(z,w)^{k-1-i}}{(1-z\overline{w})^{k-1-i}}\frac{1}{(1-z\overline{w})^{\alpha+2-k}} \succeq 0, \end{aligned}$$

since each factor is a positive kernel by the result on H^2.

(b) The relation $G_{m,\alpha}(\overline{b_1}) \hookrightarrow G_{m,\alpha}(\overline{b_1b_2})$ simply follows from

$$\beta_m(T_{b_1}) = T_{(1-|b_1|^2)^m} \le T_{(1-|b_1b_2|^2)^m} = \beta_m(T_{b_1b_2}).$$

This completes the proof. □

If $b(z) = b_1(z)b_2(z)$, where $b_1, b_2 \in (H^\infty)_1$, the following simple result illustrates a relation among $G_{k,\alpha}(b_1b_2)$, $G_{k,\alpha}(b_1)$, and $G_{k,\alpha}(b_2)$.

Corollary 3.17. *Let $b(z) = b_1(z)b_2(z)$, where $b_1, b_2 \in (H^\infty)_1$. Then on A^2_α (with $\alpha \geq 0$),*

$$G_{2,\alpha}(b_1b_2) \supseteq G_{2,\alpha}(b_1) + b_1^2 G_{2,\alpha}(b_2).$$

Proof. Note that

$$\begin{aligned}\frac{(1-b(z)\overline{b(w)})^2}{(1-z\overline{w})^{\alpha+2}} &= \frac{\Big(1-b_1(z)\overline{b_1(w)}+b_1(z)\big(1-b_2(z)\overline{b_2(w)}\big)\overline{b_1(w)}\Big)^2}{(1-z\overline{w})^{\alpha+2}} \\ &= \frac{(1-b_1(z)\overline{b_1(w)})^2}{(1-z\overline{w})^{\alpha+2}} + \frac{b_1(z)^2\big(1-b_2(z)\overline{b_2(w)}\big)^2\overline{b_1(w)}^2}{(1-z\overline{w})^{\alpha+2}} \\ &\quad + 2b_1(z)\overline{b_1(w)}\frac{1-b_1(z)\overline{b_1(w)}}{1-z\overline{w}}\frac{1-b_2(z)\overline{b_2(w)}}{(1-z\overline{w})^{\alpha+1}} \qquad (20) \\ &\succeq \frac{(1-b_1(z)\overline{b_1(w)})^2}{(1-z\overline{w})^{\alpha+2}} + \frac{b_1(z)^2\big(1-b_2(z)\overline{b_2(w)}\big)^2\overline{b_1(w)}^2}{(1-z\overline{w})^{\alpha+2}},\end{aligned}$$

where the kernel in (20) is positive semi-definite by the assumption $\alpha \geq 0$. By Lemma 3.2(iii), $G_{2,\alpha}(b_1b_2) \supseteq G_{2,\alpha}(b_1) + b_1^2 G_{2,\alpha}(b_2)$. □

Let $\mathrm{Hol}(\overline{\mathbb{D}})$ denote the set of all functions that are analytic on a domain containing the closed unit disk $\overline{\mathbb{D}}$.

Proposition 3.18. *Let $\varphi \in H^\infty$ and $b \in (H^\infty)_1$. Then for $1 \leq k < [\alpha+2]$, $G_{k,\alpha}(b)$ is invariant under T_φ and the norm of the operator $T_\varphi : G_{k,\alpha}(b) \to G_{k,\alpha}(b)$ does not exceed $\|\varphi\|_\infty$. Furthermore, if b is nonextreme, then for $\varphi \in \mathrm{Hol}(\overline{\mathbb{D}})$ and $k = [\alpha+2]$, $G_{k,\alpha}(b)$ is invariant under T_φ.*

Proof. Let $\varphi \in H^\infty$, $b \in (H^\infty)_1$, and $1 \leq k < [\alpha+2]$. Without loss of generality we may assume $\|\varphi\|_\infty = 1$. We need to show $T_\varphi G_{k,\alpha}(b) \hookrightarrow G_{k,\alpha}(b)$. By Corollary 3.5 and Theorem 3.15, it suffices to show that

$$\varphi(z)\overline{\varphi(w)}\frac{(1-b(z)\overline{b(w)})^k}{(1-z\overline{w})^{\alpha+2}} \preceq \frac{(1-b(z)\overline{b(w)})^k}{(1-z\overline{w})^{\alpha+2}}.$$

The above holds since by Lemma 3.2(ii),

$$(1-\varphi(z)\overline{\varphi(w)})\frac{(1-b(z)\overline{b(w)})^k}{(1-z\overline{w})^{\alpha+2}} = \frac{1-\varphi(z)\overline{\varphi(w)}}{1-z\overline{w}}\frac{(1-b(z)\overline{b(w)})^k}{(1-z\overline{w})^{\alpha+1}} \succeq 0,$$

where for $1 \leq k < [\alpha+2]$,

$$\frac{1-\varphi(z)\overline{\varphi(w)}}{1-z\overline{w}} \succeq 0 \quad \text{and} \quad \frac{(1-b(z)\overline{b(w)})^k}{(1-z\overline{w})^{\alpha+1}} \succeq 0.$$

If b is nonextreme and $\varphi \in \mathrm{Hol}(\overline{\mathbb{D}})$, then φ is a multiplier of de Branges–Rovnyak space $\mathcal{H}(b)$ by Theorem 24.6 of [24]. Hence by Corollary 3.5,

$$\varphi(z)\overline{\varphi(w)}\frac{1-b(z)\overline{b(w)}}{1-z\overline{w}} \preceq \gamma\frac{1-b(z)\overline{b(w)}}{1-z\overline{w}}$$

for some positive constant γ. Therefore, multiplying the above relation by an appropriate kernel, we have

$$\varphi(z)\overline{\varphi(w)}\frac{(1-b(z)\overline{b(w)})^{[\alpha+2]}}{(1-z\overline{w})^{\alpha+2}} \preceq \gamma\frac{(1-b(z)\overline{b(w)})^{[\alpha+2]}}{(1-z\overline{w})^{\alpha+2}}.$$

That is, $G_{[\alpha+2],\alpha}(b)$ is invariant under T_φ. □

We remark the above result is sharp in the sense that T_φ does not necessarily map $G_{k,\alpha}(b)$ into $G_{k,\alpha}(b)$ for $k = [\alpha + 2]$. For example, when α is a nonnegative integer, $G_{\alpha+2,\alpha}(b)$ is finite dimensional in the case b is a finite Blaschke product (see Theorem 4.18), so that it is clear that T_z does not map $G_{\alpha+2,\alpha}(b)$ into $G_{\alpha+2,\alpha}(b)$.

When α is an integer, the reproducing kernel $F^b_{\alpha+2,\alpha}(z,w)$ is simply the power of the reproducing kernel of $\mathcal{H}(b)$ of H^2.

Corollary 3.19. *When $\alpha = m \geq 0$ is an integer, then*

$$F^b_{m+2,m}(z,w) = \left(\frac{1 - b(z)\overline{b(w)}}{1 - z\overline{w}}\right)^{m+2}.$$

For two reproducing kernel Hilbert spaces, if one kernel is the power of the other kernel, then it may seem these two spaces will be similar. But the classical examples of Hardy space H^2 and Bergman space A^2 tell a different story in that the functions theory of these two spaces and related operators on them such as Toeplitz and Hankel operators are significantly different. Therefore it is significant to consider the higher-order sub-Bergman spaces $G_{k,\alpha}(b)$, in particular when b is an inner function. For example, in view of the rich function theory of $\mathcal{H}(b)$ [24] [39], it is interesting to study the function theory of $G_{k,\alpha}(b)$.

The $G_{k,\alpha}(\overline{b})$ is also a reproducing kernel space on $\mathbb{D}$ with the following kernel.

Proposition 3.20. *Let $b \in (H^\infty)_1$. For all $k \geq 1$, the reproducing kernel of $G_{k,\alpha}(\overline{b})$, denoted by $F^{\overline{b}}_{k,\alpha}(z,w)$, is*

$$F^{\overline{b}}_{k,\alpha}(z,w) = \int_{\mathbb{D}} \frac{(1 - |b(u)|^2)^k}{(1 - z\overline{u})^{\alpha+2}(1 - u\overline{w})^{\alpha+2}}\, dA_\alpha(u).$$

Proof. The proof is similar to Proposition 3.3 in [43]. By the idea in (I-3) of [39] or Theorem 16.13 in [24] and (9), the reproducing kernel of $G_{k,\alpha}(\overline{b})$ is

$$\begin{aligned} F^{\overline{b}}_{k,\alpha}(z,w) &= \beta_k(T_b)K_\alpha(z,w) \\ &= T_{(1-|b|^2)^k}K_\alpha(z,w) = P_\alpha\big((1 - |b(z)|^2)^k K_\alpha(z,w)\big). \end{aligned}$$

Now the result follows from (18). □

In the study of de Branges–Rovnyak spaces on H^2 [24], [39], the connection between $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ plays an important role. Similarly, on the Bergman space A^2 [12], [41], [43], [44], the connection between $\mathcal{A}(b)$ and $\mathcal{A}(\overline{b})$ also plays a crucial role.

It was shown by Zhu [43] that on the Bergman space

$$\mathcal{A}(b) = G_{1,0}(b) \supseteq \mathcal{A}(\overline{b}) = G_{1,0}(\overline{b}) \supseteq H^\infty. \tag{21}$$

Based on Zhu's characterization of multipliers on $\mathcal{A}(b)$ and $\mathcal{A}(\overline{b})$, Chu [12] observed that

$$\mathcal{A}(b) \approx \mathcal{A}(\overline{b}).$$

That is, $\mathcal{A}(b)$ and $\mathcal{A}(\overline{b})$ are equal as a set and have equivalent norms. Recently, the relation (21) is strengthened by Chu [13] to

$$\mathcal{A}(b) \approx \mathcal{A}(\overline{b}) \supseteq H^2.$$

The similar relation between $G_{k,\alpha}(b)$ and $G_{k,\alpha}(\overline{b})$ does not hold for $k \geq 2$. In fact we will see that $G_{k,\alpha}(b)$ could be finite dimensional when b is a finite Blaschke product (Theorem 4.18) and $G_{k,\alpha}(\overline{b})$ is always infinite dimensional if b is not a constant (Theorem 5.7).

3.3 Mixed higher-order de Branges–Rovnyak spaces

Let $A^* \in \mathcal{B}(H)$ be a 2-hypercontraction. By Lemma 2.11, A maps $G_1(A)$ into $G_1(A)$ and A as an operator in $\mathcal{B}(G_1(A))$ is a contraction. This gives rise to the second order de Branges–Rovnyak space $G_2(A)$. Surprisingly, A also maps $G_1(A^*)$ into $G_1(A^*)$ and A as an operator in $\mathcal{B}(G_1(A^*))$ is a contraction. This will give rise to the space $\mathcal{H}(\mathcal{H}(H, A^*), A)$. We may call this space a *mixed second order de Branges–Rovnyak space.*

Theorem 3.21. *Let $A \in \mathcal{B}(H)$ be a contraction. If $B \in \mathcal{B}(H)$ is a contraction such that $BG_1(A) \hookrightarrow G_1(A)$ and $BA = AB$, then $BG_1(A^*) \hookrightarrow G_1(A^*)$.*

Proof. Let $h \in G_1(A^*) = \mathcal{H}(A^*)$. Then by Theorem 16.18 in [24], $Ah \in G_1(A)$. By assumption $BG_1(A) \hookrightarrow G_1(A)$, so that $BAh \in G_1(A)$. Now $A(Bh) = BAh \in G_1(A)$ and Theorem 16.18 in [24] again imply $Bh \in G_1(A^*)$. To see that $BG_1(A^*) \hookrightarrow G_1(A^*)$, by Theorem 16.18 in [24], for $h \in G_1(A^*)$,

$$\|Bh\|^2_{G_1(A^*)} = \|Bh\|^2_H + \|BAh\|^2_{G_1(A)} \leq \|h\|^2_H + \|Ah\|^2_{G_1(A)} = \|h\|^2_{G_1(A^*)}\,,$$

where the inequality follows from the fact that B is a contraction on H and B is a contraction as an operator on $G_1(A)$. □

Applying Theorem 3.21 with $B = A$ yields what we remarked just before Theorem 3.21.

Corollary 3.22. *Let $A^* \in \mathcal{B}(H)$ be a 2-hypercontraction. Then $AG_1(A^*) \hookrightarrow G_1(A^*)$.*

Remark 3.23. *The above corollary can be stated in terms of operator inequalities as follow: for $A \in \mathcal{B}(H)$,*

$$(I - AA^*) \geq 0 \text{ and } \left(I - 2AA^* + A^2A^{*2}\right) \geq 0 \implies A\,(I - A^*A)\,A^* \leq (I - A^*A)\,.$$

The following result generalizes Theorem 1.1 in [12] where this result is proved in the context of sub-Bergman space inside A^2.

Proposition 3.24. *If $A^* \in \mathcal{B}(H)$ is a 2-hypercontraction and A is hyponormal, then $G_1(A) = G_1(A^*)$ as a set and for $h \in G_1(A^*)$,*

$$\|h\|_{G_1(A)} \leq \|h\|_{G_1(A^*)} \leq \sqrt{2}\,\|h\|_{G_1(A)}\,.$$

Proof. The first inequality follows from $G_1(A^*) \hookrightarrow G_1(A)$ by (3) since A is hyponormal. Now

$$\|h\|^2_{G_1(A^*)} = \|h\|^2_H + \|Ah\|^2_{G_1(A)} \leq \|h\|^2_{G_1(A)} + \|h\|^2_{G_1(A)} = 2\,\|h\|^2_{G_1(A)}\,,$$

where the first equality follows from 16.18 in [24] and the second result follows from $G_1(A) \hookrightarrow H$ and $AG_1(A) \hookrightarrow G_1(A)$. □

If $A \in \mathcal{B}(H)$ is such that both A and A^* are 2-hypercontraction, then there are four second order de Branges–Rovnyak spaces,

$$G_2(A),\ G_2(A^*),\ \mathcal{H}(\mathcal{H}(H, A), A^*),\ \mathcal{H}(\mathcal{H}(H, A^*), A).$$

For example, since by Corollary 3.8 and Proposition 3.12, for $b \in (H^\infty)_1$, both T_b and T_b^* are 2-hypercontractions on the Bergman space A^2, there are four second order sub-Bergman spaces,

$$\mathcal{H}(\mathcal{H}(A^2, T_b), T_b),\ \mathcal{H}(\mathcal{H}(A^2, T_b^*), T_b^*),\ \mathcal{H}(\mathcal{H}(A^2, T_b^*), T_b),\ \mathcal{H}(\mathcal{H}(A^2, T_b), T_b^*).$$

It is natural to ask if $A \in \mathcal{B}(H)$ is such that both A and A^* are k-hypercontractions, which of the following 2^k order-k de Branges–Rovnyak spaces are defined,

$$\mathcal{H}(\mathcal{H}(\cdots \mathcal{H}(\mathcal{H}(H, S_1), S_2), \cdots), S_k)$$

where $\mathcal{H}$ is repeated k-times and S_i is either A or A^*.

4 Higher-order sub-Bergman spaces associated to a finite Blaschke product

4.1 Identifying $G_{k,\alpha}(b)$ when b is a finite Blaschke product

When b is a finite Blaschke product, $\mathcal{H}(b)$ is the finite dimensional model space inside H^2 and $\mathcal{H}(\bar{b}) = \{0\}$. On the Bergman space, Zhu [44] proved that

$$\mathcal{A}(b) \approx \mathcal{A}(\bar{b}) \approx H^2, \tag{22}$$

when b is a finite Blaschke product. The above result was extended to weighted Bergman spaces A^2_α for any $\alpha > -1$ in [41]:

$$\mathcal{A}_\alpha(b) \approx \mathcal{A}_\alpha(\bar{b}) \approx A^2_{\alpha-1} \quad (\alpha > -1)$$

(see also a different proof of the result on A^2_α in [1]).

In this section, we prove and extend the results in [41] and [44] to $G_{k,\alpha}(b)$. In fact we obtain a surprisingly general result for a k-hypercontraction T which reduces the identification of $G_k(b(T))$ to $G_k(T)$. We first recall a formula which is essentially contained in [26]. Let $a \in \mathbb{D}$ and let $\varphi_a(z)$ be the automorphism of the disk,

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}. \tag{23}$$

Let $a_i \in \mathbb{D}$ for $1 \le i \le n$. We do not require a_i's to be distinct. The proof of the following lemma is obtained by a direct computation and is similar to the proof of Lemma 2.2 in [26]. The proof also sets up notation for future use.

Lemma 4.1. *Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}(z)$. Then the following holds for $m \ge 1$:*

$$\left(\frac{1 - b(z)\overline{b(w)}}{1 - z\overline{w}} \right)^m = \sum_{m_1 + \cdots + m_n = m} g_{(m_1,\cdots,m_n)}(z)\overline{g_{(m_1,\cdots,m_n)}(w)}, \tag{24}$$

where

$$g_{(m_1,\cdots,m_n)}(z) := \sqrt{c(m_1,\cdots,m_n)\gamma(m_1,\cdots,m_n)} \left(\prod_{i=1}^{n} b_i^{m_i}(z)(1 - \overline{a_i}z)^{-m_i} \right), \tag{25}$$

where $c(m_1,\cdots,m_n)$ and $\gamma(m_1,\cdots,m_n)$ are constants defined by

$$c(m_1,\cdots,m_n) := \binom{m}{m_1,\cdots,m_n} \quad \text{and} \quad \gamma(m_1,\cdots,m_n) := \prod_{i=1}^{n} \left(1 - |a_i|^2\right)^{m_i} \tag{26}$$

and $b_i(z)$ is defined by

$$b_i(z) = \begin{cases} 1 & (i = 1), \\ \prod_{j=1}^{i-1} \varphi_{a_j}(z) & (i = 2,\cdots,n). \end{cases} \tag{27}$$

Proof. Note that

$$1 - \varphi_a(z)\overline{\varphi_a(w)} = \left(1 - |a|^2\right)(1 - z\overline{w})(1 - \bar{a}z)^{-1}(1 - a\overline{w})^{-1}.$$

Raising the above identity to the power m, we have

$$\left(\frac{1-\varphi_a(z)\overline{\varphi_a(w)}}{1-z\overline{w}}\right)^m = \left(1-|a|^2\right)^m (1-\overline{a}z)^{-m}(1-a\overline{w})^{-m}. \tag{28}$$

Now for a finite Blaschke product $b(z)=\prod_{i=1}^{n}\varphi_{a_i}(z)$,

$$\begin{aligned} 1-b(z)\overline{b(w)} &= 1-\prod_{i=1}^{n}\varphi_{a_i}(z)\overline{\varphi_{a_i}(w)} \\ &= \sum_{i=1}^{n}\left(\prod_{j=1}^{i-1}\varphi_{a_j}(z)\right)\left(1-\varphi_{a_i}(z)\overline{\varphi_{a_i}(w)}\right)\left(\prod_{j=1}^{i-1}\overline{\varphi_{a_j}(w)}\right) \\ &= \sum_{i=1}^{n} b_i(z)\left(1-\varphi_{a_i}(z)\overline{\varphi_{a_i}(w)}\right)\overline{b_i(w)}, \end{aligned}$$

where b_i is given by (27). By the multinomial formula,

$$\begin{aligned} &\left(\frac{1-b(z)\overline{b(w)}}{1-z\overline{w}}\right)^m \\ &= \sum_{m_1+\cdots+m_n=m} c(m_1,\cdots,m_n)\prod_{i=1}^{n} b_i^{m_i}(z)\left(\frac{1-\varphi_{a_i}(z)\overline{\varphi_{a_i}(w)}}{1-z\overline{w}}\right)^{m_i}\overline{b_i^{m_i}(w)}. \end{aligned} \tag{29}$$

By (28), we also have

$$\prod_{i=1}^{n}\left(\frac{1-\varphi_{a_i}(z)\overline{\varphi_{a_i}(w)}}{1-z\overline{w}}\right)^{m_i} = \gamma\, d(z)\overline{d(w)},$$

where $\gamma := \gamma(m_1,\cdots,m_n)$ and $d(z) := d_{(m_1,\cdots,m_n)}(z) = \prod_{i=1}^{n}(1-\overline{a_i}z)^{-m_i}$. The lemma now follows from (29). $\square$

Let $T\in\mathcal{B}(H)$ be a contraction. Lemma 2.2 in Gu [26] expresses $\beta_m(b(T))$ in terms of $\beta_m(T)$. Below if $g(z)$ is an analytic function in the neighborhood of $\sigma(T)$ (the spectrum of T), then $g(T)$ is the Riesz–Dunford functional calculus of T.

Lemma 4.2. [26] *Let $T\in\mathcal{B}(H)$ be a contraction and $b(z)=\prod_{i=1}^{n}\varphi_{a_i}(z)$. Then*

$$\begin{aligned} \beta_m(b(T)) &= \sum_{m_1+\cdots+m_n=m} g_{(m_1,\cdots,m_n)}(T)^*\beta_m(T)g_{(m_1,\cdots,m_n)}(T); \\ \beta_m(b(T)^*) &= \sum_{m_1+\cdots+m_n=m} g_{(m_1,\cdots,m_n)}(T)\beta_m(T^*)g_{(m_1,\cdots,m_n)}(T^*). \end{aligned}$$

Corollary 4.3. [26] *If $T\in\mathcal{B}(H)$ is a k-hypercontraction, then for any finite Blaschke product b, $b(T)\in\mathcal{B}(H)$ is a k-hypercontraction.*

We conjecture that if $T\in\mathcal{B}(H)$ is a k-hypercontraction, then for any $b\in(H^\infty)_1$ such that $b(T)\in\mathcal{B}(H)$, $b(T)$ is a k-hypercontraction. See Theorem 3.7 above and also Theorem 4.6 of [26].

The above lemma also immediately implies the following result on A^2_α. Note that if $T=T_z$ on A^2_α, then $b(T)=b(T_z)=T_{b(z)}$.

Corollary 4.4. *Let $b(z)=\prod_{i=1}^{n}\varphi_{a_i}(z)$. Then on A_α^2,*

$$\beta_m(T_b)=\sum_{m_1+\cdots+m_n=m} T^*_{g_{(m_1,\cdots,m_n)}(z)}\beta_m(T_z)T_{g_{(m_1,\cdots,m_n)}(z)};$$
$$\beta_m(T_b^*)=\sum_{m_1+\cdots+m_n=m} T_{g_{(m_1,\cdots,m_n)}(z)}\beta_m(T_z^*)T^*_{g_{(m_1,\cdots,m_n)}(z)}.$$

Next we are going to find the relationship between $G_m(b(T))$ and $G_m(T)$. We first prove a lemma on the sum of operator range spaces which is similar to Theorem 16.22 in [24].

Lemma 4.5. *Let $A_i\in\mathcal{B}(H)$ for $1\le i\le n$. Set*

$$C:=(A_1A_1^*+\cdots+A_nA_n^*)^{1/2}.$$

Then $\mathcal{M}(C)=\mathcal{M}(D)$, where $D:H\oplus\cdots\oplus H\to H$ is the row operator defined by

$$D\begin{bmatrix}h_1\\ \vdots\\ h_n\end{bmatrix}=\begin{bmatrix}A_1 & \cdots & A_n\end{bmatrix}\begin{bmatrix}h_1\\ \vdots\\ h_n\end{bmatrix}=A_1h_1+\cdots+A_nh_n\quad(h_i\in H).$$

In particular, $\mathcal{M}(C)=\bigvee_{i=1}^{n}\mathcal{M}(A_i)$ as a set.

Proof. Note that $D^*:H\to H\oplus\cdots\oplus H$ is the column operator defined by

$$D^*h=\begin{bmatrix}A_1^*\\ \vdots\\ A_n^*\end{bmatrix}h=\begin{bmatrix}A_1^*h\\ \vdots\\ A_n^*h\end{bmatrix}.$$

Then

$$CC^*=A_1A_1^*+\cdots+A_nA_n^*=DD^*.$$

By Corollary 16.8 of [24], $\mathcal{M}(C)=\mathcal{M}(D)$. □

Lemma 4.6. *If $T\in\mathcal{B}(H)$ is a k-hypercontraction and $b(z)=\prod_{i=1}^{n}\varphi_{a_i}(z)$, then for $1\le m\le k$,*

$$(1-\overline{a_1}T)^{-m}G_m(T)\subseteq G_m(b(T))=\bigvee_{m_1+\cdots+m_n=m} g_{(m_1,\cdots,m_n)}(T)G_m(T);$$
$$(1-a_1T^*)^{-m}G_m(T^*)\subseteq G_m(b(T)^*)=\bigvee_{m_1+\cdots+m_n=m} g_{(m_1,\cdots,m_n)}(T)^*G_m(T^*).$$

Proof. By Lemma 4.2,

$$\Delta_m(b(T)^*)=\left(\sum_{m_1+\cdots+m_n=m} g_{(m_1,\cdots,m_n)}(T)\beta_m(T^*)g_{(m_1,\cdots,m_n)}(T^*)\right)^{1/2}.$$

It thus follows from Lemma 4.5 that

$$G_m(b(T))=\Delta_m(b(T)^*)H=\bigvee_{m_1+\cdots+m_n=m} g_{(m_1,\cdots,m_n)}(T)\Delta_m(T^*)H.$$

This proves the equality in the lemma. For the inclusion, note that

$$g_{(m,0,\cdots,0)}(T)\Delta_m(T^*)H=g_{(m,0,\cdots,0)}(T)G_m(T),$$

where $g_{(m,0,\cdots,0)}(T)=\mu(1-\overline{a_1}T)^{-m}$ for some positive constant μ and $g_{(m,0,\cdots,0)}(T)\in\mathcal{B}(H)$ is an invertible operator. □

Theorem 4.7. *Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}(z)$ be a finite Blaschke product. If $T^* \in \mathcal{B}(H)$ is a k-hypercontraction $(k \geq 1)$, then*

$$G_m(b(T)) = G_m(T) \quad \text{for } 1 \leq m \leq k-1.$$

If $TG_k(T) \subseteq G_k(T)$ and $(I-\overline{a_i}T)^{-1}G_k(T) \subseteq G_k(T)$ for $1 \leq i \leq n$, we also have $G_k(b(T)) = G_k(T)$. Similarly, if $T \in \mathcal{B}(H)$ is a k-hypercontraction, then

$$G_m(b(T)^*) = G_m(T^*) \quad \text{for } 1 \leq m \leq k-1.$$

If $T^*G_k(T^*) \subseteq G_k(T^*)$ and $(I - a_iT^*)^{-1}G_k(T) \subseteq G_k(T)$ for $1 \leq i \leq n$, we also have $G_k(b(T)^*) = G_k(T^*)$.

Proof. By Lemma 2.11, for $1 \leq m \leq k-1$, $TG_m(T) \hookrightarrow G_m(T)$. Hence each $g_{(m_1,\cdots,m_n)}(T)$ is a bounded operator on $G_m(T)$ and

$$g_{(m_1,\cdots,m_n)}(T)G_m(T) \subseteq G_m(T) \quad \text{and} \quad G_m(b(T)) \subseteq G_m(T),$$

Since $g_{(m,0,\cdots,0)}(T)G_m(T) \subseteq G_m(T)$ and $g_{(m,0,\cdots,0)}(T)$ is invertible, $g_{(m,0,\cdots,0)}(T)G_m(T) = G_m(T)$ and $G_m(b(T)) \supseteq G_m(T)$.

If $TG_k(T) \subseteq G_k(T)$ and $(I - \overline{a_i}T)^{-1}G_k(T) \subseteq G_k(T)$ for $1 \leq i \leq n$, then again each $g_{(m_1,\cdots,m_n)}(T)$ is a bounded operator on $G_k(T)$. Similarly, we also have $G_k(b(T)) = G_k(T)$. $\square$

If the condition $TG_k(T) \subseteq G_k(T)$ in the above theorem is not satisfied, then $G_k(b(T))$ could be different from $G_k(T)$ as we will see below (cf. Theorem 4.16 and Proposition 4.17).

Corollary 4.8. *Let $b(z)$ be a finite Blaschke product. If $T \in \mathcal{B}(H)$ is a 2-hypercontraction, then*

$$\mathcal{H}(b(T)) \approx \mathcal{H}(T).$$

Corollary 4.9. *Let $\varphi \in (H^\infty)_1$ be nonextreme and let $b(z)$ be a finite Blaschke product. Then*

$$\mathcal{H}(b \circ \varphi) \approx \mathcal{H}(\varphi).$$

Proof. Let $T = T_\varphi$ on H^2. If $\varphi \in (H^\infty)_1$ is nonextreme, then $T_\varphi \mathcal{H}(\varphi) \subseteq \mathcal{H}(\varphi)$ [24], [39]. The result follows from the previous theorem. $\square$

We remark that the relation $\mathcal{H}(\overline{b \circ \varphi}) \approx \mathcal{H}(\overline{\varphi})$ holds for any $\varphi \in (H^\infty)_1$ since $T_{\overline{\varphi}}\mathcal{H}(\overline{\varphi}) \subseteq \mathcal{H}(\overline{\varphi})$ always holds.

Theorem 4.10. *Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}(z)$. Then on A^2_α $(\alpha > -1)$,*

(i) *For $1 \leq k \leq [\alpha + 2] - 1$,*

$$G_{k,\alpha}(b) \approx A^2_{\alpha-k}.$$

(ii) *If α is not an integer, then for $k = [\alpha + 2]$,*

$$G_{k,\alpha}(b) \approx A^2_{\alpha-k}.$$

(iii) *If α is an integer, then for $k = \alpha + 2$, $G_{k,\alpha}(b)$ is a finite dimensional Hilbert space.*

Proof. By Corollary 3.8, T_b^* is an $[\alpha+2]$-hypercontraction. Thus it follows from Theorem 4.7 that $G_{k,\alpha}(b) \approx G_{k,\alpha}(z)$ for $1 \le k \le [\alpha+2]-1$. For (i), by Theorem 3.15, the reproducing kernel of $G_{k,\alpha}(z)$ is $1/(1-z\overline{w})^{\alpha+2-k}$. Hence $G_{k,\alpha}(b) \approx A^2_{\alpha-k}$ for $1 \le k \le [\alpha+2]-1$.

We now prove (ii). In this case, $0 < \alpha+2-k < 1$, so the set of multipliers of $A^2_{\alpha-k}$ is a proper subset of H^∞. However since $g_{(m_1,\cdots,m_n)}(z)$ is a rational function, it is easy to see that $g_{(m_1,\cdots,m_n)}(z)$ is still a multiplier of $A^2_{\alpha-k}$. It is also easy to see that $1/g(z)$ is a multiplier of $A^2_{\alpha-k}$. Hence the proof of (ii) is the same as the proof of (i) with the modification about multipliers.

In the case (iii), $\alpha+2-k=0$. The reproducing kernel of $G_{k,\alpha}(b)$ is (24), which is an integer power of the reproducing kernel of $\mathcal{H}(b)$, which is finite dimensional. Thus (iii) follows at once from Lemma 3.3. □

Similarly, Theorem 4.7 gives that $G_{k,\alpha}(\overline{b}) \approx G_{k,\alpha}(\overline{z})$ which we shall compute in Theorem 5.7 below.

The above theorem can be strengthened, as this was done on A^2 for $\mathcal{A}(b)$ [44] and on A^2_α for $\mathcal{A}_\alpha(b)$ [13] by using the key result of Lemma 1 in [44].

Theorem 4.11. *Let $b \in (H^\infty)_1$. Then on A^2_α $(\alpha > -1)$, any one of the following three statements holds if and only if b is a finite Blaschke product:*

(i) *For $1 \le k \le [\alpha+2]-1$, $G_{k,\alpha}(b) \approx A^2_{\alpha-k}$;*

(ii) *If α is not an integer, then for $k=[\alpha+2]$, $G_{k,\alpha}(b) \approx A^2_{\alpha-k}$;*

(iii) *If α is an integer, then for $k=\alpha+2$, $G_{k,\alpha}(b)$ is a finite dimensional Hilbert space.*

Proof. We will prove (ii) since the proof of (i) is similar. Assume α is not an integer, and for $k=[\alpha+2]$, $G_{k,\alpha}(b) \approx A^2_{\alpha-k}$. By Lemma 3.2(iii) and Theorem 3.15,

$$\frac{(1-b(z)\overline{b(w)})^{[\alpha+2]}}{(1-z\overline{w})^{\alpha+2}} \preceq \gamma \frac{1}{(1-z\overline{w})^{\alpha+2-[\alpha+2]}}$$

for some positive constant γ. Now

$$\frac{(1-|b(z)|^2)^{[\alpha+2]}}{(1-|z|^2)^{\alpha+2}} \le \gamma \frac{1}{(1-|z|^2)^{\alpha+2-[\alpha+2]}}.$$

Equivalently,

$$\frac{1-|b(z)|^2}{1-|z|^2} \le \gamma^{1/[\alpha+2]} \quad \text{for } z \in \mathbb{D}.$$

By Lemma 1 in [44], this implies that b is a finite Blaschke product.

Next we prove (iii). By Corollary 3.19, the reproducing kernel of $G_{\alpha+2,\alpha}(b)$ is simply an integer $(=\alpha+2)$ power of the reproducing kernel of de Branges–Rovnyak space $\mathcal{H}(b)$. Since $\mathcal{H}(b)$ is finite dimensional if and only if b is a finite Blaschke product, it is easy to see that $G_{\alpha+2,\alpha}(b)$ is finite dimensional if and only if b is a finite Blaschke product. □

The above result says that $G_{k,\alpha}(b)$ is finite dimensional if and only if α is an integer, $k=\alpha+2$, and b is a finite Blaschke product. We remark that the proof of above result and Theorem 4.12 also demonstrate a different way of obtaining Theorem 4.10.

Next we extend the relation $\mathcal{A}_\alpha(b) = G_{1,\alpha}(b) \supseteq A^2_{\alpha-1}$ for $\alpha \ge 0$ which is Theorem 3.1 in Chu [13]. This also gives a completely different proof of the result of Chu.

Theorem 4.12. *Let $b \in (H^\infty)_1$ be not a constant. Then for $1 \le k < [\alpha+2]$, $G_{k,\alpha}(b) \supseteq A^2_{\alpha-k}$. In particular, for $1 \le k < [\alpha+2]$, $G_{k,\alpha}(b) \supseteq H^2$.*

Proof. Note that T_b^* is a $[\alpha+2]$-hypercontraction by Corollary 3.8. Thus by Theorem 4.7, for $1 \leq k < [\alpha+2]$, $G_{k,\alpha}(b) \approx G_{k,\alpha}(\varphi(b))$ for any finite Blaschke product $\varphi(z)$. In particular, if $\varphi = (z-b(0))/(1-\overline{b(0)}z)$, then $\varphi(b(z)) = zb_1(z)$ for some $b_1 \in (H^\infty)_1$. By Corollary 3.16,

$$G_{k,\alpha}(b) \approx G_{k,\alpha}(\varphi(b)) = G_{k,\alpha}(zb_1) \supseteq G_{k,\alpha}(z).$$

Now the result follows since the reproducing kernel of $G_{k,\alpha}(z)$ is the same as the reproducing kernel of $A^2_{\alpha-k}$ by Theorem 3.15. □

See Theorem 5.8 for an analogue of Theorem 4.12 for $G_{k,\alpha}(\bar{b})$ which is valid for all $k \geq 1$. The short proof of Theorem 5.8 is similar to the proof of Theorem 4.12 which demonstrates the power of the abstract Theorem 4.7.

4.2 Finite dimensional higher-order sub-Bergman spaces

The case (iii) of Theorem 4.11 and (24) naturally leads to the following question.

Problem 4.13. *For $m \geq 0$ an integer, is*

$$G := \left\{g_{(m_1,\cdots,m_n)}(z) : m_1 + \cdots + m_n = m+2\right\}$$

an orthonormal basis of $G_{m+2,m}(b)$?

By the multinomial formula, the cardinality of the set G is $\binom{m+n+1}{m+2}$. We first note that the space $G_{m+2,m}(b)$ is spanned by the above set G.

Proposition 4.14. *Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}(z)$. The following holds:*

$$G_{m+2,m}(b) = \operatorname{Span}\left\{g_{(m_1,\cdots,m_n)}(z) : m_1 + \cdots + m_n = m+2\right\}.$$

Proof. Recall the reproducing kernel of $G_{m+2,m}(b)$ is given by (24). Thus $G_{m+2,m}(b)$ is contained in the right-hand side of the above equation. On the other hand, it is clear that

$$\left(\frac{1-b(z)\overline{b(w)}}{1-z\overline{w}}\right)^m \succeq g_{(m_1,\cdots,m_n)}(z)\overline{g_{(m_1,\cdots,m_n)}(w)}.$$

By Lemma 3.2(iv), $g_{(m_1,\cdots,m_n)}(z) \in G_{m+2,m}(b)$. The proof is complete. This result can also be derived from Lemma 5.5 below. □

But it seems difficult to figure out an orthonormal basis of $G_{m+2,m}(b)$ or even the dimension of $G_{m+2,m}(b)$. So even though the reproducing kernel of $G_{m+2,m}(b)$ is just the power of the reproducing kernel of $\mathcal{H}(b)$, it is more difficult to study $G_{m+2,m}(b)$.

By Proposition 4.14, the set

$$\left\{g_{(m_1,\cdots,m_n)}(z) : m_1 + \cdots + m_n = m+2\right\} \tag{30}$$

is a candidate for an orthonormal basis of $G_{m+2,m}(b)$. By Lemma 3.3, we only need to check whether the functions in this set are linearly independent. But it seems difficult to directly work with the above set, and instead we try to find a more convenient linearly independent basis of $G_{m+2,m}(b)$. The following observation is useful.

Lemma 4.15. *Let $H(K)$ be a finite dimensional reproducing kernel Hilbert space of dimension n. Let $\{f_i(z) : 1 \le i \le n\}$ be any algebraic (i.e., vector space) basis of $H(K)$. Then there exist two positive constants γ and δ such that*

$$\gamma K(z,w) \preceq \sum_{i=1}^{n} f_i(z)\overline{f_i(w)} \preceq \delta K(z,w). \tag{31}$$

Hence, moreover, for any natural number m,

$$\gamma^m K(z,w)^m \preceq \left(\sum_{i=1}^{n} f_i(z)\overline{f_i(w)} \right)^m \preceq \delta^m K(z,w)^m. \tag{32}$$

Proof. Let

$$K_1(z,w) = \sum_{i=1}^{n} f_i(z)\overline{f_i(w)}.$$

Then $K_1(z,w)$ is a reproducing kernel and hence, by Lemma 3.3, the dimension of $H(K_1)$ is n. Since two complex Hilbert spaces of dimension n are norm equivalent, the formula (31) follows from Lemma 3.2(iii). The formula (32) follows by repeated applying Lemma 3.2(ii). We demonstrate the proof for $m = 2$.

$$\begin{aligned} \delta K(z,w) \cdot \delta K(z,w) &\succeq \left(\sum_{i=1}^{n} f_i(z)\overline{f_i(w)} \right) \cdot \delta K(z,w) \\ &\succeq \left(\sum_{i=1}^{n} f_i(z)\overline{f_i(w)} \right) \cdot \left(\sum_{i=1}^{n} f_i(z)\overline{f_i(w)} \right). \end{aligned}$$

The proof is complete. $\square$

Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}^{l_i}(z)$, where all the a_i's are distinct and nonzero. Then it is known [24] that the following set is an algebraic basis of $\mathcal{H}(b)$:

$$\left\{ \frac{1}{(1-\overline{a_i}z)^{j_i}} : 1 \le i \le n,\ 1 \le j_i \le l_i \right\}.$$

We can prove more:

Theorem 4.16. *Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}(z)$, where all the a_i's are distinct and nonzero. Then, the set $Q(a_1, \cdots, a_n)$ is an algebraic basis of $G_{m+2,m}(b)$, where*

$$Q(a_1,\cdots,a_n) = \left\{ \prod_{i=1}^{n} \frac{1}{(1-\overline{a_i}z)^{m_i}} : m_1 + \cdots + m_n = m+2 \right\}. \tag{33}$$

Hence the set (30) is an orthonormal basis of $G_{m+2,m}(b)$. The dimension of $G_{m+2,m}(b)$ is $\binom{m+n+1}{m+2}$.

Proof. Note that the numbers of functions in (33) and (30) are the same. So once we prove that $Q(a_1, \cdots, a_n)$ is an algebraic basis of $G_{m+2,m}(b)$, by Lemma 3.3, the set (30) is an orthonormal basis of $G_{m+2,m}(b)$.

Suppose that all the a_i's are distinct and nonzero. It follows from Lemma 3.2(iv), Corollary 3.19 and Lemma 4.15 that

$$\gamma\left(\frac{1-b(z)\overline{b(w)}}{1-z\overline{w}}\right)^m \preceq \left(\sum_{i=1}^n \frac{1}{(1-\overline{a_i}z)}\frac{1}{(1-a_i\overline{w})}\right)^m \preceq \delta\left(\frac{1-b(z)\overline{b(w)}}{1-z\overline{w}}\right)^m$$

for some positive constants γ and δ. By Lemma 3.2(iii), as a vector space,

$$\operatorname{Span} Q(a_1,\cdots,a_n)\subseteq G_{m+2,m}(b).$$

Thus, by Lemma 3.3, it suffices to show that $Q(a_1,\cdots,a_n)$ are linearly independent. Let

$$f(z)=\sum_{i=1}^{m+1} h_i(z)\frac{1}{(1-\overline{a_n}z)^i}+\frac{c_{m+2}}{(1-\overline{a_n}z)^{m+2}}+h(z)\in G_{m+2,m}(b)$$

for some $h(z),h_i(z)\in \operatorname{Span} Q(a_1,\cdots,a_n)$. We need to show that

$$f(z)=0$$

implies that $h_i(z)=c_{m+2}=h(z)=0$. Suppose that $c_{m+2}\neq 0$. Then the function

$$\sum_{i=1}^{m+1} h_i(z)\frac{1}{(1-\overline{a_n}z)^i}+h(z)$$

has a pole of order $m+2$ at $z=\frac{1}{\overline{a_n}}$, a contradiction. Similarly, we can show that $h_{m+1}(z)=0$, $\ldots$, $h_1(z)=0$ and $h(z)=0$. This completes the proof. □

In the case when all the a_i's are distinct and $a_n=0$, similarly, we have

$$G_{m+2,m}(b)=\operatorname{Span}\left\{\prod_{i=1}^n\frac{z^{m_n}}{(1-\overline{a_i}z)^{m_i}} : m_1+\cdots+m_n=m+2\right\}.$$

Thus the set (30) is still an orthonormal basis of $G_{m+2,m}(b)$.

One may ask if the above theorem holds without the assumption on the a_i's. The next simple result shows that the above theorem does not extend, that is, the set (30) in general is not an orthonormal basis of $G_{m+2,m}(b)$.

Proposition 4.17. *Let $b(z)=z^n$. On the A^2_m, there exist positive numbers c_i's such that*

$$\left\{\sqrt{c_i}z^i : 0\le i\le (m+2)(n-1)\right\}$$

is an orthonormal basis of $G_{m+2,m}(z^n)$. The dimension of $G_{m+2,m}(z^n)$ is $(m+2)(n-1)+1$.

Proof. Note that

$$\left(\frac{1-z^n\overline{w}^n}{1-z\overline{w}}\right)^{m+2}=\left(1+z\overline{w}+\cdots+z^{n-1}\overline{w}^{n-1}\right)^{m+2}=\sum_{i=0}^{(m+2)(n-1)} c_i\left(z\overline{w}\right)^i$$

for some positive c_i. The result now follows from Lemma 3.3. □

Here is a closed formula for c_i. Write

$$\begin{aligned}\left(\frac{1-z^n\overline{w}^n}{1-z\overline{w}}\right)^{m+2} &= (1-z^n\overline{w}^n)^{m+2}\frac{1}{(1-z\overline{w})^{m+2}} \\ &= \sum_{j=0}^{m+2}(-1)^j\binom{m+2}{j}(z\overline{w})^{nj}\sum_{q=0}^{\infty}\binom{q+m+1}{m+1}(z\overline{w})^q.\end{aligned}$$

Then for $0 \leq i \leq (m+2)(n-1)$,

$$c_i = \sum_{j=0}^{[i/n]}(-1)^j\binom{m+2}{j}\binom{i-nj+m+1}{m+1}.$$

Next we show that a slightly different way of expanding the reproducing kernel of $G_{m+2,m}(b)$ indeed leads to an orthonormal basis of $G_{m+2,m}(b)$. For simplicity, we only demonstrate the idea on the Bergman space.

Theorem 4.18. *Let $b(z) = \prod_{i=1}^{n}\varphi_{a_i}^{l_i}(z)$ where all the a_i's are distinct. Then the dimension of $G_{2,0}(b)$ is*

$$\sum_{i=1}^{n}(2l_i-1) + \sum_{1\leq i<j\leq n} l_i\, l_j. \tag{34}$$

Proof. Let $b_1(z) = 1$ and

$$b_i(z) = \prod_{j=1}^{i-1}\varphi_{a_j}^{l_j}(z), \quad i = 2,\cdots,n.$$

Write

$$\begin{aligned}1-b(z)\overline{b(w)} &= 1-\prod_{i=1}^{n}\varphi_{a_i}^{l_i}(z)\overline{\varphi_{a_i}^{l_i}(w)} \\ &= \sum_{i=1}^{n}\prod_{j=1}^{i-1}\varphi_{a_j}^{l_j}(z)\left(1-\varphi_{a_i}^{l_i}(z)\overline{\varphi_{a_i}^{l_i}(w)}\right)\prod_{j=1}^{i-1}\overline{\varphi_{a_j}^{l_j}(w)} \\ &= \sum_{i=1}^{n} b_i(z)\left(1-\varphi_{a_i}^{l_i}(z)\overline{\varphi_{a_i}^{l_i}(w)}\right)\overline{b_i(w)}.\end{aligned}$$

Then,

$$\begin{aligned}\left(\frac{1-b(z)\overline{b(w)}}{1-z\overline{w}}\right)^2 &= \sum_{i=1}^{n} b_i^2(z)\left(\frac{1-\varphi_{a_i}^{l_i}(z)\overline{\varphi_{a_i}^{l_i}(w)}}{1-z\overline{w}}\right)^2\overline{b_i^2(w)} \\ &+ 2\sum_{1\leq i<j\leq n} b_i(z)b_j(z)\left(\frac{1-\varphi_{a_i}^{l_i}(z)\overline{\varphi_{a_i}^{l_i}(w)}}{1-z\overline{w}}\right)\left(\frac{1-\varphi_{a_j}^{l_j}(z)\overline{\varphi_{a_j}^{l_j}(w)}}{1-z\overline{w}}\right)\overline{b_i(w)b_j(w)}.\end{aligned} \tag{35}$$

Note that

$$\begin{aligned}
&\left(\frac{1-\varphi_{a_i}^{l_i}(z)\overline{\varphi_{a_i}^{l_i}(w)}}{1-z\overline{w}}\right)^2 \\
&= \left(1+\cdots+\varphi_{a_i}^{l_i-1}(z)\overline{\varphi_{a_i}^{l_i-1}(w)}\right)^2\left(\frac{1-\varphi_{a_i}(z)\overline{\varphi_{a_i}(w)}}{1-z\overline{w}}\right)^2 \\
&= \left(1+c_1\varphi_{a_i}(z)\overline{\varphi_{a_i}(w)}+\cdots+c_{2(l_i-1)}\varphi_{a_i}^{2(l_i-1)}(z)\overline{\varphi_{a_i}^{2(l_i-1)}(w)}\right)\left(\frac{1-\varphi_{a_i}(z)\overline{\varphi_{a_i}(w)}}{1-z\overline{w}}\right)^2
\end{aligned}$$

for some positive constants c_i's. Thus the first summation in (35) contains $\sum_{i=1}^{n}(2l_i-1)$ terms, and the second summation contains

$$\sum_{1\le i<j\le n} l_i\, l_j$$

terms. Thus the dimension of $G_{2,0}(b)$ is less than or equal to the value in (34).

By using the idea as in the proof of Theorem 4.16, (assume all a_i's are not zero), we study the kernel

$$\begin{aligned}
&\left(\sum_{i=1}^{n}\sum_{k=1}^{l_i}\frac{1}{(1-\overline{a_i}z)^k}\frac{1}{(1-a_i\overline{w})^k}\right)^2 \\
&= \sum_{i=1}^{n}\left(\sum_{j=1}^{l_i}\frac{1}{(1-\overline{a_i}z)^j}\frac{1}{(1-a_i\overline{w})^j}\right)^2 \\
&+2\sum_{1\le i<j\le n}\left(\sum_{k=1}^{l_i}\frac{1}{(1-\overline{a_i}z)^k}\frac{1}{(1-a_i\overline{w})^k}\right)\left(\sum_{k=1}^{l_j}\frac{1}{(1-\overline{a_j}z)^k}\frac{1}{(1-a_j\overline{w})^k}\right)
\end{aligned}$$

and find the following algebraic basis of $G_{2,0}(b)$

$$\begin{aligned}
&\left\{\frac{1}{(1-\overline{a_i}z)^j}: 1\le i\le n,\ 2\le j\le 2l_i\right\} \\
&\cup\left\{\frac{1}{(1-\overline{a_i}z)^j\,(1-\overline{a_j}z)^k}: 1\le i<j\le n,\ 1\le j\le l_i,\ 1\le k\le l_j\right\}.
\end{aligned}$$

We omit the proof of the above set of functions (using partial fraction expansion) being linearly independent. It is clear that the cardinality of the above set is (34). Therefore, the expansion method of (35) will produce an orthonormal basis of $G_{2,0}(b)$. □

We give an explicit example.

Example 4.19. *Let $b(z)=\varphi_a^2(z)\varphi_b^3(z)$, where a and b in $\mathbb{D}$ are distinct and nonzero. Then, by Theorem 4.18, the dimension of $G_{2,0}(b)$ is 14. In this case, the cardinality of (30) is $(5+1)(5)/2=15$. Note that the following set*

$$\begin{aligned}
&\left\{\frac{\varphi_a^i(z)}{(1-\overline{a}z)^2}: i=0,1,2\right\}\cup\left\{\frac{\varphi_a^2(z)\varphi_b^i(z)}{(1-\overline{b}z)^2}: i=0,1,2,3,4\right\} \\
&\cup\left\{\frac{\varphi_a^2(z)\varphi_a^i(z)\varphi_b^j(z)}{(1-\overline{a}z)\left(1-\overline{b}z\right)}: i=0,1 \text{ and } j=0,1,2\right\}
\end{aligned}$$

is an orthogonal basis (we omit some constants) of $G_{2,0}(b)$.

Corollary 4.20. *Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}^{l_i}(z)$ where all the a_i's are distinct. The set (30) with $m = 0$ is an orthonormal basis of $G_{2,0}(b)$ if and only if $1 \le l_i \le 2$ for $1 \le i \le n$.*

Proof. The cardinality of the set (30) with $m = 0$ is

$$\frac{1}{2}\left(\sum_{i=1}^{n} l_i + 1\right)\left(\sum_{i=1}^{n} l_i\right) = \frac{1}{2}\sum_{i=1}^{n} l_i^2 + \sum_{1\le i<j\le n} l_i\, l_j + \frac{1}{2}\sum_{i=1}^{n} l_i.$$

We need to know when the above number is the same as (34). Equivalently

$$\frac{1}{2}\sum_{i=1}^{n} l_i^2 + \frac{1}{2}\sum_{i=1}^{n} l_i = \sum_{i=1}^{n}(2l_i - 1).$$

The above equality holds if and only if $1 \le l_i \le 2$ for $1 \le i \le n$. □

In the general case of $G_{m+2,m}(b)$, we change the powers in (35) and subsequent formulas from 2 to $m+2$. We state the following result but omit the combinatorial proof.

Theorem 4.21. *Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}^{l_i}(z)$, where all the a_i's are distinct. Then the dimension of $G_{m+2,m}(b)$ is*

$$\sum_{\substack{m_1+\cdots+m_n=m+2\\ 1\le m_1\le\cdots\le m_n}} \left(\prod_{i=1}^{n} [m_i(l_i-1)+1]\right).$$

The above idea of obtaining an orthonormal basis of $G_{m+2,m}(b)$ from an orthonormal basis of $\mathcal{H}(b)$ by using Lemma 3.3 also applies for the more general b. We demonstrate this idea for $G_{2,0}(b)$, where b is an infinite Blaschke product with simple zeros.

Proposition 4.22. *Let $b(z) = \prod_{i=1}^{\infty} \varphi_{a_i}(z)$ where all a_i's are distinct and nonzero. Then the following set K is an orthonormal basis of $G_{2,0}(b)$:*

$$K := \left\{b_i^2 e_{\lambda_i}^2 : i \ge 1\right\} \cup \left\{\sqrt{2} b_i b_j e_{\lambda_i} e_{\lambda_j} : j > i \ge 1\right\}, \tag{36}$$

$$\text{where} \quad b_1(z) = 1, \quad b_i(z) = \prod_{j=1}^{i-1} \varphi_{a_j}(z)\ (i \ge 2), \quad e_\lambda(z) = \frac{\sqrt{1-|\lambda|^2}}{1-z\overline{\lambda}}.$$

Proof. By [24, Theorem 14.7], $\{b_i e_{\lambda_i} : i \ge 1\}$ is an orthonormal basis of $\mathcal{H}(b)$ (called a Takenaka–Malmquist–Walsh basis in [25]). That is,

$$\frac{1 - b(z)\overline{b(w)}}{1 - z\overline{w}} = \sum_{i=1}^{\infty} b_i(z) e_{\lambda_i}(z) \overline{b_i(w) e_{\lambda_i}(w)}.$$

Square both sides of the above equation and apply Theorem 4.16 and Lemma 3.3 to see that K is an orthonormal basis of $G_{2,0}(b)$. □

5 Identifying $G_{k,\alpha}(\bar{b})$ when b is a finite Blaschke product

In previous sections, we study the operator range spaces for the operators $\beta_m(T_b^*)^{1/2}$ and $\beta_m(T_b)^{1/2}$, where b is a finite Blaschke product. These operators are not easy to compute, see for example, the computation of $\beta_m(T_b)^{1/2}$ in the special case where $m = 1$ and b is a single Blaschke factor [41]. By Lemma 4.2, we can express $\beta_m(b(T))$ in terms of $\beta_m(T)$. Hence in principle we can compute $\beta_m(T_b^*)$ and $\beta_m(T_b)$ in terms of $\beta_m(T_z^*)$ and $\beta_m(T_z)$ which are diagonal Hilbert–Schmidt operators. In this section, we take a digression to compute some examples of $\beta_m(T_b^*)$ and $\beta_m(T_b)$ which maybe of independent interest and then, by Theorem 4.7, we will identify $G_{k,\alpha}(\bar{b})$.

For convenience, we first recall the following corollary.

Corollary 5.1. *Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}(z)$. Then on A^2_α,*

$$\beta_m(T_b^*) = \sum_{m_1+\cdots+m_n=m} T_{g_{(m_1,\cdots,m_n)}(z)}\beta_m(T_z^*)T^*_{g_{(m_1,\cdots,m_n)}(z)},$$

$$\beta_m(T_b) = \sum_{m_1+\cdots+m_n=m} T^*_{g_{(m_1,\cdots,m_n)}(z)}\beta_m(T_z)T_{g_{(m_1,\cdots,m_n)}(z)}.$$

Since T_z on A^2_α is a weighted shift, it is easy to see that $\beta_m(T_z^*)$ and $\beta_m(T_z)$ are diagonal operators. These diagonal operators can be displayed in a straightforward fashion and the diagonals are computed in terms of $c_{i,\alpha}$ as in (18). In this sense, the above lemma gives an explicit formula for $\beta_m(T_b^*)$ and $\beta_m(T_b)$. Below we just compute $\beta_1(T_b^*)$ and $\beta_1(T_b)$ to illustrate the general case and we think these results are of independent interest.

Lemma 5.2. *The following formulas hold on A^2_α:*

$$\beta_1(T_z^*) = I - T_zT_z^* = e_0 \otimes e_0 + \sum_{i=0}^{\infty} \frac{\alpha+1}{i+\alpha+2} e_{i+1} \otimes e_{i+1},$$

$$\beta_1(T_z) = I - T_z^*T_z = \sum_{i=0}^{\infty} \frac{\alpha+1}{i+\alpha+2} e_i \otimes e_i, \tag{37}$$

where $e_i(z) = \sqrt{c_{i,\alpha}}z^i$ and $\{e_i : i \geq 0\}$ is an orthonormal basis of A^2_α as noted in (17).

Proof. By (17), $\{e_i : i \geq 0\}$ is an orthonormal basis of A^2_α. Note that

$$T_ze_i = \sqrt{c_{i,\alpha}}z^{i+1} = \frac{\sqrt{c_{i,\alpha}}}{\sqrt{c_{i+1,\alpha}}}\sqrt{c_{i+1,\alpha}}z^{i+1} = \frac{\sqrt{c_{i,\alpha}}}{\sqrt{c_{i+1,\alpha}}}e_{i+1},$$

$$T_z^*e_0 = 0, \quad T_z^*e_{i+1} = \frac{\sqrt{c_{i,\alpha}}}{\sqrt{c_{i+1,\alpha}}}e_i, \quad i \geq 0.$$

Hence

$$\begin{aligned} I - T_zT_z^* &= e_0 \otimes e_0 + \sum_{i=0}^{\infty}\left(1 - \frac{c_{i,\alpha}}{c_{i+1,\alpha}}\right) e_{i+1} \otimes e_{i+1} \\ &= e_0 \otimes e_0 + \sum_{i=0}^{\infty} \frac{\alpha+1}{i+\alpha+2} e_{i+1} \otimes e_{i+1}, \end{aligned}$$

where the second equality follows from

$$1 - \frac{c_{i,\alpha}}{c_{i+1,\alpha}} = 1 - \frac{i+1}{i+\alpha+2} = \frac{\alpha+1}{i+\alpha+2}.$$

Similarly,

$$I - T_z^* T_z = \sum_{i=0}^{\infty} \left(1 - \frac{c_{i,\alpha}}{c_{i+1,\alpha}}\right) e_i \otimes e_i = \sum_{i=0}^{\infty} \frac{\alpha+1}{i+\alpha+2} e_i \otimes e_i.$$

The proof is complete. ☐

Proposition 5.3. *Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}(z)$. Then on A^2_α,*

$$\beta_1(T_b^*) = \sum_{j=1}^{n} g_j(z)e_0 \otimes g_j(z)e_0 + \sum_{j=1}^{n}\sum_{i=0}^{\infty} \frac{\alpha+1}{i+\alpha+2} g_j(z)e_{i+1} \otimes g_j(z)e_{i+1},$$

$$\beta_1(T_b) = \sum_{j=1}^{n}\sum_{i=0}^{\infty} \frac{\alpha+1}{i+\alpha+2} T_{g_j}^* e_{i+1} \otimes T_{g_j}^* e_{i+1},$$

where $g_j(z) := \sqrt{1-|a_j|^2}\, b_j(z)(1-\overline{a_j}z)^{-1}$ with b_j defined in (27).

Proof. Specializing Corollary 5.1 to $m = 1$, we have

$$g_j(z) = g_{(m_1,\cdots,m_n)}(z) := \sqrt{1-|a_j|^2}\, b_j(z)(1-\overline{a_j}z)^{-1},$$

where $(m_1, \cdots, m_n) = (0, \cdots, 0, 1, 0, \cdots, 0)$ and the 1 is in the j-th position. Now the result follows from Corollary 5.1 and Lemma 5.2. ☐

Corollary 5.4. *Let $b(z) = \varphi_a(z)$ for $a \in \mathbb{D}$. Then on A^2_α,*

$$\beta_1(T_b^*) = g(z)e_0 \otimes g(z)e_0 + \sum_{i=0}^{\infty} \frac{\alpha+1}{i+\alpha+2} g(z)e_{i+1} \otimes g(z)e_{i+1},$$

$$\beta_1(T_b) = T_{1-|\varphi_a(z)|^2} = \sum_{i=0}^{\infty} \frac{\alpha+1}{i+\alpha+2} T_{g_j}^* e_{i+1} \otimes T_{g_j}^* e_{i+1}, \tag{38}$$

where $g(z) = \sqrt{1-|a|^2}\,(1-\overline{a}z)^{-1}$.

Proof. It follows at once from Proposition 5.3. ☐

The formula (38) is similar to the results of Proposition 2.2 and Proposition 2.3 in [41]. We also note the following result about finite rank $\beta_m(T_b^*)$.

Lemma 5.5. *Let $b(z) = \prod_{i=1}^{n} \varphi_{a_i}(z)$. When $\alpha = m \geq 0$ is an integer, $\beta_{m+2}(T_b^*)$ on A^2_α is of finite rank. In fact,*

$$\beta_{m+2}(T_b^*) = \sum_{m_1+\cdots+m_n=m+2} g_{(m_1,\cdots,m_n)}(z) \otimes g_{(m_1,\cdots,m_n)}(z).$$

Proof. By a more general result [28], see the proof on pages 488–489 in [28], $\beta_{m+2}(T_z^*) = e_0 \otimes e_0$ on A^2_α. The result now follows from Corollary 5.1. This lemma also follows (implicitly) from Corollary 3.19 and Lemma 4.1. ☐

The rank of $\beta_2(T_b^*)$ on the Bergman space A^2 is given by Theorem 4.18. The rank of $\beta_{m+2}(T_b^*)$ on the Bergman space A^2_α for $\alpha = m$ is given by Theorem 4.21.

Proposition 5.6. *On A^2_α, for $k \geq 1$,*

$$\beta_k(T_z) = T_{(1-|z|^2)^k} = \sum_{i=0}^{\infty} \left(\prod_{j=0}^{k-1} \frac{\alpha+1+j}{i+\alpha+2+j} \right) e_i \otimes e_i. \tag{39}$$

Hence $\beta_k(T_z)$ is a diagonal operator with diagonals (asymptotically) $\{(i+1)^{-k} : i \geq 0\}$.

Proof. We prove the result by using induction on k. For $k = 1$, (39) is just (37). Assume (39) holds for k. Note that

$$\begin{aligned} T_z^* \beta_k(T_z) T_z e_i &= T_z^* \beta_k(T_z) \frac{\sqrt{c_{i,\alpha}}}{\sqrt{c_{i+1,\alpha}}} e_{i+1} = \frac{\sqrt{c_{i,\alpha}}}{\sqrt{c_{i+1,\alpha}}} \prod_{j=0}^{k-1} \frac{\alpha+1+j}{i+1+\alpha+2+j} T_z^* e_{i+1} \\ &= \frac{c_{i,\alpha}}{c_{i+1,\alpha}} \prod_{j=0}^{k-1} \frac{\alpha+1+j}{i+1+\alpha+2+j} e_i = \frac{i+1}{i+\alpha+2} \prod_{j=0}^{k-1} \frac{\alpha+1+j}{i+1+\alpha+2+j} e_i \\ &= \frac{i+1}{i+\alpha+2+k} \prod_{j=0}^{k-1} \frac{\alpha+1+j}{i+\alpha+2+j} e_i, \end{aligned}$$

where the second equality follows from the induction hypothesis. By (5),

$$\begin{aligned} \beta_{k+1}(T_z) e_i &= \beta_k(T_z) e_i - T_z^* \beta_k(T_z) T_z e_i \\ &= \prod_{j=0}^{k-1} \frac{\alpha+1+j}{i+\alpha+2+j} e_i - \frac{i+1}{i+\alpha+2+k} \prod_{j=0}^{k-1} \frac{\alpha+1+j}{i+\alpha+2+j} e_i \\ &= \left(\prod_{j=0}^{k-1} \frac{\alpha+1+j}{i+\alpha+2+j} \right) \left(1 - \frac{i+1}{i+\alpha+2+k} \right) e_i = \prod_{j=0}^{k} \frac{\alpha+1+j}{i+\alpha+2+j} e_i. \end{aligned}$$

The proof is complete. □

In order to identify $G_{k,\alpha}(\bar{b})$, we recall the following scale of Dirichlet type spaces D_γ [9]. Let

$$D_\gamma := \left\{ f \in \mathrm{Hol}(\mathbb{D}) : f = \sum_{i=0}^{\infty} f_i z^i,\ \|f\|_\gamma^2 = \sum_{i=0}^{\infty} (1+i)^\gamma |f_i|^2 < \infty \right\},$$

where $\mathrm{Hol}(\mathbb{D})$ is the set of all holomorphic functions on $\mathbb{D}$. Hence $\{(1+i)^{-\gamma/2} z^i : i \geq 0\}$ is an orthonormal basis of D_γ. If $\gamma = 1$, then D_γ is the Dirichlet space. When $\gamma > 0$, these D_γ's are called Dirichlet type spaces. For $\gamma \geq 0$, D_γ has the complete Pick property [3]. For $\gamma < 0$, $D_\gamma \approx A^2_\alpha$ with $\alpha = -\gamma - 1 > -1$. The Hardy space H^2 is either D_0 or A^2_{-1}. We will use notation D_γ for $\gamma \geq 0$ and A^2_α for $\alpha > -1$.

Theorem 5.7. *Let $b(z)$ be a finite Blaschke product. The following statements hold on A^2_α with $\alpha > -1$.*

(i) *For $1 \leq k \leq [\alpha+2] - 1$,*

$$G_{k,\alpha}(\bar{b}) \approx A^2_{\alpha-k}.$$

(ii) *If α is an integer, then for $k = \alpha + 2$,*

$$G_{k,\alpha}(\bar{b}) \approx D_1.$$

(iii) *For* $k \geq [\alpha+2]$,

$$G_{k,\alpha}(\bar{b}) \approx D_{k-\alpha-1}.$$

Proof. By Proposition 3.12 and Theorem 4.7, we can assume $b(z) = z$. We prove our result by comparing orthonormal bases of norm equivalent spaces involved. By (17),

$$\left\{(i+1)^{(\alpha+1)/2} z^i : i \geq 0\right\}$$

is an orthogonal basis of A^2_α. Since by Proposition 5.6, $\beta_k(T_z)$ is a diagonal operator with diagonals (asymptotically) $\left\{(1+i)^{-k} : i \geq 0\right\}$, $G_{k,\alpha}(\bar{b})$ is the space with orthogonal basis

$$\left\{(1+i)^{(\alpha+1-k)/2} z^i : i \geq 0\right\}.$$

Thus, $G_{k,\alpha}(\bar{b}) \approx A^2_{\alpha-k}$ for $1 \leq k \leq [\alpha+2]-1$ and $G_{k,\alpha}(\bar{b}) \approx D_{k-\alpha-1}$ for $k \geq [\alpha+2]$. If α is an integer and $k = \alpha+2$, then $k-\alpha-1 = 1$. Hence $G_{\alpha+2,\alpha}(\bar{b}) \approx D_1$. This completes the proof. □

If b is any nonconstant function in $(H^\infty)_1$, then we have:

Theorem 5.8. *Let* $b \in (H^\infty)_1$ *and* b *be not a constant. The following statements hold on* A^2_α *with* $\alpha > -1$.

(i) *For* $1 \leq k \leq [\alpha+2]-1$,

$$G_{k,\alpha}(\bar{b}) \supseteq A^2_{\alpha-k}.$$

(ii) *If* α *is an integer, then for* $k = \alpha+2$,

$$G_{k,\alpha}(\bar{b}) \supseteq D_1.$$

(iii) *For* $k \geq [\alpha+2]$,

$$G_{k,\alpha}(\bar{b}) \supseteq D_{k-\alpha-1}.$$

Proof. By Proposition 3.12, T_b on A^2_α is a k-hypercontraction for all $k \geq 1$. Thus by Theorem 4.7, $G_{k,\alpha}(\bar{b}) \approx G_{k,\alpha}\left(\overline{\varphi(b)}\right)$ for any finite Blaschke product $\varphi(z)$. In particular, if $\varphi = (z - b(0))/(1 - \overline{b(0)}z)$, then $\varphi(b(z)) = z b_1(z)$ for some $b_1 \in (H^\infty)_1$. By Corollary 3.16,

$$G_{k,\alpha}(\bar{b}) \approx G_{k,\alpha}\left(\overline{\varphi(b)}\right) = G_{k,\alpha}\left(\overline{zb_1}\right) \supseteq G_{k,\alpha}\left(\bar{z}\right).$$

Now the result follows from Theorem 5.7, where $G_{k,\alpha}\left(\bar{z}\right)$ is identified. □

We would like to ask if there is an analogue of Theorem 4.11 for $G_{k,\alpha}(\bar{b})$ in the sense that for example, for $1 \leq k \leq [\alpha+2]-1$, $G_{k,\alpha}(\bar{b}) \approx A^2_{\alpha-k}$ only if b is a finite Blaschke product.

We get a connection between $G_{k,\alpha}(b)$ and $G_{k,\alpha}(\bar{b})$.

Corollary 5.9. *Let* $b(z)$ *be a finite Blaschke product. Then we have:*

(i) *If* $1 \leq k \leq [\alpha+2]-1$, *then*

$$G_{k,\alpha}(b) \approx G_{k,\alpha}(\bar{b}) \approx A^2_{\alpha-k}.$$

(ii) *If* α *is an integer and* $k = \alpha+2$, *then* $G_{k,\alpha}(b)$ *is finite dimensional and*

$$G_{k,\alpha}(\bar{b}) \approx D_1.$$

(iii) *For* $k > [\alpha+2]$, $G_{k,\alpha}(b)$ *is undefined and* $G_{k,\alpha}(\bar{b}) \approx D_{k-\alpha-1}$.

Proof. It follows at once from combining Theorem 4.10 and Theorem 5.7. □

We would like to remark that the case (i) of Corollary 5.9 gives that

$$G_{m+1,m}(b) \approx G_{m+1,m}(\bar{b}) \approx H^2 \quad \text{for all } m > -1$$

and in particular, we can recapture the Zhu's theorem (22):

$$\mathcal{A}(b) = G_{1,0}(b) \approx H^2 \approx G_{1,0}(\bar{b}) \approx \mathcal{A}(\bar{b}).$$

6 Density of polynomials in $G_{k,\alpha}(\bar{b})$

Chu [12] proved that polynomials are dense in $\mathcal{A}(\bar{b})$, and showed that $\mathcal{A}(b) \approx \mathcal{A}(\bar{b})$. Thus polynomials are also dense in $\mathcal{A}(b)$ which answers a question of Zhu [43]. In this section, by a slight modification of Chu's method, we prove that polynomials are dense in $G_{k,\alpha}(\bar{b})$. We first observe that $G_{1,\alpha}(b) = \mathcal{A}_\alpha(b) \approx \mathcal{A}_\alpha(\bar{b}) = G_{1,\alpha}(\bar{b})$. Thus polynomials are also dense in $\mathcal{A}_\alpha(b)$.

Corollary 6.1. *Let* $b \in (H^\infty)_1$. *Then on* A^2_α *for* $\alpha \ge 0$, $\mathcal{A}_\alpha(b) \approx \mathcal{A}_\alpha(\bar{b})$.

Proof. This is just the special case of Proposition 3.24 with $H = A^2_\alpha$ and $A = T_b$ since T_b and T_b^* are $[\alpha+2]$-hypercontractions and T_b is subnormal. □

The following theorem represents $G_{k,\alpha}(\bar{b})$ as an analytic weighted L^2 space. This result on A^2 is Theorem 2.1 in Chu [12] which was inspired by a similar result for de Branges–Rovnyak space $\mathcal{H}(\bar{b})$, where b is nonextreme, and it is also implicitly contained in the proof of Proposition 3.5 in Zhu [43]. By slight modifications of proofs from [12], [43], we have a generalization. Let $L^2_{b,k,\alpha}$ denote the weighted L^2 space $L^2(\mathbb{D}, dA_{b,k,\alpha}(z))$, where

$$dA_{b,k,\alpha}(z) = (1-|b(z)|^2)^k dA_\alpha(z), \quad dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z).$$

We then have:

Theorem 6.2. *Let* $b \in (H^\infty)_1$. *Let* $A^2_{b,k,\alpha}$ *be the closure of polynomials in* $L^2_{b,k,\alpha}$. *Define* $S_{b,k,\alpha} g = P_\alpha((1-|b|^2)^k g)$ *for* $g \in A^2_{b,k,\alpha}$. *Then* $S_{b,k,\alpha}$ *is an isometry from* $A^2_{b,k,\alpha}$ *onto* $G_{k,\alpha}(\bar{b})$.

Proof. Set $T := S_{b,k,\alpha}$. For $g \in A^2_{b,k,\alpha}$ and $h \in A^2_\alpha$,

$$\begin{aligned} \langle Tg, h\rangle_{A^2_\alpha} &= \left\langle P_\alpha((1-|b|^2)^k g), h\right\rangle_{A^2_\alpha} = \left\langle (1-|b|^2)^k g, h\right\rangle_{A^2_\alpha} \\ &= \int_{\mathbb{D}} (1-|b|^2)^k g\bar{h}\, dA_\alpha(z) = \langle g, h\rangle_{L^2_{b,k,\alpha}}. \end{aligned} \tag{40}$$

That is, T^* is the inclusion map from A^2_α into $A^2_{b,k,\alpha}$. Therefore, for $h_1, h_2 \in A^2_\alpha$,

$$\begin{aligned} \langle TT^* h_1, h_2\rangle_{A^2_\alpha} &= \langle T^* h_1, T^* h_2\rangle_{L^2_{b,k,\alpha}} = \langle h_1, h_2\rangle_{L^2_{b,k,\alpha}} \\ &= \int_{\mathbb{D}} (1-|b|^2)^k h_1 \overline{h_2}\, dA_\alpha(z) = \left\langle T_{(1-|b|^2)^k} h_1, h_2\right\rangle_{A^2_\alpha} \\ &= \langle \beta_k(T_b) h_1, h_2\rangle_{A^2_\alpha}. \end{aligned}$$

Thus $TT^* = \beta_k(T_b)$. By Corollary 2.6 with $C = I$ on $A^2_{b,k,\alpha}$, $D = \beta_k(T_b)^{1/2}$ and $B = T$, we see that $\widetilde{T}$ is a co-isometry, where $\widetilde{T}$ stands for T viewed as an operator from $A^2_{b,k,\alpha}$ into $G_{k,\alpha}(\bar{b})$. Furthermore, if $Tg = 0$, then (40) implies that $g \in L^2_{b,k,\alpha} \ominus A^2_{b,k,\alpha}$. So $\ker(\widetilde{T}) = 0$ and hence, $\widetilde{T}$ is unitary. Hence $S_{b,k,\alpha}$ is an isometry from $A^2_{b,k,\alpha}$ onto $G_{k,\alpha}(\bar{b})$. □

Lemma 6.3. *Let $b \in (H^\infty)_1$. Let M_n denote the closure of the span of $\{z^m\}_{m=n}^\infty$ in $A^2_{b,k,\alpha}$. Set $\mathcal{P} = \bigcup_{n\geq 0} M_n^\perp$. Then $\mathcal{P}$ is dense in $A^2_{b,k,\alpha}$.*

Proof. This is Lemma 2.1 in [12] when $k = 1, \alpha = 0$. The proof here is a slight modification of the proof in [12]. For clarity, we include the slightly condensed proof. Assume b is not a constant. Let $f \in A^2_{b,k,\alpha}$ be such that $f \perp \mathcal{P}$. That is, $f \in M_n$ for all $n \geq 0$. We need to show $f = 0$. Assume $f \neq 0$. Then $f(z) = \sum_{j=m}^\infty a_j z^j$ with $a_m \neq 0$. Since $f \in M_m$, there exists a sequence of polynomials $\{p_s\}$ such that $p_s \to f$ in $A^2_{b,k,\alpha}$. Hence $p_s - a_m z^m \to f - a_m z^m \in M_{m+1}$. Now $f \in M_{m+1}$ implies that $z^m \in M_{m+1}$.

Let $g(z) = \sum_{j=m+1}^{m+N} a_j z^j \in M_{m+1}$, where $N \geq 1$. Fix $r \in (0, 1)$. Then there exists δ such that $(1 - |b(z)|^2) \geq \delta$ for all $|z| \leq r$. Now

$$\begin{aligned}\|z^m - g\|^2_{A^2_{b,k,\alpha}} &= \int_{\mathbb{D}} |z^m - g|^2 \, dA_{b,k,\alpha}(z) \geq \int_{r\mathbb{D}} |z^m - g|^2 (1 - |b(z)|^2)^k dA_\alpha(z) \\ &\geq \delta^k \int_{r\mathbb{D}} |z^m - g|^2 \, dA_\alpha(z) = \delta^k \left(\int_{r\mathbb{D}} |z^m|^2 \, dA_\alpha(z) + \int_{r\mathbb{D}} |g|^2 \, dA_\alpha(z) \right) \\ &\geq \delta^k \int_{r\mathbb{D}} |z^m|^2 \, dA_\alpha(z) > 0.\end{aligned}$$

Therefore, $z^m \notin M_{m+1}$. This is a contradiction, so $f = 0$. □

We conclude with:

Theorem 6.4. *Let $b \in (H^\infty)_1$. Polynomials are dense in $G_{k,\alpha}(\bar{b})$ for all $k \geq 1$. Furthermore, polynomials are dense in $\mathcal{A}_\alpha(b)$.*

Proof. Let $f \in G_{k,\alpha}(\bar{b})$ and $\varepsilon > 0$. By Theorem 6.2, there exists $g \in A^2_{b,k,\alpha}$ such that $f = S_{b,k,\alpha} g$. By Lemma 6.3, there exists $h \in \mathcal{P}$ such that $\|g - h\| < \varepsilon$ in $A^2_{b,k,\alpha}$. By Theorem 6.2,

$$\|f - S_{b,k,\alpha} h\|_{G_{k,\alpha}(\bar{b})} = \|S_{b,k,\alpha}(g - h)\|_{G_{k,\alpha}(\bar{b})} = \|(g - h)\|_{A^2_{b,k,\alpha}} < \varepsilon.$$

Now $h \in \mathcal{P}$ implies that $h \in M_n^\perp$ for some $n \geq 0$. That is, $\langle h, z^m \rangle_{L^2_{b,k,\alpha}} = 0$ for all $m \geq n$. By (40), $\langle S_{b,k,\alpha} h, z^m \rangle_{A^2_\alpha} = \langle h, z^m \rangle_{L^2_{b,k,\alpha}} = 0$ for all $m \geq n$. Hence, $S_{b,k,\alpha} h$ is a polynomial and the proof is complete. □

In view of the above result, we conjecture that polynomials are dense in $G_{k,\alpha}(b)$ for $1 \leq k < [\alpha + 2]$. This conjecture is true when b is a finite Blaschke product or $k = 1$ as seen before.

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References

[1] A. Abkar and B. Jafarzadeh, *Weighted sub-Bergman Hilbert spaces in the unit disk*, Czechoslovak Math. J. **60(135)** (2010), no. 2, 435–443.

[2] J. Agler, *Hypercontractions and subnormality*, J. Operator Theory **13** (1985), no. 2, 203–217.

[3] J. Agler and J. E. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002.

[4] A. Aleman and B. Malman, *Hilbert spaces of analytic functions with a contractive backward shift*, J. Funct. Anal. **277** (2019), no. 1, 157–199.

[5] A. Aleman, S. Richter, and C. Sundberg, *Beurling's theorem for the Bergman space*, Acta Math. **177** (1996), no. 2, 275–310.

[6] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[7] J. A. Ball and V. Bolotnikov, *de Branges–Rovnyak Spaces: Basics and Theory, in: Operator Theory*, New Mathematical Monographs, pp. 631–679, Springer Basel, Basel, 2015.

[8] A. Baranov, E. Fricain, and J. Mashreghi, *Weighted norm inequalities for de Branges–Rovnyak spaces and their applications*, Amer. J. Math. **132** (2010), no. 1, 125–155.

[9] L. Brown and A. L. Shields, *Cyclic vectors in the Dirichlet space*, Trans. Amer. Math. Soc. **285** (1984), no. 1, 269–303.

[10] S. Chavan and R. E. Curto, *Operators Cauchy dual to 2-hyperexpansive operators: the multivariable case*, Integral Equations Operator Theory **73** (2012), no. 4, 481–516.

[11] N. Chevrot, D. Guillot, and T. Ransford, *De Branges–Rovnyak spaces and Dirichlet spaces*, J. Funct. Anal. **259** (2010), no. 9, 2366–2383.

[12] C. Chu, *Density of polynomials in sub-Bergman Hilbert spaces*, J. Math. Anal. Appl. **467** (2018), no. 1, 699–703.

[13] ______, *Hilbert spaces contractively contained in weighted Bergman spaces on the unit disk*, J. Math. Anal. Appl. **472** (2019), no. 1, 386–394.

[14] C. C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103** (1988), no. 3, 809–812.

[15] R. E. Curto, I. S. Hwang, D. Kang, and W. Y. Lee, *Subnormal and quasinormal Toeplitz operators with matrix-valued rational symbols*, Adv. Math. **255** (2014), 562–585.

[16] B. M. Davis and J. E. McCarthy, *Multipliers of de Branges spaces*, Michigan Math. J. **38** (1991), no. 2, 225–240.

[17] L. de Branges and J. Rovnyak, *Square Summable Power Series*, Holt, Rinehart and Winston, New York-Toronto, Ont.-London, 1966.

[18] R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.

[19] R. G. Douglas, H-K. Kwon, and S. Treil, *Similarity of n-hypercontractions and backward Bergman shifts*, J. Lond. Math. Soc. (2) **88** (2013), no. 3, 637–648.

[20] J. Eschmeier, *Bergman inner functions and m-hypercontractions*, J. Funct. Anal. **275** (2018), no. 1, 73–102.

[21] G. R. Exner, I. B. Jung, and S. S. Park, *On n-contractive and n-hypercontractive operators, II*, Integral Equations Operator Theory **60** (2008), no. 4, 451–467.

[22] E. Fricain, *Bases of reproducing kernels in de Branges spaces*, J. Funct. Anal. **226** (2005), no. 2, 373–405.

[23] E. Fricain, A. Hartmann, and W. T. Ross, *Concrete examples of $\mathscr{H}(b)$ spaces*, Comput. Methods Funct. Theory **16** (2016), no. 2, 287–306.

[24] E. Fricain and J. Mashreghi, *The Theory of $\mathcal{H}(b)$ Spaces. Vol. 1 and Vol. 2*, New Mathematical Monographs, vol. 20 and vol. 21, Cambridge University Press, Cambridge, 2016.

[25] S. R. Garcia, J. Mashreghi, and W. T. Ross, *Introduction to Model Spaces and Their Operators*, Cambridge Studies in Advanced Mathematics, vol. 148, Cambridge University Press, Cambridge, 2016.

[26] C. Gu, *Functional calculus for m-isometries and related operators on Hilbert spaces and Banach spaces*, Acta Sci. Math. (Szeged) **81** (2015), no. 3-4, 605–641.

[27] ———, *On (m, p)-expansive and (m, p)-contractive operators on Hilbert and Banach spaces*, J. Math. Anal. Appl. **426** (2015), no. 2, 893–916.

[28] C. Gu and Z. Chen, *A model for (n, p)-hypercontractions on Banach space*, Indag. Math. (N.S.) **26** (2015), no. 3, 485–494.

[29] C. Gu, J. Hendricks, and D. Rutherford, *Hyponormality of block Toeplitz operators*, Pacific J. Math. **223** (2006), no. 1, 95–111.

[30] C. Gu, I. S. Hwang, and W. Y. Lee, *Hyponormality of Toeplitz operators in several variables by the weighted shifts approach*, Linear Multilinear Algebra **68** (2020), no. 8, 1695–1720.

[31] M. T. Jury, *Reproducing kernels, de Branges–Rovnyak spaces, and norms of weighted composition operators*, Proc. Amer. Math. Soc. **135** (2007), no. 11, 3669–3675.

[32] T. Kailath and A. H. Sayed, *Displacement structure: theory and applications*, SIAM Rev. **37** (1995), no. 3, 297–386.

[33] B. A. Lotto and D. Sarason, *Multiplicative structure of de Branges's spaces*, Rev. Mat. Iberoamericana **7** (1991), no. 2, 183–220.

[34] S. Luo, C. Gu, and S. Richter, *Higher order local Dirichlet integrals and de Branges–Rovnyak spaces*, Adv. Math. **385** (2021), Paper No. 107748.

[35] M. K. Ng, *Iterative Methods for Toeplitz Systems*, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2004.

[36] A. Olofsson, *An operator-valued Berezin transform and the class of n-hypercontractions*, Integral Equations Operator Theory **58** (2007), no. 4, 503–549.

[37] V. Y. Pan, *Structured Matrices and Polynomials: Unified Superfast Algorithms*, Birkhäuser Boston, Inc., Boston, MA; Springer-Verlag, New York, 2001.

[38] V. I. Paulsen and M. Raghupathi, *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces*, Cambridge Studies in Advanced Mathematics, vol. 152, Cambridge University Press, Cambridge, 2016.

[39] D. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, University of Arkansas Lecture Notes in the Mathematical Sciences, vol. 10, John Wiley & Sons, Inc., New York, 1994.

[40] ______, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices **1** (2007), no. 4, 491–526.

[41] S. Sultanic, *Sub-Bergman Hilbert spaces*, J. Math. Anal. Appl. **324** (2006), no. 1, 639–649.

[42] D. Timotin, *A short introduction to de Branges–Rovnyak spaces*, Invariant subspaces of the shift operator, Contemp. Math., vol. 638, Amer. Math. Soc., Providence, RI, 2015, pp. 21–38.

[43] K. Zhu, *Sub-Bergman Hilbert spaces on the unit disk*, Indiana Univ. Math. J. **45** (1996), no. 1, 165–176.

[44] ______, *Sub-Bergman Hilbert spaces in the unit disk. II*, J. Funct. Anal. **202** (2003), no. 2, 327–341.

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