ON THE STRUCTURE OF POLYNOMIALLY COMPACT OPERATORS

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Throughout this note, let $\mathcal{H}$ denote an infinite dimensional separable Hilbert space, let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$ and let $\mathcal{K}(\mathcal{H})$ denote the ideal of compact operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ write $\sigma(T)$ for the spectrum of T. If $T \in \mathcal{L}(\mathcal{H})$ is a Fredholm operator, that is, T has finite dimensional null space and its range of finite co-dimension, then the *index* of T, denoted $\text{ind}\,(T)$, is given by $\text{ind}\,(T) = \dim T^{-1}(0) - \dim T(H)^{\perp}$ $(= \dim T^{-1}(0) - \dim T^{*-1}(0))$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called a *Weyl* operator if it is Fredholm of index zero. An operator $T \in \mathcal{L}(\mathcal{H})$ is called a *Browder* operator if it is Fredholm "of finite ascent and descent": equivalently ([**5**, Theorem 7.9.3]) if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined by

$$\begin{aligned} \sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}; \\ \omega(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}; \\ \sigma_b(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\} : \end{aligned}$$

evidently $\sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}\,\sigma(T)$, where we write $\text{acc}\,\mathbf{K}$ for the *accumulation points* of $\mathbf{K} \subseteq \mathbb{C}$. If we write $\eta(\mathbf{K})$ for the "polynomially convex-hull" of $\mathbf{K} \subseteq \mathbb{C}$, then $\sigma_e(T) \subseteq \omega(T) \subseteq \eta\,\sigma_e(T)$, and if we write

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$$

for the *Riesz points* of $\sigma(T)$ then $\sigma(T) \subseteq \eta\,\sigma_e(T) \cup p_{00}(T)$ (cf. [**5**, Theorem 9.8.4]). An operator $T \in \mathcal{L}(\mathcal{X})$ is said to be *polynomially compact* if there exists a nonzero complex polynomial p such that $p(T)$ is compact. It is familiar that every polynomially compact operator has a nontrivial invariant subspace. The structure of polynomially compact operators was described by F. Gilfeather [**3**] and C. Olsen [**7**]. Gilfeather showed that every polynomially compact operator is the finite direct sum of translates of operators whch have

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the property that a finite power of the operator is compact. Also his structure theorem provides an information about the Weyl spectra of polynomially compact operators. By comparison, Olsen independently showed that every polynomially compact operator is the sum of an algebraic operator and a compact operator. Since every algebraic operator is unitarily equivalent to a finite (upper triangular) operator matrix of the form (cf. [**4**];[**8**])

$$\begin{pmatrix} \lambda_1 I & * & \cdots & \cdots & * \\ & \lambda_2 I & * & & \vdots \\ & & \lambda_3 I & \ddots & \vdots \\ & 0 & & \ddots & * \\ & & & & \lambda_n I \end{pmatrix},$$

Olsen's structure theorem consequently says that every polynomially compact operator T is represented by the sum

$$T = N + K + \left(\bigoplus_{i=1}^{n} \lambda_i I \right), \tag{0.1}$$

where N is nilpotent and K is compact. Now the following question is naturally raised: *Is every polynomially compact operator decomposed into the finite direct sum of nilpotent operators and translates of compact operators ?* In this note we answer the above question affirmatively and, in addition, give an equivalent property to the finite-ness of the Weyl spectrum of the operator. Our structure theorem of polynomially compact operators can be written as follows:

Theorem 1. *If $T \in \mathcal{L}(\mathcal{H})$ is polynomially compact then T is decomposed into the finite direct sum*

$$T = \bigoplus_{i=1}^{n} \big(N_i + K_i + \lambda_i I \big), \tag{1.1}$$

where the N_i are nilpotents, the K_i are compact, and $\{\lambda_1, \cdots, \lambda_n\} = \omega(T)$.

Evidently, (1.1) implies (0.1). Before proving Theorem 1, we characterize the finite-ness of the Weyl spectrum. S. Berberian [**1**] considered a relationship between the polynomial compactness of the operator and the finite-ness of its Weyl spectrum, and gave several sufficient conditions for the finite-ness of the Weyl spectrum; for example, if T is a seminormal operator then T is polynomailly compact if and only if $\omega(T)$ is finite. We recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called a *Riesz* operator if $\sigma_e(T) = \{0\}$. If $T \in \mathcal{L}(\mathcal{H})$ then the West decomposition theorem [**9**] says that

$$T \text{ is a Riesz operator} \Longleftrightarrow T = K + Q,$$

where K is compact and Q is quasinilpotent. An operator $T \in \mathcal{L}(\mathcal{H})$ will be said to be *polynomially Riesz* if there exists a nonzero complex polynomial p such that $p(T)$ is Riesz.

We then have:

Lemma 2. *If $T \in \mathcal{L}(\mathcal{H})$ is polynomially Riesz then $\omega(T)$ is finite. Also if T is a Riesz operator, then T is polynomially compact if and only if T^n is compact for some $n \in \mathbb{N}$.*

Proof. If $p(T)$ is Riesz for some nonzero polynomial p, then it follows that $p(\sigma_e(T)) = \sigma_e(p(T)) = \{0\}$, which says that $\sigma_e(T)$ is finite. But since $\sigma_e(T) \subseteq \omega(T) \subseteq \eta\sigma_e(T)$, it follows that $\omega(T)$ is finite. This proves the first assertion. For the second assertion, let p be a nonzero polynomial such that $p(T)$ is compact. Writing $p(\lambda) \equiv a_0\,(\lambda - \lambda_1)\cdots(\lambda - \lambda_m)$, we have that $a_0\,(T - \lambda_1 I)\cdots(T - \lambda_m I)$, say K, is compact. But since T is a Riesz operator and hence $T - \lambda_i I$ is Fredholm for each nonzero λ_i, there exists at least one λ_i such that $\lambda_i = 0$ $(1 \le i \le m)$: for if were not so then K would be compact and Fredholm, which contradicts to the infinite-dimensionality of $\mathcal{H}$. Thus for some $1 \le n \le m$, we have that $a_0\, T^n S = K$ with a Fredholm operator S. Therefore we can conclude that T^n is compact. The converse is evident. □

By definition and Lemma 2 we have that

$$T \text{ is polynomially compact} \implies T \text{ is polynomially Riesz} \implies \omega(T) \text{ is finite.} \tag{2.1}$$

But the first implication in (2.1) is not reversible in general. To see this we can borrow an example due to Foias and Pearcy (cf. [**2**];[**8**]): if $T : \ell_2 \to \ell_2$ is the backward weighted shift defined by

$$Te_0 = 0 \quad \text{and} \quad Te_{n+1} = \tau_n e_n, n \ge 0,$$

where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis of ℓ_2 and the weight sequence $\{\tau_n\}_{n=0}^\infty$ is given by

$$\left\{\frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{16}}, \frac{1}{2}, \frac{1}{2^4}, \frac{1}{2}, \frac{1}{2^{64}}, \frac{1}{2}, \frac{1}{2^4}, \cdots\right\},$$

then T is quasinilpotent and hence Riesz, while T^n is not compact for any $n \in \mathbb{N}$, so that by Lemma 2, T is not polynomailly compact.

By contrast, the second implication in (2.1) is reversible; the following gives the structure of polynomially Riesz operators on a Hilbert space.

Lemma 3. *An operator $T \in \mathcal{L}(\mathcal{H})$ is polynomially Riesz if and only if $\omega(T)$ is finite. In this case, T is decomposed into the finite direct sum*

$$T = \bigoplus_{i=1}^{n} \big(K_i + Q_i + \lambda_i I\big), \tag{3.1}$$

where

(i) *the K_i are compact operators;*
(ii) *the Q_i are quasinilpotent operators;*
(iii) *$\omega(T) = \{\lambda_1, \cdots, \lambda_n\}$.*

Proof. Suppose $\omega(T)$ is finite. We first claim that

$$\operatorname{acc}\sigma(T) \subseteq \omega(T) : \tag{3.2}$$

To see this suppose $\lambda \in \sigma(T) \setminus \omega(T)$ and hence $T - \lambda I$ is Fredholm but not invertible. Let $\partial\mathbf{K}$ denote the topological boundary of $\mathbf{K} \subseteq \mathbb{C}$. Then we have that $\lambda \in \partial\,\sigma(T)$: for if it were not so then $\lambda \in \operatorname{int}\sigma(T)$ and hence $\sigma_e(T)$ would be infinite because $\sigma(T) \subseteq$

$\eta\,\sigma_e(T)\cup p_{00}(T)$ for every $T\in\mathcal{L}(\mathcal{H})$. Therefore by the punctured neighborhood theorem, i.e., $\partial\,\sigma(T)\setminus\sigma_e(T)\subseteq \text{iso}\,\sigma(T)$ (cf. [**5**],[**6**]), it follows that $\lambda\in\text{iso}\,\sigma(T)$, which proves (3.2). We may thus assume that $\omega(T)=\sigma_e(T)=\{\lambda_1,\cdots,\lambda_n\}$, and $\text{acc}\,\sigma(T)\subseteq\{\lambda_1,\cdots,\lambda_n\}$. Then we can find a collection $\{\Delta_1,\cdots,\Delta_n\}$ of closed subsets of $\sigma(T)$ satisfying

(i) $\cup_{i=1}^n\Delta_i=\sigma(T)$;
(ii) $\Delta_i\cap\Delta_j=\emptyset$ if $i\neq j$;
(iii) $\lambda_i\in\Delta_i$ for $i=1,\cdots,n$.

If for every $i=1,\cdots,n$, N_i is a neighborhood of Δ_i which contains no other points of $\sigma(T)$, then by using the spectral projections $P=\frac{1}{2\pi i}\int_{\partial N_i}(\lambda I-T)^{-1}\,d\lambda$ corresponding to Δ_i $(i=1,\cdots,n)$, we can decompose T as

$$T=\bigoplus_{i=1}^n(T_i+\lambda_i I),$$

where $\sigma_e(T_i)=\{0\}$, i.e., T_i is a Riesz operator for every $i=1,\cdots,n$. Then by the West decomposition theorem each T_i is the sum of a compact operator K_i and a quasinilpotent operator Q_i. This proves (3.1). Writing $p(\lambda)\equiv(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$, we have that

$$p(T)=\bigoplus_{j=1}^n\left[\prod_{i=1}^n\Big(K_j+Q_j+(\lambda_j-\lambda_i)I\Big)\right]. \tag{3.3}$$

Note that for every $j=1,\cdots,n$, the j-th direct summand in (3.3) is the sum of a compact operator and an operator of the form

$$V_j:=Q_j^n+\mu_{n-1}Q_j^{n-1}+\cdots+\mu_1Q_j\quad\text{for some } \mu_i\in\mathbb{C}\ (i=1,\cdots,n-1). \tag{3.4}$$

Remembering ([**5**, Theorem 7.4.3]) that if a and b are commuting quasinilpotents in a normed algebra then $a+b$ and ab are both quasinilpotents, we can see that V_j is a quasinilpotent for every $j=1,\cdots,n$. Therefore $\sigma_e(p(T))=\sigma_e\left(\bigoplus_{j=1}^n V_j\right)=\{0\}$, which says that $p(T)$ is a Riesz operator. The converse follows from Lemma 2. □

Lemma 4([7, Theorem 2.3]). *If $A,B\in\mathcal{L}(\mathcal{H})$ are such that AB is compact then there exists a projection $P\in\mathcal{L}(\mathcal{H})$ such that AP and $(I-P)B$ are both compact.*

Lemma 5. *If $T\in\mathcal{L}(\mathcal{H})$ then*

$$T^n \text{ is compact for some } n\in\mathbb{N}\Longleftrightarrow T=N+K, \tag{5.1}$$

where N is a nilpotent operator and K is a compact operator.

Proof. For the forward implication in (5.1), we will use an induction on n. If $n=1$ then it is clear. Assume that if T^{n-1} is compact then T is the sum of a nilpotent operator and a compact operator. Now suppose T^n is compact. Then by Lemma 4, there exists a projection $P\in\mathcal{L}(\mathcal{H})$ such that $T^{n-1}P$ and $(I-P)T$ are both compact. Then T can be represented as the following 2×2 operator matrix relative to the direct sum $P(\mathcal{H})\oplus(I-P)(\mathcal{H})$:

$$T=\begin{pmatrix} PTP & PT(I-P)\\ (I-P)TP & (I-P)T(I-P)\end{pmatrix}.$$

Note that $(I-P)TP$ and $(I-P)T(I-P)$ are both compact. Observe

$$(PTP)^{n-1} = (PT)^{n-1}P = \big(T-(I-P)T\big)^{n-1}P \in T^{n-1}P + \mathcal{K}(\mathcal{H}),$$

which is a compact operator. Then by the induction hypothesis, we can write

$$PTP = N_1 + K_1 \quad \text{with } N_1 \text{ a nilpotent operator and } K_1 \text{ a compact operator.}$$

Therefore if we put

$$N := \begin{pmatrix} N_1 & PT(I-P) \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad K := \begin{pmatrix} K_1 & 0 \\ (I-P)TP & (I-P)T(I-P) \end{pmatrix},$$

then N is nilpotent and K is compact, and $T = N+K$. This proves the forward implication in (5.1). The backward implication is evident. □

Corollary 6. *If $T \in \mathcal{L}(\mathcal{H})$ is a Riesz operator then*

(6.1) $$T \text{ is polynomially compact} \Longleftrightarrow T = N + K,$$

where N is a nilpotent operator and K is a compact operator.

Proof. This immediately follows from Lemma 2 and Lemma 5. □

We are ready for proving Theorem 1.

Proof of Theorem 1. If T is polynomially compact then by Lemma 3 passing to the West decomposition theorem, T is a finite direct sum of translates of Riesz operators:

$$T = \bigoplus_{i=1}^{n} \big(T_i + \lambda_i I\big),$$

where the T_i are Riesz operators. But since T is polynomially compact, it follows that $T_i + \lambda_i I$ is also polynomially compact and hence so is T_i for every $i = 1, \cdots, n$. Thus by Corollary 6, we can write $T_i = N_i + K_i$, where N_i is nilpotent and K_i is compact for every $i = 1, \cdots, n$. This completes the proof of Theorem 1. □

Let $\mathcal{C}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denote the Calkin algebra and let π denote the Calkin homomorphism of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{C}(\mathcal{H})$. Then $T \in \mathcal{L}(\mathcal{H})$ is a Riesz operator if and only if $\pi(T)$ is a quasinilpotent element of $\mathcal{C}(\mathcal{H})$. Thus the West decomposition theorem is equivalent to the following: if $\mathcal{Q}_{\mathcal{L}(\mathcal{H})}$ and $\mathcal{Q}_{\mathcal{C}(\mathcal{H})}$ denote the sets of quasinilpotent elements of $\mathcal{L}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$, respectively, then

(6.2) $$\pi\left(\mathcal{Q}_{\mathcal{L}(\mathcal{H})}\right) = \mathcal{Q}_{\mathcal{C}(\mathcal{H})}.$$

By contrast, we have:

Corollary 7. *If $\mathcal{N}_{\mathcal{L}(\mathcal{H})}$ and $\mathcal{N}_{\mathcal{C}(\mathcal{H})}$ denote the sets of nilpotent elements of $\mathcal{L}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$, respectively, then*

(7.1) $$\pi\left(\mathcal{N}_{\mathcal{L}(\mathcal{H})}\right) = \mathcal{N}_{\mathcal{C}(\mathcal{H})}.$$

Proof. This immediately follows from (5.1). □

It remains still open whether the West decomposition theorem survives in the Banach space setting: i.e., *Is the equality (6.2) true if $\mathcal{H}$ is a Banach space ?* In fact we don't know even if the equality (7.1) is true when $\mathcal{H}$ is a Banach space. But if Lemma 4 were true when $\mathcal{H}$ is a Banach space then (7.1) would be true for Banach spaces $\mathcal{H}$ because Lemma 5 doesn't use the Hilbert space properties except using Lemma 4.

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