A canonical decomposition of strong $L^2$-functions

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Abstract The aim of this paper is to establish a canonical decomposition of operator-valued strong $L^2$-functions by the aid of the Beurling-Lax-Halmos Theorem which characterizes the shift-invariant subspaces of vector-valued Hardy space. This decomposition reduces to the Douglas-Shapiro-Shields factorization if the flip of a strong $L^2$-function is of bounded type. To consider a converse of the Beurling-Lax-Halmos Theorem, we introduce a notion of the “Beurling degree” for inner functions by employing a canonical decomposition of strong $L^2$-functions induced by the given inner functions. Eventually, we establish a deep connection between the Beurling degree of the given inner function and the spectral multiplicity of the truncated backward shift on the corresponding model space.

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1. Introduction

The celebrated Beurling Theorem [Beu] characterizes the shift-invariant subspaces of the Hardy space. P.D. Lax [La] has extended the Beurling Theorem to the case of finite multiplicity, which is so called the Beurling-Lax Theorem. P.R. Halmos [Ha1] has given a beautiful proof for the case of infinite multiplicity, which is so called the Beurling-Lax-Halmos Theorem. Since then, the Beurling-Lax-Halmos Theorem has been extended and applied in the various settings and in the connection with the model theory, the system theory and the interpolation problem, etc. by many authors (cf. [ADR], [AS], [BH1], [BH2], [BH3], [Ca], [dR], [Hed], [Po], [Ri], [SFBK], etc). In this paper, we will closely analyze the matrix-valued theory related to the Beurling-Lax-Halmos Theorem. Throughout the paper, whenever we deal with operator-valued functions $\Phi$ on the unit circle $\mathbb{T}$, we assume that for almost all $z \in \mathbb{T}$, $\Phi(z)$ belongs to the set of all bounded linear operators, denoted by $B(D, E)$, between separable complex Hilbert spaces $D$ and $E$. We write simply $B(E)$ for $B(E, E)$. Unless otherwise stated, the functions in this paper are operator-valued functions. For a separable complex Hilbert space $E$, if $S_E$ is the shift operator on the $E$-valued Hardy space $H_E^2$, i.e.,

$$(S_E f)(z) := zf(z) \quad \text{for each } f \in H_E^2,$$

then the Beurling-Lax-Halmos Theorem states that every invariant subspace $M$ under $S_E$ (i.e., a closed subspace $M$ of $H_E^2$ such that $S_E f \in M$ for all $f \in M$) is of the form $\Delta H_E^2$, where $E'$ is a closed subspace of $E$ and $\Delta$ is an (possibly one-sided) inner function, i.e., $\Delta(z)^* \Delta(z)$ is the identity operator $I_E$ for almost all $z \in \mathbb{T}$, but $\Delta(z)\Delta^*(z)$ is not required to be the identity operator $I_E$. Equivalently, if a closed subspace $M$ of $H_E^2$ is invariant for the backward (or the adjoint) shift operator $S_E^*$, then $M = \mathcal{H}(\Delta) := H_E^2 \ominus \Delta H_E^2$ for some inner function $\Delta$, where $\mathcal{H}(\Delta)$ is often called a model space or a de Branges-Rovnyak space [dR], [Sa], [SFBK]. Thus, for a subset $F$ of $H_E^2$, if $E_F^*$ denotes the smallest $S_E^*$-invariant subspace containing $F$, i.e.,

$$E_F^* := \bigvee \{ S_E^* F : n \geq 0 \},$$

where $\bigvee$ denotes the closed linear span, then $E_F^* = \mathcal{H}(\Delta)$ for some inner function $\Delta$. Now, given a backward shift-invariant subspace $\mathcal{H}(\Delta)$, we may ask:

What is the smallest number of vectors in $F$ satisfying $\mathcal{H}(\Delta) = E_F^*$?

More generally, we are interested in the problem of describing the set $F$ in $H_E^2$ such that $\mathcal{H}(\Delta) = E_F^*$. This problem invites us to consider a bigger class than the set of operator-valued $L^2$-functions. This question is closely related to a canonical decomposition of (operator-valued) strong $L^2$-functions, where a strong $L^2$-function $\Phi$ is a $B(D, E)$-valued function $\Phi$ defined almost everywhere on the unit circle $\mathbb{T}$ such that $\Phi(\cdot)x \in L_E^2$ for each $x \in D$: indeed, every operator-valued $L^p$-function ($p > 2$) is a strong $L^2$-function (cf. p.11). Following to V. Peller [Pe], we write $L_E^2(\mathcal{B}(D, E))$ for the set of strong $L^2$-functions with values in $\mathcal{B}(D, E)$. The set $L_E^2(\mathcal{B}(D, E))$ is nicely served as general symbols of vectorial Hankel operators (see [Pe]). Similarly, we write $H_E^2(\mathcal{B}(D, E))$ for the set of strong $L^2$-functions with values in $\mathcal{B}(D, E)$ such that $\Phi(\cdot)x \in H_E^2$ for each $x \in D$. Of course, $H_E^2(\mathcal{B}(D, E))$ contains all $\mathcal{B}(D, E)$-valued $H^2$-functions. The terminology a “strong
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$H^2$-function” is reserved for the operator-valued functions on the unit disk $\mathbb{D}$, following to N.K. Nikolskii [Ni1]: A function $\Phi : \mathbb{D} \to \mathcal{B}(D, E)$ is called a strong $H^2$-function if $\Phi(x) \in H^2_E$ for each $x \in D$. In general, the boundary values of strong $H^2$-functions do not need to be bounded linear operators (see p.12). Thus we do not guarantee that every strong $H^2$-function belongs to $H^2_E(\mathcal{B}(D, E))$. On the other hand, we may regard $\Phi \in H^2_E(\mathcal{B}(D, E))$ as an (linear, but not necessarily bounded) operator-valued function defined on the unit disk $\mathbb{D}$ (cf. p.13). Thus if $\dim D < \infty$, then every function in $H^2_E(\mathcal{B}(D, E))$ is a strong $H^2$-function. In particular, if $\dim D < \infty$ and $\dim E < \infty$, then $H^2_E(\mathcal{B}(D, E))$ coincides with the set of strong $H^2$-functions (see Lemma 3.3).

In this paper we explore a canonical decomposition of strong $L^2$-functions. We first observe that if $\Phi$ is an operator-valued $L^\infty$-function, then the kernel of the Hankel operator $H_{\Phi^*}$ is shift-invariant. Thus by the Beurling-Lax-Halmos Theorem, the kernel of a Hankel operator $H_{\Phi^*}$ is of the form $\Delta H^2_E$, for some inner function $\Delta$. If the kernel of a Hankel operator $H_{\Phi^*}$ is trivial, take $E^\prime = \{0\}$. Of course, $\Delta$ need not be a two-sided inner function. In fact, we can show that if $\Phi$ is an operator-valued $L^\infty$-function and $\Delta$ is a two-sided inner function, then the kernel of a Hankel operator $H_{\Phi^*}$ is $\Delta H^2_E$, if and only if $\Phi$ is expressed in the form

$$\Phi = \Delta A^\ast,$$

(1)

where $A$ is an operator-valued $H^{\infty}$-function such that $\Delta$ and $A$ are right coprime (see Lemma 3.13). The expression (1) is called the Douglas-Shapiro-Shields factorization of an operator-valued $L^\infty$-function $\Phi$ (see [DSS], [FB], [Fu2]; in particular, [Fu2] contains many important applications of the Douglas-Shapiro-Shields factorization to the linear system theory). We recall that a meromorphic function $\varphi : \mathbb{D} \to \mathbb{C}$ is said to be of bounded type (or in the Nevanlinna class) if it is a quotient of two bounded analytic functions. A matrix function of bounded type is defined by a matrix-valued function whose entries are of bounded type. Very recently, a systematic study on matrix-valued functions of bounded type was done in a monograph [CHL3]. It was also known that every matrix-valued $L^\infty$-function whose adjoint is of bounded type admits the expression (1) (cf. [GHR]). In fact, if we extend the notion of “bounded type” for operator-valued $L^\infty$-functions (we will do this in section 3.4 for a more bigger class), then we may say that the expression (1) is a monopoly for $L^\infty$-functions whose flips are of bounded type, where the flip $\tilde{\Phi}$ of $\Phi$ is defined by $\tilde{\Phi}(z) := \Phi(\overline{z})$. From this viewpoint, we may ask whether there exists an appropriate decomposition corresponding to general $L^\infty$-functions, more generally, to strong $L^2$-functions. The following problem is the first object of this paper:

Find a canonical decomposition of strong $L^2$-functions.

For an answer to this problem, we should make several programs. This will be done in section 3. Firstly, to understand the smallest $S^\infty_E$-invariant subspace containing a subset $F \subseteq H^2_E$, we need to consider the kernels of the adjoints of unbounded Hankel operators with strong $L^2$-symbols involved with $F$. Thus we will deal with unbounded Hankel operators $H_{\Phi^*}$ with strong $L^2$-symbols $\Phi$. However, the adjoint of the unbounded Hankel operator need not be a Hankel operator. Of course, if $\Phi$ is an $L^\infty$-function then $H_{\Phi^*} = H_{\Phi^*}$. Thus for an $L^\infty$-symbol $\Phi$, we may use the notations $H_{\Phi^*}$ and $H_{\Phi^*}$ interchangeably with one another. By contrast, $H_{\Phi^*}$ may not be equal to $H_{\Phi^*}$ for a strong $L^2$-function $\Phi$. In particular, the kernel of an unbounded Hankel operator $H_{\Phi^*}$ is liable to be trivial because it is defined in the subset of polynomials in $H^2_D$. In spite of it, since the kernel of the adjoint of a densely defined operator is always closed, we can show that via the
Beurling-Lax-Halmos Theorem, the kernel of $H_\Phi^*$ with strong $L^2$-symbol $\Phi$ is still of the form $\Delta H_\Phi^*$ (see Corollary 3.15). Therefore, we will consider $H_\Phi^*$ rather than $H_\Phi$ for a strong $L^2$-function $\Phi$.

To establish a canonical decomposition of strong $L^2$-functions, we need to introduce a new notion - the "complementary factor", denoted by $\Delta_c$, of an inner function $\Delta$ with values in $B(D,E)$. This notion is defined by using the kernel of $\Delta^*$, denoted by $\ker \Delta^*$, which is defined by the set of vectors $f$ in $H_\Phi^*$ such that $\Delta^* f = 0$ a.e. on $T$. Moreover, the kernel of $H_{\Delta_c}$ can be computed by complementing the complementary factor $\Delta_c$ to $\Delta$ (see Lemma 3.16). We also employ a notion of "degree of non-cyclicity" on the set of all subsets (or vectors) of $H_\Phi^*$, which is a complementary notion of "degree of cyclicity" defined by V.I. Vasyunin and N.K. Nikolskii [VN]. The degree of non-cyclicity, denoted by $\text{nc}(F)$, of subsets $F$, $F \subseteq H_\Phi^*$, is defined by the number

$$\text{nc}(F) := \sup_{\zeta \in D} \dim \left\{ g(\zeta) : g \in H_\Phi^* \otimes E_F \right\}.$$ 

Thus, in comparison with the degree of cyclicity, the degree of non-cyclicity admits $\infty$, which is more beneficial to understand the Beurling-Lax-Halmos Theorem. If $\Phi$ is a strong $L^2$-function with values in $B(D,E)$ and $\{d_k\}_{k \geq 1}$ is an orthonormal basis for $D$, then we define $\{\Phi\} := \{\Phi d_k\}_{k \geq 1}$, which may be regarded as the set of "column" vectors of $\Phi$ because we may think of $\Phi$ as an infinite matrix-valued function. Then we can show that if the kernel of $H_\Phi^*$ is $\Theta H_\Phi^*$ for some inner function $\Theta$, then the dimension of $E'$ can be computed by the degree of non-cyclicity of $\{\Phi_*\}$, where $\Phi_*$ denotes the analytic part of $\Phi$ (Lemma 3.19). Thus, the degree of non-cyclicity of $\{\Phi_*\}$ is independent of a particular choice of the orthonormal basis of $D$, so that $\text{nc}(\Phi_*)$ is well-defined. As a corollary, we may also obtain an extension of Abrahamse’s Lemma ([Ab, Lemma 4]) to $L^2$-functions: if $\phi \in L^2$, then $\phi$ is of bounded type if and only if $\ker H_\phi^* \neq \{0\}$ (see Remark 3.20(3)).

On the other hand, if $\Delta$ is an inner matrix function, we may ask when we complement $\Delta$ to a two-sided inner function by aid of an inner matrix function $\Omega$, in other words, $[\Delta, \Omega]$ is two-sided inner, where $[\Delta, \Omega]$ is understood as an $1 \times 2$ operator matrix. In fact, this question can be answered by the Complementing Lemma (see [VN] or [N11]). To understand this question more precisely, we should remark that if $\Delta$ is an inner matrix function, $\Delta^*$ need not be of bounded type (see Example 3.4). In fact, $\Delta^*$ is of bounded type if and only if $\Delta$ is a complemented to a two-sided inner function: in this case, if $\Delta_c$ is the complementary factor of $\Delta$, then $[\Delta, \Delta_c]$ is two-sided inner (see Remark 3.20(4)).

The following question asks a more general complementation: If $\Delta$ is an $n \times r$ inner matrix function, which condition on $\Delta$ allows us to complement $\Delta$ to an $n \times (r + q)$ inner matrix function $[\Delta, \Omega]$ by aid of an $n \times q$ inner matrix function $\Omega$? An answer to this question is also subject to the degree of non-cyclicity of $\{\Delta\}$ (see Proposition 3.21).

Now, for a canonical decomposition of strong $L^2$-functions $\Phi$, we are tempted to guess that $\Phi$ can be factorized as the form $\Delta A^*$ (where $\Delta$ is a possibly one-sided inner function) like the Douglas-Shapiro-Shields factorization, in which $\Delta$ is two-sided inner. But this is not such a case. In fact, we can see that a canonical decomposition is affected by the kernel of $\Delta^*$ through some examples (see p. 31). Indeed, we recognize that this is not accidental. The following main theorem realizes the idea inside those examples:
Theorem A. (A canonical decomposition of strong $L^2$-functions) If $\Phi$ is a strong $L^2$-function with values in $\mathcal{B}(D,E)$, then $\Phi$ can be expressed in the form

$$\Phi = \Delta A^* + B,$$

(2)

where

(i) $\Delta$ is an inner function with values in $\mathcal{B}(E',E)$;
(ii) $\tilde{A}$ belongs to $H^2_s(B(D,E'))$ such that $\Delta$ and $A$ are right coprime;
(iii) $B$ is a strong $L^2$-function with values in $\mathcal{B}(D,E)$ such that $\Delta^* B = 0$;
(iv) $\text{nc} \{\Phi_+\} \leq \dim E'$.

In particular, if $\dim E' < \infty$ (more specially, $\dim E < \infty$), then the expression (2) is unique (up to a unitary constant right factor): in this case, $\Delta$ is given by the equation

$$\ker H^s_{\Phi} = \Delta H^2_{E'}.$$ 

Moreover, if $\tilde{\Delta}$ is of bounded type then $B$ in (2) is given by $B = \Delta_c \Delta^* \Phi$, where $\Delta_c$ is the complementary factor of $\Delta$, with values in $\mathcal{B}(E'',E)$.

Definition. The expression (2) will be called a canonical decomposition of a strong $L^2$-function $\Phi$.

The proof of Theorem A (p. 32) shows that the inner function $\Delta$ in a canonical decomposition (2) of a strong $L^2$-function $\Phi$ can be obtained from equation $\ker H^s_{\Phi} = \Delta H^2_{E'}$ which is guaranteed by the Beurling-Lax-Halmos Theorem (see Corollary 3.15). In this case, the expression (2) will be called the BLH-canonical decomposition of $\Phi$ in the viewpoint that $\Delta$ comes from the Beurling-Lax-Halmos Theorem. However, if $\dim E' = \infty$ (even though $\dim D < \infty$), then it is possible to get another inner function $\Theta$ of a canonical decomposition (2) for the same function: in this case, $\ker H^s_{\Phi} \neq \Theta H^2_{E'}$. Therefore the canonical decomposition of a strong $L^2$-function is not unique in general (see Remark 4.3). But the second assertion of Theorem A says that if the codomain of $\Phi(z)$ is finite-dimensional (in particular, if $\Phi$ is a matrix-valued $L^2$-function), then the canonical decomposition (2) of $\Phi$ is unique, in other words, the inner function $\Delta$ in (2) should be obtained from the equation $\ker H^s_{\Phi} = \Delta H^2_{E'}$. Thus the unique canonical decomposition (2) of matrix-valued $L^2$-functions is just the BLH-canonical decomposition.

If the flip $\tilde{\Phi}$ of $\Phi$ is of bounded type (or equivalently, $\ker H^s_{\Phi} = \Delta H^2_{E'}$ for some two-sided inner function $\Delta$), then we have $B = 0$ in the BLH-canonical decomposition (2) of $\Phi$, which reduces to the Douglas-Shapiro-Shields factorization. In fact, the Douglas-Shapiro-Shields factorization was given for $L^\infty$-functions, while the case of $B = 0$ in the BLH-canonical decomposition (2) is an extended version to strong $L^2$-functions. Prior to this result, we notice that the notion of “bounded type” for matrix-valued functions is not appropriate for operator-valued functions, i.e., the statement “each entry of the matrix is of bounded type” is not a natural extension to operator-valued functions even though it may have a meaning for infinite matrices (remember that we treat operators between separable complex Hilbert spaces). Thus we need to introduce an appropriate notion of “bounded type” for strong $L^2$-functions. We will do this in section 3.4. Let $D_\infty \equiv \{z : 1 < |z| \leq \infty\}$, the complement of the closed unit disk in the extended complex plane $\mathbb{C}_\infty$. To guarantee the statement “each entry is of bounded type,” we adopt
the notion of “meromorphic pseudocontinuation of bounded type” in $D_e$ for strong $L^2$-functions (cf. [Fu1]) - by definition, the meromorphic pseudocontinuation $\Phi$ of a $B(D, E)$-valued strong $L^2$-function $\Phi$ is of the form $\Phi = \frac{\theta}{\Delta}$ (where $\theta$ is a scalar inner function in $D_e$) and $G$ is a strong $H^2$-function defined in $D_e$, with values in $B(D, E)$) such that $\Phi$ is the nontangential SOT limit of $\Phi$, which is collapsed to the notion of “bounded type” for the flips of matrix-valued functions (see Corollary 3.31).

On the other hand, we recall that the spectral multiplicity for a bounded linear operator $T$ acting on a separable complex Hilbert space $E$ is defined by the number $\mu_T$: \[
\mu_T := \inf \dim F,
\]
where $F \subseteq E$, the infimum being taken over all generating subspaces $F$, i.e., subspaces such that $M_F = \sqrt{\{T^n F : n \geq 0\}} = E$. In the definition of the spectral multiplicity, $F$ may be taken as a subset rather than a subspace. In this case, we may regard $\mu_T = \inf \dim \{f : f \in F\}$ such that $M_F = E$. Unless this leads to ambiguity, we will deal with $M_F$ for subsets $F \subseteq E$. If $S_E$ is the shift operator on $H^2_E$, then it was known that $\mu_S_E = \dim E$. By contrast, if $S^\perp_E$ is the backward shift operator on $H^2_E$, then $S^\perp_E$ has a cyclic vector, i.e., $\mu_{S^\perp_E} = 1$. Moreover, the cyclic vectors of $S^\perp_E$ form a dense subset of $H^2_E$ (see [Ha2], [Ni1], [Wo]). We here observe that the problem of finding $\inf \dim \{F \subseteq E : \mathcal{H}(\Delta) = E\}$ is identical to the problem of finding the spectral multiplicity of the truncated backward shift operator $S^\perp_E|_{\mathcal{H}(\Delta)}$, i.e., the restriction of $S^\perp_E$ to its invariant subspace $\mathcal{H}(\Delta)$. The truncated backward shift operator has been studied by many authors because it is the functional model for a contraction operator $T \in B(\mathcal{H})$ for a complex Hilbert space $\mathcal{H}$, satisfying that $T^n x \to 0$ as $n \to \infty$ for each $x \in \mathcal{H}$; i.e., if $||T|| \leq 1$ and $T^n x \to 0$ for each $x \in \mathcal{H}$, then $T$ is unitarily equivalent to $S^\perp_E|_{\mathcal{H}(\Delta)}$ for an inner function $\Delta$ (cf. [Ni1], [SFBK]). Thus the truncated backward shift operator is often called the model operator. The second object of this paper is to show that this problem has a deep connection with a canonical decomposition of strong $L^2$-functions involved with the inner function $\Delta$.

To proceed, we need to consider a question. By the Beurling-Lax-Halmos Theorem, we saw that the kernel of the adjoint of a Hankel operator with a strong $L^2$-symbol is of the form $\Delta H^2_\Phi$, for some inner function $\Delta$. In view of its converse, we may ask the following question: Is every shift-invariant subspace $\Delta H^2_\Phi$, represented by the kernel of $H^* \Phi$ with some strong $L^2$-symbol $\Phi$ with values in $B(D, E)$? This question is to ask whether a strong $L^2$-solution $\Phi$, with values in $B(D, E)$, always exists for the equation ker $H^* \Phi = \Delta H^2_\Phi$, for a given inner function $\Delta$ with values in $B(E', E)$. Indeed, we give an affirmative answer to this question (see Lemma 5.1). The matrix-valued version of this version is as follows (see Corollary 5.2): for a given $n \times r$ inner matrix function $\Delta$, there exists at least a solution $\Phi \in L^2_{M_{n \times m}}$ (namely, $m \leq r + 1$) of the equation ker $H^* \Phi = \Delta H^2_\Phi$. Thus it is reasonable to ask whether such a solution $\Phi \in L^2_{M_{n \times m}}$ exists for each $m = 1, 2, \ldots$ even though it exists for some $m$. But the answer to this question is negative (see Remark 5.4). From this viewpoint, we may ask how to determine a possible dimension of $D$ for which there exists a strong $L^2$-solution $\Phi$, with values in $B(D, E)$, of the equation ker $H^* \Phi = \Delta H^2_\Phi$. In fact, we would like to ask what is the infimum of $\dim D$ such that there exists a strong $L^2$-solution $\Phi$ of the equation ker $H^* \Phi = \Delta H^2_\Phi$. To find a way to determine such an infimum, we introduce a notion of the “Beurling degree” for an inner function by employing the canonical decomposition of a strong $L^2$-function induced by the given inner function: if
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$\Delta$ is an inner function with values in $\mathcal{B}(E', E)$, then the Beurling degree, denoted by $\deg_B(\Delta)$, of $\Delta$ is defined by the infimum of the dimension of nonzero space $D$ for which there exists a pair $(A, B)$ such that $\Phi \equiv \Delta A^* + B$ is a canonical decomposition of a strong $L^2$-function $\Phi$ with values in $\mathcal{B}(D, E)$ (Definition 5.5).

Now we can prove that if $\Delta$ is an inner function with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$ (in particular, $\Delta$ is an inner matrix function) then the spectral multiplicity of the model operator, i.e., the truncated backward shift on $\mathcal{H}(\Delta)$ is equal to the Beurling degree of the inner function $\Delta$ - this is the second object of this paper.

**Theorem B.** (The spectral multiplicity and the Beurling degree) Given an inner function $\Delta$ with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$, let $T := S^*_E|_{\mathcal{H}(\Delta)}$. Then

$$\mu_T = \deg_B(\Delta).$$ (3)

On the other hand, given a special interest to the case of $\deg_B(\Delta) = 1$ for an inner function $\Delta$. In view of Theorem B, if $\dim E' < \infty$ and $T := S^*_E|_{\mathcal{H}(\Delta)}$, then this case is equal to the case of $\mu_T = 1$, in other words, $T$ has a cyclic vector. We may thus consider the following question: For which inner function $\Delta$, does it follow

$$\deg_B(\Delta) = 1?$$ (4)

To get an answer to the question (4), we need to consider the notion of “characteristic scalar inner functions” for inner functions, which is a generalization of the cases of twosided inner matrix functions (often, we call it square-inner matrix function) (cf. [Hel], [SFBK], [CHL3]). On the other hand, if $\Delta$ is an inner function and $\Delta_c$ is the complementary factor of $\Delta$, we write $\Delta_{cc} \equiv (\Delta_c)c$, $\Delta_{ccc} \equiv (\Delta_{cc})c$, etc. for iterated complementary factors of $\Delta$. Then the key idea for an answer to the question (4) is given in the following: If an inner function $\Delta$ has a meromorphic pseudocontinuation of bounded type in $D_\epsilon$ and if $\tilde{\Delta}$ is an outer function then $\Delta_{cc} = \Delta$ (see Lemma 6.13). We can then get an answer to the question (4):

**Theorem C.** Let $\Delta$ be an inner function having a meromorphic pseudocontinuation of bounded type in $D_\epsilon$. If $\tilde{\Delta}$ is an outer function, then $\deg_B(\Delta) = 1$.

Since for an inner matrix function $\Delta$, the condition “$\Delta$ has a meromorphic pseudocontinuation of bounded type” in $D_\epsilon$ is equivalent to the condition “$\tilde{\Phi}$ is of bounded type” (see Corollary 3.31), the matrix-valued version of Theorem C can be rephrased as: If $\Delta$ is an inner matrix function whose flip $\tilde{\Delta}$ is of bounded type and if $\Delta'$, the transpose of $\Delta$, is an outer function, then $\deg_B(\Delta) = 1$ (see Corollary 6.14).

On the other hand, we may ask whether the converse of the key idea (Lemma 6.13) for Theorem C is true: i.e., if $\Delta$ is an inner function having a meromorphic pseudocontinuation of bounded type in $D_\epsilon$ and $\Delta_{cc} = \Delta$, does it follow that $\tilde{\Delta}$ is an outer function? We can show that the answer to this question is affirmative when $\Delta$ is an inner matrix function: i.e., if $\Delta_{cc} = \Delta$, then $\tilde{\Delta}$ is an outer function when $\Delta$ is an inner matrix function whose flip $\tilde{\Delta}$ is of bounded type (see Corollary 6.15).

The organization of this paper is as follows. The main results of this paper are Theorem A, Theorem B and Theorem C. In section 2, we provide notations and preliminary notions, which will be used in this paper. To prove the main theorems, we need to
consider several auxiliary lemmas, and new notions of the “complementary factor” of an inner function, the “degree of non-cyclicity”, “bounded type” strong $L^2$-functions, and the “Beurling degree” of an inner function. In section 3, we provide auxiliary lemmas to prove the main results. In section 3.1, we review vector-valued Hardy classes, and then prove some properties which will be used in the sequel. In section 3.2, we introduce the Beurling-Lax-Halmos Theorem and the Douglas-Shapiro-Shields factorization, and in section 3.3, we coin the notions of the complementary factor of an inner function and the degree of non-cyclicity. In section 3.4, we introduce strong $L^2$-functions of bounded type, which is collapsed to the functions whose entries are of bounded type for the matrix-valued functions. In section 4, we establish a canonical decomposition of a strong $L^2$-functions and give a proof of Theorem A. In section 5, we establish a connection between the spectral multiplicity of the truncated backward shift on $H(\Delta)$ and the Beurling degree of the inner function $\Delta$ and give a proof of Theorem B. As an application of the preceding results, in section 6.1, we consider the characteristic scalar inner function of an inner function, by using results of section 3.4, and then in section 6.2, we analyze the case of the Beurling degree 1 and give a proof of Theorem C.

2. Preliminaries

In this section we provide notations and preliminary notions, which will be used in this paper.

(a) Write $\mathbb{D}$ for the open unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T}$ for the unit circle in $\mathbb{C}$. To avoid a confusion, we will write $z$ for points on $\mathbb{T}$ and $\zeta$ for points in $\mathbb{D}$.

(b) For $\phi \in L^2$, write
\[ \hat{\phi}(z) := \phi(\overline{z}) \quad \text{and} \quad \check{\phi}(z) := \overline{\phi(\overline{z})}. \]
For $\phi \in L^2$, write
\[ \phi_+ := P_+ \phi \quad \text{and} \quad \phi_- := P_- \phi, \]
where $P_+$ and $P_-$ are the orthogonal projections from $L^2$ onto $H^2$ and $L^2 \ominus H^2$, respectively. Then we may write $\phi = \hat{\phi}_- + \check{\phi}_+.$

(c) Throughout the paper, we assume that

- $X$ and $Y$ are complex Banach spaces;
- $D$ and $E$ are separable complex Hilbert spaces.

Write $B(X, Y)$ for the set of all bounded linear operators from $X$ to $Y$ and abbreviate $B(X, X)$ to $B(X)$. For a complex Banach space $X$, write $X^*$ for its dual and $\langle x, \phi \rangle$ for $\phi(x)$ for $x \in X$ and $\phi \in X^*$.

(d) If $A : D \to E$ is a linear operator whose domain is a subspace of $D$, then $A$ is also a linear operator from the closure of the domain of $A$ into $E$. So we will only consider those $A$ such that the domain of $A$ is dense in $D$. Such an operator $A$ is said to be densely defined. If $A : D \to E$ is densely defined, we write dom $A$, ker $A$, and ran $A$ for the domain, the kernel, and the range of $A$, respectively. It is well known from the unbounded operator theory (cf. [Go], [Con]) that if $A$ is densely defined, then ker $A^* = (\text{ran } A)^\perp$, so that ker $A^*$ is closed even though ker $A$ is not closed.
(e) We recall ([Ab], [Co2], [GHR], [Ni1]) that a meromorphic function \( \phi : \mathbb{D} \to \mathbb{C} \) is said to be of \textit{bounded type} (or in the Nevanlinna class \( \mathcal{N} \)) if there are functions \( \psi_1, \psi_2 \in H'^\infty \) such that

\[
\phi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.
\]

It was known that \( \phi \) is of bounded type if and only if \( \phi = \frac{\psi_1}{\psi_2} \) for some \( \psi_1 \in H^p \) \((p > 0, i = 1, 2)\). If \( \psi_2 = \psi^i \psi^e \) is the inner-outer factorization of \( \psi_2 \), then \( \phi = \overline{\psi^a e^a} \). Thus if \( \phi \in L^2 \) is of bounded type, then \( \phi \) can be written as

\[
\phi = \frac{\psi_1}{\psi_2},
\]

where \( \theta \) is inner, \( a \in H^2 \) and \( \theta \) and \( a \) are coprime.

(f) Write \( D_{e} := \{ z : 1 < |z| \leq \infty \} \). For a function \( g : D_{e} \to \mathbb{C} \), define a function \( g_{D} : \mathbb{D} \to \mathbb{C} \) by

\[
g_D(\zeta) := g(1/\zeta) \quad (\zeta \in \mathbb{D}).
\]

For a function \( g : D_{e} \to \mathbb{C} \), we say that \( g \) belongs to \( H^p(D_{e}) \) if \( g_D \in H^p \) \((1 \leq p \leq \infty)\). A function \( g : D_{e} \to \mathbb{C} \) is said to be of \textit{bounded type} if \( g_D \) is of bounded type.

(g) If \( f \in H^2 \), then the function \( \tilde{f} \) defined in \( D_{e} \) is called a \textit{pseudocontinuation} of \( f \) if \( \tilde{f} \) is a function of bounded type and \( \tilde{f}(z) = f(z) \) for almost all \( z \in \mathbb{T} \). Then we can easily show that \( \tilde{f} \) is of bounded type if and only if \( f \) has a pseudocontinuation \( \tilde{f} \). In this case, \( f_D(z) = \overline{f(z)} \) for almost all \( z \in \mathbb{T} \). In particular,

\[
\phi = \tilde{\phi}_- + \phi_+ \in L^2 \text{ is of bounded type } \iff \phi_- \text{ has a pseudocontinuation.} \quad (5)
\]

(h) Write \( M_{n \times m} \) for the set of \( n \times m \) complex matrices and abbreviate \( M_{n \times m} \) to \( M_n \).

(i) Write \( \text{g.c.d.}(\cdot) \) and \( \text{l.c.m.}(\cdot) \) for the greatest common inner divisor and the least common inner multiple, respectively. Also, write \( \text{left-g.c.d.}(\cdot) \) and \( \text{left-l.c.m.}(\cdot) \) for the greatest common left inner divisor and the least common left inner multiple, respectively.

(j) Let \( (\Omega, \mathcal{M}, \mu) \) be a positive \( \sigma \)-finite measure space and \( X \) be a complex Banach space. A function \( f : \Omega \to X \) of the form \( f = \sum_{k=1}^{\infty} x_k \chi_{\sigma_k} \) (where \( x_k \in X \), \( \sigma_k \in \mathcal{M} \) and \( \sigma_k \cap \sigma_j = \emptyset \) for \( k \neq j \)) is said to be \textit{countable-valued}. A function \( f : \Omega \to X \) is called \textit{weakly measurable} if the map \( s \mapsto \langle f(s), \phi \rangle \) is measurable for all \( \phi \in X^* \) and is called \textit{strongly measurable} if there exist countable-valued functions \( f_n \) such that \( f(s) = \lim_n f_n(s) \) for almost all \( s \in \Omega \). It is known that when \( X \) is separable,

(a) If \( f \) is weakly measurable, then \( \|f\| \) is measurable;

(b) \( f \) is strongly measurable if and only if it is weakly measurable.

A countable-valued function \( f = \sum_{k=1}^{\infty} x_k \chi_{\sigma_k} \) is called \textit{(Bochner) integrable} if

\[
\int_{\Omega} \|f(s)\|d\mu(s) < \infty
\]

and its integral is defined by

\[
\int_{\Omega} f d\mu := \sum_{k=1}^{\infty} x_k \mu(\sigma_k).
\]
A function $g : \Omega \to X$ is called integrable if there exist countable-valued integrable functions $g_n$ such that $g(s) = \lim_n g_n(s)$ for almost all $s \in \Omega$ and $\lim_n \int_\Omega |g - g_n|d\mu = 0$. Then $\int_\Omega g d\mu \equiv \lim_n \int_\Omega g_n d\mu$ exists and $\int_\Omega g d\mu$ is called the (Bochner) integral of $g$. If $f : \Omega \to X$ is integrable, then we can see that

$$T \left( \int_\Omega f d\mu \right) = \int_\Omega (Tf) d\mu \text{ for each } T \in \mathcal{B}(X,Y).$$

(6)

(k) For a function $\Phi : \mathbb{T} \to \mathcal{B}(D,E)$, write $\Phi^*(z) := \Phi(z)^*$ for $z \in \mathbb{T}$.

### 3. Auxiliary lemmas

To prove the main theorems, we need to set up the new notions of the “complementary factor” of an inner function, the “degree of non-cyclicity” and “bounded type” strong $L^2$-functions and then establish several auxiliary lemmas.

#### 3.1. Vector-valued Hardy classes

In this section, we consider vector-valued $L^p$- and $H^p$-functions, using [FF], [Ni1], [Ni2], [Pe], [Sa] for general references and then derive some properties, which will be used in the sequel.

Let $m$ denote the normalized Lebesgue measure on $\mathbb{T}$. For a complex Banach space $X$ and $1 \leq p \leq \infty$, let

$$L^p_X \equiv L^p(\mathbb{T}, X) := \{ f : \mathbb{T} \to X : f \text{ is strongly measurable and } ||f||_p < \infty \},$$

where

$$||f||_p \equiv ||f||_{L^p_X} := \begin{cases} \left( \int_\mathbb{T} ||f(z)||_X^p dm(z) \right)^{\frac{1}{p}} & (1 \leq p < \infty); \\ \text{ess sup}_{z \in \mathbb{T}} ||f(z)||_X & (p = \infty). \end{cases}$$

Then we can see that $L^p_X$ forms a Banach space. For $f \in L^1_X$, the $n$-th Fourier coefficient of $f$, denoted by $\widehat{f}(n)$, is defined by

$$\widehat{f}(n) := \int_\mathbb{T} z^n f(z) dm(z) \text{ for each } n \in \mathbb{Z}.$$ 

Also, $H^p_X \equiv H^p(\mathbb{T}, X)$ is defined by the set of $f \in L^p_X$ with $\widehat{f}(n) = 0$ for $n < 0$. Also we write $H^2(\mathbb{D}, X)$ for the set of all analytic functions $f : \mathbb{D} \to X$ satisfying

$$||f||_{H^2(\mathbb{D}, X)} := \sup_{0<r<1} \left( \int_\mathbb{T} ||f(rz)||_X^2 dm(z) \right)^{\frac{1}{2}} < \infty.$$ 

If $E$ is a separable Hilbert space then we conventionally identify $H^2(\mathbb{D}, E)$ with $H^2_E(\mathbb{T}, E)$: in this case, as in the scalar-valued case, if $f \in H^2(\mathbb{D}, E)$, then “nontangential limit $bf$ exists a.e. on $\mathbb{T}$ and $f$ can be recaptured by the Poisson integral $bf \ast P_r$ (cf. [Ni2, Theorem 3.11.7]). For $f, g \in L^2_E$ with a separable complex Hilbert space $E$, the inner product $\langle f, g \rangle$ is defined by

$$\langle f, g \rangle \equiv \langle f(z), g(z) \rangle_{L^2_E} := \int_\mathbb{T} \langle f(z), g(z) \rangle_E dm(z).$$
If \( f, g \in L^2_X \) with \( X = M_{n \times m} \), then \( \langle f, g \rangle = \int_T \text{tr}(g^*f)dm \).

A function \( \Phi : T \to B(X,Y) \) is called SOT measurable if \( z \mapsto \Phi(z)x \) is strongly measurable for every \( x \in X \) and is called WOT measurable if \( z \mapsto \Phi(z)x \) is weakly measurable for every \( x \in X \). We can easily check that if \( \Phi : T \to B(X,Y) \) is strongly measurable, then \( \Phi \) is SOT-measurable and if \( D \) and \( E \) are separable complex Hilbert spaces then \( \Phi : T \to B(D, E) \) is SOT measurable if and only if \( \Phi \) is WOT measurable.

We then have:

**Lemma 3.1.** If \( \Phi : T \to B(D, E) \) is WOT measurable, then so is \( \Phi^* \).

*Proof.* Suppose that \( \Phi \) is WOT measurable. Then the function

\[
\langle \Phi^*(z)y, x \rangle = \langle x, \Phi^*(z)y \rangle = \langle \Phi(z)x, y \rangle.
\]

is measurable for all \( x \in D \) and \( y \in E \). Thus the function \( z \mapsto \langle \Phi^*(z)y, x \rangle \) is measurable for all \( x \in D \) and \( y \in E \). \( \square \)

Let \( \Phi : T \to B(D, E) \) be a WOT measurable function. Then \( \Phi \) is called WOT integrable if \( \langle \Phi(x), y \rangle \in L^1 \) for every \( x \in D \) and \( y \in E \), and there exists an operator \( U \in B(D, E) \) such that \( \langle Ux, y \rangle = \int_T \langle \Phi(z)x, y \rangle dm(z) \). Also \( \Phi \) is called SOT integrable if \( \Phi(x) \) is integrable for every \( x \in D \). In this case, the operator \( V : x \mapsto \int_T \Phi(z)x dm(z) \) is bounded, i.e., \( V \in B(D, E) \). If \( \Phi : T \to B(D, E) \) is SOT integrable, then it follows from (6) that for every \( x \in D \) and \( y \in E \),

\[
\int_T \langle \Phi(z)x dm(z), y \rangle = \int_T \langle \Phi(z)x, y \rangle dm(z),
\]

which implies that \( \Phi \) is WOT integrable and that the SOT integral of \( \Phi \) is equal to the WOT integral of \( \Phi \).

We can say more:

**Lemma 3.2.** For \( \Phi \in L^1_{B(D, E)} \), the Bochner integral of \( \Phi \) is equal to the SOT integral of \( \Phi \), i.e.,

\[
\left( \int_T \Phi(z) dm(z) \right) x = \int_T \Phi(z) x dm(z) \quad \text{for all } x \in D.
\]

*Proof.* This follows from a straightforward calculation. \( \square \)

Write \( L^\infty(B(D, E)) \) for the set of all bounded WOT measurable \( B(D, E) \)-valued functions on \( T \). For \( \Phi \in L^\infty(B(D, E)) \), define

\[
||\Phi||_{\infty} := \text{ess sup}_{z \in T} ||\Phi(z)||.
\]

Following to V. Peller [Pe], for \( 1 \leq p < \infty \), we define the class \( L^p_B(B(D, E)) \) by the set of all WOT measurable \( B(D, E) \)-valued functions \( \Phi \) on \( T \) such that \( \Phi(x) \in L^p_E \) for each \( x \in D \), i.e.,

\[
\int_T ||\Phi(z)x||^p dm(z) < \infty \quad \text{for each } x \in D.
\]

We can easily check that

\[
L^p_{B(D, E)} \subseteq L^p(B(D, E));
\]

\[
L^\infty_{B(D, E)} \subseteq L^\infty(B(D, E)) \subseteq L^p(B(D, E)).
\]
If $\Phi \in L^1(B(D,E))$ and $x \in D$, then $\Phi(\cdot)x \in L^1_B$. Thus the $n$-th Fourier coefficient $\widehat{\Phi(\cdot)x}(n)$ of $\Phi(\cdot)x$ is given by

$$\widehat{\Phi(\cdot)x}(n) = \int_\mathbb{T} \pi^n \Phi(z)x \, dm(z).$$

We now define the $n$-th Fourier coefficient of $\Phi \in L^1(B(D,E))$, denoted by $\widehat{\Phi}(n)$, by

$$\widehat{\Phi}(n)x := \Phi(\cdot)x(n) \quad (n \in \mathbb{Z}, \ x \in D).$$

A function $\Phi \in L^2(B(D,E))$ will be called a strong $L^2$-function with values in $B(D,E)$. We also define

$$H^2_s(B(D,E)) := \{ \Phi \in L^2(B(D,E)) : \widehat{\Phi}(n) = 0 \text{ for } n < 0 \},$$

in other words, a function $\Phi \in H^2_s(B(D,E))$ is a $B(D,E)$-valued function defined almost everywhere on the unit circle such that $\Phi(\cdot)x \in H^2_E$ for each $x \in D$. If $\Phi \in L^\infty(B(D,E))$, then $\Phi \in L^1(B(D,E))$. Thus we may define

$$H^\infty(B(D,E)) := \{ \Phi \in L^\infty(B(D,E)) : \widehat{\Phi}(n) = 0 \text{ for } n < 0 \}.$$ 

By (8), we have $L^1_{B(D,E)} \subseteq L^1(B(D,E))$. Thus if $\Phi \in L^1_{B(D,E)}$, then there are two kinds of definitions for the $n$-th Fourier coefficient of $\Phi$. However, we can, by Lemma 3.2, see that the $n$-th Fourier coefficient of $\Phi$ as an element of $L^1_{B(D,E)}$ coincides with the $n$-th Fourier coefficient of $\Phi$ as an element of $L^1(B(D,E))$. Also we note that if dim $D = m < \infty$ and dim $E = n < \infty$, then $L^2(B(D,E))$ and $H^2_s(B(D,E))$ are the spaces of matrix-valued functions, i.e.,

$$L^2(B(D,E)) = L^2_{M_{m \times m}} \text{ and } H^2_s(B(D,E)) = H^2_{M_{m \times m}}.$$ 

The terminology a “strong $H^2$-function” is reserved for the operator-valued functions on the unit disk $\mathbb{D}$, following to N.K. Nikolskii [Ni1]: A function $\Phi : \mathbb{D} \to B(D,E)$ is called a strong $H^2$-function if $\Phi(\cdot)x \in H^2(\mathbb{D},E)$ for each $x \in D$. In general, the boundary values of strong $H^2$-functions do not need to be bounded linear operators (defined almost everywhere on the unit circle $\mathbb{T}$). Thus we do not guarantee that the boundary value of a strong $H^2$-function belongs to $H^2_s(B(D,E))$. For example, if $\Phi$ is defined on the unit disk $\mathbb{D}$ by

$$\Phi(\zeta) = \begin{bmatrix} 1 & \zeta & \zeta^2 & \cdots \end{bmatrix} : \ell^2 \to \mathbb{C} \quad (\zeta \in \mathbb{D}),$$

then $\Phi(\zeta)$ is a bounded linear operator for each $\zeta \in \mathbb{D}$ and $\Phi(\zeta)x \in H^2(D,\mathbb{C})$ for each $x \equiv (x_n) \in \ell^2$. Thus $\Phi$ is a strong $H^2$-function with values in $B(\ell^2,\mathbb{C})$. However, the boundary value

$$\Phi(z) = \begin{bmatrix} 1 & z & z^2 & \cdots \end{bmatrix} : \ell^2 \to \mathbb{C} \quad (z \in \mathbb{T})$$

is not bounded for all $z \in \mathbb{T}$ because for any $z_0 \in \mathbb{T}$, if we let

$$x_0 := \begin{pmatrix} 1, \frac{x_0^2}{2}, \frac{x_0^3}{3}, \cdots \end{pmatrix}^t \in \ell^2,$$

then

$$\Phi(z_0)x_0 = 1 + \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

which shows that $\Phi \notin H^2_s(B(D,E))$. In spite of it, there are useful relations between the set of strong $H^2$-functions and the set $H^2_s(B(D,E))$. To see this, let $\Phi \in H^2_s(B(D,E))$. Then $\Phi(z) \in B(D,E)$ for almost all $z \in \mathbb{T}$ and $\Phi(z)x \in H^2_E$ for each $x \in D$. We now
define a (function-valued with domain $D$) function $p\Phi$ on the unit disk $D$ by the Poisson integral in the strong sense:

$$p\Phi(re^{i\theta})x := (\Phi(\cdot)x * P_r)(e^{i\theta}) \quad (x \in D)$$

$$= \int_0^{2\pi} P_r(\theta - t)\Phi(e^{it})x \, dm(t) \in E,$$

where $P_r(\cdot)$ is the Poisson kernel. Then $p\Phi(\zeta)x \in H^2(D, E)$ for all $\zeta \in D$. Thus, for all $\zeta \in D$, $p\Phi(\zeta)$ can be viewed as a function from $D$ into $E$. A straightforward calculation shows that $p\Phi(\zeta)$ is a linear map for each $\zeta \in \mathbb{D}$. Since $p\Phi(\zeta)x \in H^2(D, E)$ is the Poisson integral of $\Phi(z)x \in H^2_E$ we will conventionally identify $\Phi(z)x$ and $p\Phi(\zeta)x$ for each $x \in D$. From this viewpoint, we will also regard $\Phi \in H^2_B(D, E)$ as an (linear, but not necessarily bounded) operator-valued function defined on the unit disk. Thus if $\dim D < \infty$, then every function $\Phi \in H^2_B(D, E)$ becomes a strong $H^2$-function with values in $B(D, E)$.

We thus have:

**Lemma 3.3.** The following hold:

(a) If $\dim D < \infty$, then every function in $H^2_B(D, E)$ is a strong $H^2$-function.

(b) Every function in $H^2_B(B(D, E)) \cup H^\infty(B(D, E))$ is a strong $H^2$-function.

(c) If $\dim D < \infty$ and $\dim E < \infty$, then $H^2_B(B(D, E))$ coincides with the set of all strong $H^2$-functions with values in $B(D, E)$.

**Proof.** (a) This follows from the preceding remark.

(b) Let $\Phi \in H^2_B(B(D, E)) \cup H^\infty(B(D, E))$. Then we first claim that there exists $M > 0$ such that

$$\sup \left\{ ||\Phi(\cdot)x||_{L^2_E} : x \in D \text{ with } ||x|| = 1 \right\} < M,$$

which proves the claim (10). Now, in view of the preceding remark, it suffices to show $\Phi(\zeta) \in B(D, E)$ for all $\zeta \in \mathbb{D}$. Let $\zeta = re^{it} \in \mathbb{D}$ and $x \in D$ with $||x|| = 1$. Then for $y \in E$ with $||y||_{E} \leq 1$,

$$\left| \langle \Phi(re^{i\theta})x, y \rangle_E \right| = \left| \left\langle \int_0^{2\pi} P_r(\theta - t)\Phi(e^{it})x \, dm(t), y \right\rangle_E \right|$$

$$\leq \int_0^{2\pi} \left| P_r(\theta - t)\Phi(e^{it})x \right| \, dm(t) \leq \int_0^{2\pi} \left| \Phi(e^{it})x \right| \, dm(t),$$

(by (7))

$$\leq \frac{1+r}{1-r} \int_0^{2\pi} \left| \langle \Phi(e^{it})x, y \rangle_E \right| \, dm(t).$$
which implies, by our assumption,

$$
\|\Phi(\zeta)x\|_E \leq \frac{1 + r}{1 - r} \int_0^{2\pi} \|\Phi(e^{it})x\|_S dm(t)
$$

$$
= \frac{1 + r}{1 - r} \|\Phi(x)\|_{L^1_S} < \infty,
$$

which shows that $\Phi(\zeta) \in \mathcal{B}(D, E)$ for all $\zeta \in \mathbb{D}$. Thus $\Phi$ is a strong $H^2$-function.

(c) If $\dim D = m < \infty$ and $\dim E = n < \infty$, then $H^2_2(\mathcal{B}(D, E)) = H^2_{M_{n \times m}}$. Also by (b), $H^2_{M_{n \times m}}$ is contained in the set of all strong $H^2$-functions with values in $M_{n \times m}$. Conversely, if $\Phi$ is a strong $H^2$-function with values in $M_{n \times m}$, then $\langle \Phi(\zeta)x, y \rangle$ is a $H^2$-function for all $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$. Thus $\Phi \in H^2_{M_{n \times m}}$.

A function $\Delta \in H^\infty(\mathcal{B}(D, E))$ is called an inner function with values in $\mathcal{B}(D, E)$ if $\Delta(z)$ is an isometric operator from $D$ into $E$ for almost all $z \in \mathbb{T}$, i.e., $\Delta^*\Delta = I_D$ a.e. on $\mathbb{T}$ and is called a two-sided inner function if $\Delta\Delta^* = I_E$ a.e. on $\mathbb{T}$ and $\Delta^*\Delta = I_D$ a.e. on $\mathbb{T}$. If $\Delta$ is an inner function with values in $\mathcal{B}(D, E)$, we may assume that $D$ is a subspace of $E$, and if further $\Delta$ is two-sided inner then we may assume that $D = E$.

We write $\mathcal{P}_D$ for the set of all polynomials with values in $D$. If $F$ is a strong $H^2$-function with values in $\mathcal{B}(D, E)$, then the function $Fp$ belongs to $H^2_E$ for all $p \in \mathcal{P}_D$. The strong $H^2$-function $F$ is called outer if $e^{Fp} = H^2_E$. We then have an analogue of the scalar factorization theorem (called the inner-outer factorization): Every strong $H^2$-function $F$ with values in $\mathcal{B}(D, E)$ can be expressed in the form

$$
F = F^o F^e,
$$

where $F^e$ is an outer function with values in $\mathcal{B}(D, E')$ and $F^o$ is an inner function with values in $\mathcal{B}(E', E)$ for some subspace $E'$ of $E$ (cf. [Ni1, Corollary I. 9]). For a function $\Phi : \mathbb{T} \rightarrow \mathcal{B}(D, E)$, write

$$
\bar{\Phi}(z) := \Phi(\bar{z}), \quad \Phi^* := \bar{\Phi}^*.
$$

Often, $\hat{\Phi}$ is called the flip of $\Phi$. For $\Phi \in L^2_2(\mathcal{B}(D, E))$, we denote by $\overline{\Phi}^+ \equiv P_+ \Phi$ and $\Phi^+ \equiv P_+ \Phi$ the function

$$
(\overline{\Phi}^+(\cdot))x := P_+(\Phi(\cdot)x) \quad \text{a.e. on } \mathbb{T} \quad (x \in D);
$$

$$
(\Phi^+(\cdot))x := P_+(\Phi(\cdot)x) \quad \text{a.e. on } \mathbb{T} \quad (x \in D),
$$

where $P_+$ and $P_-$ are the orthogonal projections from $L^2_E$ onto $H^2_E$ and $L^2_E \ominus H^2_E$, respectively. Then we may write $\Phi \equiv \Phi^+ + \Phi^-$. Note that if $\Phi \in L^2_2(\mathcal{B}(D, E))$, then $\Phi^+, \Phi^- \in H^2_2(\mathcal{B}(D, E))$.

In the sequel, we will often encounter the adjoints of inner matrix functions. If $\Delta$ is a two-sided inner matrix function, it is easy to show that $\Delta^*$ is of bounded type, i.e., all entries of $\Delta^*$ are of bounded type (see p. 3). We also guess that if $\Delta$ is an inner matrix function then $\Delta^*$ is of bounded type. However the following example shows that this is not such a case.

**Example 3.4.** Let $h(z) := e^{1/z}$. Then $h \in H^\infty$ and $\overline{h}$ is not of bounded type. Let

$$
f(z) := \frac{h(z)}{\sqrt{2}||h||_\infty}.
$$

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Clearly, \( T \) is not of bounded type. Let \( h_1(z) := \sqrt{1 - |f(z)|^2} \). Then \( h_1 \in L^\infty \) and \( |h_1| \geq \frac{1}{\sqrt{2}} \). Thus there exists an outer function \( g \) such that \( |h_1| = |g| \) a.e. on \( T \) (see [Do1, Corollary 6.25]). Put

\[
\Delta := \begin{pmatrix} f \\ g \end{pmatrix} \quad (f, g \in H^\infty).
\]

Then \( \Delta^* \Delta = |f|^2 + |g|^2 = |f|^2 + |h_1|^2 = 1 \) a.e. on \( T \), which implies that \( \Delta \) is an inner function. Note that \( \Delta^* \) is not of bounded type.

For a function \( \Phi \in H^2_\sigma(B(D, E)) \), we say that an inner function \( \Delta \) with values in \( B(D', E) \) is a left inner divisor of \( \Phi \) if \( \Phi = \Delta A \) for \( A \in H^2_\sigma(B(D, D')) \). For \( \Phi \in H^2_\sigma(B(D_1, E)) \) and \( \Psi \in H^2_\sigma(B(D_2, E)) \), we say that \( \Phi \) and \( \Psi \) are left coprime if the only common left inner divisor of both \( \Phi \) and \( \Psi \) is a unitary operator. Also, for \( \Phi \in H^2_\sigma(B(E, D_1)) \) and \( \Psi \in H^2_\sigma(B(E, D_2)) \), we say that \( \Phi \) and \( \Psi \) are right coprime if \( \bar{\Phi} \) and \( \bar{\Psi} \) are left coprime. Left or right coprime-ness seems to be somewhat delicate problem. Left or right coprime-ness for matrix-valued functions was developed in [CHKL], [CHL1], [CHL2], [CHL3], and [FF].

**Lemma 3.5.** If \( \Theta \) is a two-sided inner function, then any left inner divisor of \( \Theta \) is two-sided inner.

**Proof.** Suppose that \( \Theta \) is a two-sided inner function with values in \( B(E) \) and \( \Delta \) is a left inner divisor, with values in \( B(E', E) \), of \( \Theta \). Then we may write \( \Theta = \Delta A \) for some \( A \in H^2_\sigma(B(E, E')) \). Since \( \Theta \) is two-sided inner, it follows that \( I_E = \Theta \Theta^* = \Delta AA^* \Delta^* \) a.e. on \( T \), so that \( I_E = \Delta^* \Delta = AA^* \) a.e. on \( T \). Thus \( I_E = \Delta \Delta^* \) a.e. on \( T \), and hence \( \Delta \) is two-sided inner. \( \square \)

**Lemma 3.6.** If \( \Phi \in L^\infty(B(D, E)) \), then \( \Phi^* \in L^\infty(B(E, D)) \). In this case,

\[
\widehat{\Phi^*}(-n) = \widehat{\Phi}(n) = \hat{\Phi}(n)^* \quad (n \in \mathbb{Z}).
\] (11)

In particular, \( \Phi \in H^\infty(B(D, E)) \) if and only if \( \Phi \in H^\infty(B(E, D)) \).

**Proof.** Suppose \( \Phi \in L^\infty(B(D, E)) \). Then \( \text{ess sup}_{z \in \mathbb{T}}||\Phi^*(z)|| = \text{ess sup}_{z \in \mathbb{T}}||\Phi(z)|| < \infty \), which together with Lemma 3.1 implies \( \Phi^* \in L^\infty(B(E, D)) \). The first equality of the assertion (11) comes from the definition. For the second equality, observe that for each \( x \in D \), \( y \in E \) and \( n \in \mathbb{Z} \),

\[
\langle \widehat{\Phi}(n), y \rangle = \langle \int_T \tau^n \Phi(z)x dm(z), y \rangle = \langle \int_T (\tau^n \Phi(z)x, y) dm(z) \rangle \quad (\text{by (7)})
\]

\[
= \int_T (x, \tau^n \Phi(z)y) dm(z)
\]

\[
= \langle x, \widehat{\Phi}(n)y \rangle.
\]

\( \square \)
Let $E$ be a separable complex Hilbert space. For a function $f : T \to E$, define $[f] : T \to B(C, E)$ by $[f](z)\alpha := \alpha f(z)$ ($\alpha \in \mathbb{C}$).

(12) If $g : T \to E$ is a countable-valued function of the form

$$ g = \sum_{k=1}^{\infty} x_k \chi_{\sigma_k} \quad (x_k \in E), $$

then for each $\alpha \in \mathbb{C}$,

$$ \left( \sum_{k=1}^{\infty} |x_k| \chi_{\sigma_k} \right) \alpha = \sum_{k=1}^{\infty} \alpha x_k \chi_{\sigma_k} = \alpha g = [g] \alpha, $$

which implies that $[g]$ is a countable-valued function of the form $[g] = \sum_{k=1}^{\infty} [x_k] \chi_{\sigma_k}$.

We then have:

**Lemma 3.7.** Let $E$ be a separable complex Hilbert space and $1 \leq p \leq \infty$. Define $\Gamma : L^p_E \to L^p_{B(C, E)}$ by

$$ \Gamma(f)(z) = [f](z), $$

where $[f](z) : C \to E$ is given by $[f](z)\alpha := \alpha f(z)$. Then

(a) $\Gamma$ is unitary, and hence $L^p_E \cong L^p_{B(C, E)}$;

(b) $L^p_{B(C, E)} = L^p_B(B(C, E))$ for $1 \leq p < \infty$;

(c) For each $f \in L^p_E$, $[f](n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$.

In particular, $H^p_E \cong H^p_{B(C, E)} = H^p(B(C, E))$ for $1 \leq p < \infty$.

**Proof.** (a) Let $f \in L^p_E$ ($1 \leq p \leq \infty$) be arbitrary. We first show that $[f] \in L^p_{B(C, E)}$. Since $f$ is strongly measurable, there exist countable-valued functions $f_n$ such that $f(z) = \lim_n f_n(z)$ for almost all $z \in T$. Observe that for almost all $z \in T$,

$$ ||[f](z)||_{B(C, E)} = \sup_{|\alpha|=1} ||[f](z)\alpha||_E = ||f(z)||_E. $$

Thus we have that

$$ ||f_n(z) - [f](z)||_{B(C, E)} = ||f_n(z) - f(z)||_E \to 0 \quad \text{as} \quad n \to \infty, $$

which implies that $[f]$ is strongly measurable and $||[f]||_{L^p_{B(C, E)}} = ||f||_{L^p_E}$. Thus $\Gamma$ is an isometry. For $h \in L^p_E$, let $g(z) := h(z)1 \in L^p_E$. Then for all $\alpha \in \mathbb{C}$, we have

$$ \Gamma(g)(z)\alpha = \alpha h(z)1 = h(z)\alpha, $$

which implies that $\Gamma$ is a surjection from $L^p_E$ onto $L^p_{B(C, E)}$. Thus $\Gamma$ is unitary, so that $L^p_E \cong L^p_{B(C, E)}$. This proves (a).

(b) Suppose $h \in L^p_B(B(C, E))$ ($1 \leq p < \infty$). If $g(z) := h(z)1 \in L^p_E$, then $h = [g] \in L^p_{B(C, E)}$. The converse is clear.

(c) Let $f \in L^p_E$. Then for all $\alpha \in \mathbb{C}$ and $n \in \mathbb{Z}$,

$$ \hat{f}(n)\alpha = \int_T \tau^n [f](z)\alpha dm = \alpha \int_T \tau^n f(z) dm = \alpha f(n) = [f(n)]\alpha, $$

which gives (c).

The last assertion follows at once from (b) and (c). \qed
A canonical decomposition of strong $L^2$-functions

**Remark 3.8.** By Lemma 3.7, we can see that if $1 \leq p < \infty$ and $\dim D < \infty$, then
\[ L^p_{B(D,E)} = L^p_{\ast}(B(D,E)) \quad \text{and} \quad H^p_{B(D,E)} = H^p_{\ast}(B(D,E)). \]

**Lemma 3.9.** Let $1 \leq p < \infty$. If $\Phi \in L^\infty(B(D,E))$, then $\Phi L^p_{\ast}(B(E', D)) \subseteq L^p_{\ast}(B(E', E))$. Also, if $\Phi \in H^\infty(B(D, E))$, then $\Phi H^p_{\ast}(B(E', D)) \subseteq H^p_{\ast}(B(E', E))$.

**Proof.** Suppose that $\Phi \in L^\infty(B(D, E))$ and $A \in L^p_{\ast}(B(E', D))$. Let $x \in E'$ be arbitrary. Then we have $A(z)x \in L^p_{\ast}(D)$. Let $\{d_k\}_{k \geq 1}$ be an orthonormal basis for $D$. Then we may write
\[ A(z)x = \sum_{k \geq 1} \langle A(z)x, d_k \rangle d_k \quad \text{for almost all } z \in T. \]

Thus it follows that for all $y \in E$,
\[ \langle \Phi(z)A(z)x, y \rangle = \sum_{k \geq 1} \langle A(z)x, d_k \rangle \langle \Phi(z) d_k, y \rangle, \]

which implies that $\Phi A$ is WOT measurable. On the other hand, since $\Phi \in L^\infty(B(D, E))$, it follows that
\[ \int_T \| \Phi A(z)x \|^p_{\ast} dm(z) \leq \| \Phi \|^p_{\infty} \int_T \| A(z)x \|^p_{\ast} dm(z) < \infty \quad (x \in E'), \]

which implies that $\Phi A \in L^p_{\ast}(B(E', E))$. This proves the first assertion. For the second assertion, suppose $\Phi \in H^\infty(B(D, E))$ and $A \in H^p_{\ast}(B(E', D))$. Then $\Phi A \in L^p_{\ast}(B(E', E))$. Assume to the contrary that $\Phi A \notin H^p_{\ast}(B(E', E))$. Thus, there exists $n_0 > 0$ such that $\hat{\Phi} A(-n_0) \neq 0$. Thus for some $x_0 \in E'$,
\[ \int_T z^{n_0} \Phi(z)A(z)x_0 dm(z) \neq 0. \]

Then by (7), there exists a nonzero $y_0 \in E$ such that
\[ 0 \neq \left( \int_T z^{n_0} \Phi(z)A(z)x_0 dm(z), y_0 \right) = \int_T \langle A(z)x_0, \varpi^{n_0} \Phi^*(z)y_0 \rangle dm(z). \]

On the other hand, since $\Phi \in H^\infty(B(D, E))$, it follows from Lemma 3.6 that $\hat{\Phi^*}(n_0) = \hat{\Phi}(-n_0)^* = 0$. Thus it follows from (7) that
\[ 0 = \langle \hat{\Phi^*}(n_0)y_0, A(z)x_0 \rangle = \int_T \varpi^{n_0} \Phi^*(z)y_0, A(z)x_0 dm(z), \]

a contradiction. \hfill \Box

**Corollary 3.10.** Let $1 \leq p < \infty$. If $\Phi \in L^\infty(B(D, E))$, then $\Phi L^p_{\ast(D)} \subseteq L^p_{\ast(E)}$. Also, if $\Phi \in H^\infty(B(D, E))$, then $\Phi H^p_{\ast(D)} \subseteq H^p_{\ast(E)}$.

**Proof.** Suppose that $\Phi \in L^\infty(B(D, E))$. Then for $f \in L^p_{\ast(D)}$, we can see that $[\Phi f] = \Phi [f]$. The result thus follows from Lemma 3.7 and Lemma 3.9. \hfill \Box

For an inner function $\Delta \in H^\infty(B(D, E))$, $\mathcal{H}(\Delta)$ denotes the orthogonal complement of the subspace $\Delta H^2_{\ast(D)}$ in $H^2_{\ast(E)}$, i.e.,
\[ \mathcal{H}(\Delta) := H^2_{\ast(E)} \ominus \Delta H^2_{\ast(D)}. \]

We then have:
Corollary 3.11. Let $\Delta$ be an inner function with values in $\mathcal{B}(D,E)$. Then $f \in \mathcal{H}(\Delta)$ if and only if $f \in H^2_E$ and $\Delta^* f \in L^2_D \ominus H^2_D$.

Proof. Let $f \in H^2_E$. By Lemma 3.6 and Corollary 3.10, $\Delta^* f \in L^2_D$. Then $f \in \mathcal{H}(\Delta)$ if and only if $\langle f, \Delta g \rangle = 0$ for all $g \in H^2_D$ if and only if $\langle \Delta^* f, g \rangle = 0$ for all $g \in H^2_D$, which gives the result. $\square$

3.2. The Beurling-Lax-Halmos Theorem

We first review a few essential facts for (vectorial) Toeplitz operators and (vectorial) Hankel operators, and for that we will use [BS], [Do1], [Do2], [MR], [Ni1], [Ni2], and [Pe]. For $\Phi \in L^2(B(D,E))$, a Hankel operator $H_\Phi : H^2_D \rightarrow H^2_E$ is a densely defined operator defined by

$$H_\Phi p := J P_{-}(\Phi p) \quad (p \in \mathcal{P}_D),$$

where $J$ denotes the unitary operator from $L^2_E$ to $L^2_E$ given by $(Jg)(z) := \overline{g(\overline{z})}$ for $g \in L^2_E$. Also a Toeplitz operator $T_\Phi : H^2_D \rightarrow H^2_E$ is a densely defined operator defined by

$$T_\Phi p := P_{+}(\Phi p) \quad (p \in \mathcal{P}_D).$$

The following lemma gives a characterization of the bounded Hankel operators on $H^2_D$.

Lemma 3.12. [Pe, Theorem II.2.2] Let $\Phi$ be a strong $L^2$-function with values in $\mathcal{B}(D,E)$. Then $H_\Phi$ is extended to a bounded operator on $H^2_D$ if and only if there exists a function $\Psi \in L^\infty(B(D,E))$ such that $\hat{\Psi}(n) = \hat{\Phi}(n)$ for $n < 0$ and

$$||H_\Phi|| = \text{dist}_{L^\infty}(\Psi, \mathcal{H}^\infty(B(D,E))).$$

The following basic properties can be easily derived: If $D, E,$ and $D'$ are separable complex Hilbert spaces and $\Phi \in L^\infty(B(D,E))$, then

$$T_\Phi = T_{\Phi^*}, \quad H_\Phi = H_\Phi; \quad (16)$$

$$H_\Phi T_\Psi = H_{\Phi \Psi} \quad \text{if } \Psi \in \mathcal{H}^\infty(B(D',D)); \quad (17)$$

$$H_{\Psi \Phi} = T_{\Psi^*} H_\Phi \quad \text{if } \Psi \in \mathcal{H}^\infty(B(E,D')); \quad (18)$$

A shift operator $S_E$ on $H^2_E$ is defined by

$$(S_E f)(z) := zf(z) \quad \text{for each } f \in H^2_E.$$

Thus we may write $S_E = T_z I_E$.

The following theorem is the fundamental result in the modern operator theory.

The Beurling-Lax-Halmos Theorem. [Beu], [La], [Ha1], [Pe] A subspace $M$ of $H^2_E$ is invariant for the shift operator $S_E$ on $H^2_E$ if and only if

$$M = \Delta H^2_E,$$

where $E'$ is a subspace of $E$ and $\Delta$ is an inner function with values in $\mathcal{B}(E',E)$. Furthermore, $\Delta$ is unique up to a unitary constant right factor, i.e., if $M = \Theta H^2_E$, where $\Theta$ is an inner function with values in $\mathcal{B}(E'',E)$, then $\Delta = \Theta V$, where $V$ is a unitary operator from $E'$ onto $E''$. 
Lemma 3.13. If \( \Phi \in L^\infty(\mathcal{B}(D, E)) \) and \( \Delta \) is a two-sided inner function with values in \( \mathcal{B}(E) \), then the following are equivalent:

(a) \( \ker H_{\Phi^*} = \Delta H_E^2 \);
(b) \( \Phi = \Delta A^* \), where \( A \in H^\infty(\mathcal{B}(E, D)) \) is such that \( \Delta \) and \( A \) are right coprime.

Proof. Let \( \Phi \in L^\infty(\mathcal{B}(D, E)) \) and \( \Delta \) be a two-sided inner function with values in \( \mathcal{B}(E) \).

(a) \( \Rightarrow \) (b): Suppose \( \ker H_{\Phi^*} = \Delta H_E^2 \). If we put \( A := \Phi^* \Delta \in H^\infty(\mathcal{B}(E, D)) \), then \( \Phi = \Delta A^* \). We now claim that \( \Delta \) and \( A \) are right coprime. To see this, suppose \( \Omega \) is a common left inner divisor, with values in \( \mathcal{B}(E', E') \), of \( \Delta \) and \( \hat{A} \). Then we may write \( \Delta = \Omega \Delta_1 \) and \( \hat{A} = \Theta \hat{A}_1 \), where \( \Delta_1 \in H^\infty(\mathcal{B}(E, E')) \) and \( \hat{A}_1 \in H^\infty(\mathcal{B}(D, E')) \). Since \( \Delta \) is two-sided inner, it follows that \( \Delta_1 \) is two-sided inner. Since \( \Phi = \Delta_1 A_1^* \), by Lemma 3.6, we have

\[
\Delta_1 H_E^2 \subseteq \ker H_{\Phi^*} = \Delta H_E^2 = \Delta_1 \hat{\Omega} H_E^2,
\]

which implies \( H_E^2 = \hat{\Omega} H_E^2 \). Thus by the Beurling-Lax-Halmos Theorem, \( \hat{\Omega} \) is a unitary constant and so is \( \Omega \). Therefore, \( \Delta \) and \( A \) are right coprime.

(b) \( \Rightarrow \) (a): Suppose (b) holds. Clearly, \( \Delta H_E^2 \subseteq \ker H_{\Phi^*} \). By the Beurling-Lax-Halmos Theorem, \( \ker H_{\Phi^*} = \Theta H_E^2 \) for some inner function \( \Theta \), so that \( \Delta H_E^2 \subseteq \Theta H_E^2 \). Thus \( \Theta \) is a left inner divisor of \( \Delta \) (cf. [FF], [Pe]), so that we may write \( \Delta = \Theta \Delta_0 \Delta_0^* \) for some two-sided inner function \( \Delta_0 \) with values in \( \mathcal{B}(E, E') \). Put \( G := \Phi^* \Theta \). Then it follows from Lemma 3.9 that \( G \) belongs to \( H_E^2(\mathcal{B}(E', E)) \). Then \( \Delta = \bigwedge \Delta_0 \Delta_0^* \), and hence, \( \hat{A} = \Delta_0 \hat{G} \). But since \( \Delta \) and \( A \) are right coprime, \( \Delta_0 \) is a unitary operator, and so is \( \Delta_0 \). Therefore \( \ker H_{\Phi^*} = \Delta H_E^2 \), which proves (a). \( \square \)

We recall that the factorization (b) in Lemma 3.13 is called the Douglas-Shapiro-Shields factorization of \( \Phi \in L^\infty(\mathcal{B}(D, E)) \) (see [DSS], [FB], [Fu2]). Consequently, Lemma 3.13 says that \( \Phi \in L^\infty(\mathcal{B}(D, E)) \) admits a Douglas-Shapiro-Shields factorization if and only if \( \ker H_{\Phi^*} = \Delta H_E^2 \) for some two-sided inner function \( \Delta \in H^\infty(\mathcal{B}(E)) \).

The following lemma will be frequently used in the sequel.

Complementing Lemma. [N1, p. 49, p. 53] Let \( \Psi \in H^\infty(\mathcal{B}(E', E)) \) with \( E' \subseteq E \) and \( \dim E' < \infty \), and let \( \theta \) be a scalar inner function. Then the following are equivalent:

(a) There exists a function \( G \) in \( H^\infty(\mathcal{B}(E, E')) \) such that \( G \Psi = \theta I_{E'} \);
(b) There exist functions \( \Phi \) and \( \Omega \) in \( H^\infty(\mathcal{B}(E)) \) with \( \Phi|_{E'} = \Psi \), \( \Phi|_{E \cap E'} \) being an inner function such that \( \Omega \Phi = \Phi \Omega = \theta I_{E'} \).
In addition, if $\dim E < \infty$, then (a) and (b) are equivalent to the following:

(c) $\operatorname{ess \ inf}_{z \in \mathbb{T}} \min \left\{ ||\Psi(z)x|| : ||x|| = 1 \right\} > 0$.

We recall that if $\Phi$ is a strong $H^2$-function with values in $\mathcal{B}(D, E)$, with $\dim E < \infty$, the local rank of $\Phi$ is defined by (cf. [Ni1])

$$\operatorname{Rank} \Phi := \max_{\zeta \in D} \operatorname{rank} \Phi(\zeta),$$

where $\operatorname{rank} \Phi(\zeta) := \dim \Phi(\zeta)(D)$.

As we have remarked in the introduction, if $\Phi$ is a strong $L^2$-function with values in $\mathcal{B}(D, E)$, then $H^*_\Phi$ need not be a Hankel operator. Of course, if $\Phi \in L^\infty(\mathcal{B}(D, E))$, then $H^*_\Phi$ is defined in the subset of polynomials in $H^2_D$. Thus it is much better to deal with $H^*_\Phi$ in place of $H^*_\Phi$. Even though $H^*_\Phi$ need not be a Hankel operator, we can show that the kernel of $H^*_\Phi$ is still of the form $\Delta H^2_E$ for some inner function $\Delta$. To see this, we observe:

**Lemma 3.14.** Let $\Phi$ be a strong $L^2$-function with values in $\mathcal{B}(D, E)$. Then,

$$\ker H^*_\Phi = \left\{ f \in H^2_E : \int_{\mathbb{T}} \langle \Phi(z)x, z^n f(z) \rangle_E dm(z) = 0 \text{ for all } x \in D \text{ and } n = 1, 2, 3, \cdots \right\}.$$

**Proof.** Observe that

$$f \in \ker H^*_\Phi \iff \langle H_\Phi p, f \rangle_{L^2_E} = 0 \text{ for all } p \in \mathcal{P}_D$$

$$\iff \langle \Phi(z)p(z), (Jf)(z) \rangle_{L^2_E} = 0 \text{ for all } p \in \mathcal{P}_D$$

$$\iff \int_{\mathbb{T}} \langle \Phi(z)x z^k, \overline{z} f(z) \rangle_{E'} dm(z) = 0 \text{ for all } x \in D \text{ and } k = 0, 1, 2, \cdots$$

$$\iff \int_{\mathbb{T}} \langle \Phi(z)x, z^n f(z) \rangle_E dm(z) = 0 \text{ for all } x \in D \text{ and } n = 1, 2, 3, \cdots,$$

which gives the result. \(\square\)

We then have:

**Corollary 3.15.** If $\Phi$ is a strong $L^2$-function with values in $\mathcal{B}(D, E)$, then

$$\ker H^*_\Phi = \Delta H^2_{E'},$$

where $E'$ is a subspace of $E$ and $\Delta$ is an inner function with values in $\mathcal{B}(E', E)$.

**Proof.** By Lemma 3.14, if $f \in \ker H^*_\Phi$, then $zf \in \ker H^*_\Phi$. Since $H^*_\Phi$ is always closed, it follows that $\ker H^*_\Phi$ is an invariant subspace for $S_E$. Thus, by the Beurling-Lax-Halmos Theorem, there exists an inner function $\Delta$ with values in $\mathcal{B}(E', E)$ such that $\ker H^*_\Phi = \Delta H^2_{E'}$ for a subspace $E'$ of $E$. \(\square\)
3.3. The complementary factor of an inner function

For \( \Phi \in L^\infty(B(D, E)) \), we symbolically define the kernel of \( \Phi \) by

\[
\ker \Phi := \{ f \in H^2_D : \Phi(z)f(z) = 0 \text{ for almost all } z \in \mathbb{T}\}.
\]

Note that the kernel of \( \Phi \) consists of functions in \( H^2_D \), but not in \( L^2_D \), such that \( \Phi f = 0 \) a.e. on \( \mathbb{T} \).

Let \( \Delta \) be an inner function with values in \( B(D, E) \). If \( g \in \ker \Delta^* \), then \( g \in H^2_D \), so that by Lemma 3.3 and Lemma 3.7, \( [g] \) is a strong \( H^2 \)-function with values in \( B(C, E) \) (see p.16 for the definition of \([g]\)). Write

\[
[g] = [g]^i[g]^e \quad \text{ (inner-outer factorization),}
\]

where \([g]^e\) is an outer function with values in \( B(C, E') \) and \([g]^i\) is an inner function with values in \( B(E', E) \) for some subspace \( E' \) of \( E \). If \( g \neq 0 \), then \([g]^e : C \to E' \) is a nonzero outer function, so that \( E' = C \). Thus, \([g]^i \in H^\infty(B(C, E)) \). If instead \( g = 0 \), then \( g = [g]^e : C \to E' \), so that \( E' = \{0\} \). Therefore, in this case, \([g]^i \in H^\infty(B(\{0\}, E)) \).

We then have:

**Lemma 3.16.** Let \( \Delta \) be an inner function with values in \( B(D, E) \). Put

\[
\Delta_c := \text{left-g.c.d.} \{ [g]^i : g \in \ker \Delta^* \}.
\]  
(19)

Then

(a) \( \ker \Delta^* = \Delta_c H^2_{D'} \) for some subspace \( D' \) of \( E \);
(b) \( [\Delta, \Delta_c] \) is an inner function with values in \( B(D \oplus D', E) \);
(c) \( \ker H_{\Delta_c} = [\Delta_c, \Delta] H^2_{B \oplus D'} = \Delta H^2_D \oplus \Delta_c H^2_{D'} \),

where \([\Delta, \Delta_c] \) is obtained by complementing \( \Delta_c \) to \( \Delta \), in other words, \([\Delta, \Delta_c] \) is regarded as a \( 1 \times 2 \) operator matrix.

**Definition.** The inner function \( \Delta_c \) defined in (19) will be called the complementary factor of an inner function \( \Delta \).

**Proof.** If \( \ker \Delta^* = \{0\} \), then (a) and (b) are trivial. Suppose that \( \ker \Delta^* \neq \{0\} \). Since \( \ker \Delta^* \) is invariant for the shift operator \( S_E \) on \( H^2_D \), by the Beurling-Lax-Halmos Theorem, there exists an inner function \( \Omega \) with values in \( B(D', E) \) such that

\[
\ker \Delta^* = \Omega H^2_{D'},
\]  
(20)

where \( D' \) is a nonzero subspace of \( E \). Put

\[
\Delta_c := \text{left-g.c.d.} \{ [g]^i : g \in \ker \Delta^* \} \in H^\infty(B(D'', E)),
\]  
(21)

where \( D'' \) is a nonzero subspace of \( E \). If \( g \in \ker \Delta^* \), then \( \Delta^*[g] = 0 \). Thus we have that

\[
\Delta_c H^2_{D''} = \bigvee \{ [g]^i H^2 : g \in \ker \Delta^* \} = \bigvee \{ [g]^i \mathcal{P}_C : g \in \ker \Delta^* \} \subseteq \ker \Delta^* = \Omega H^2_{D'},
\]

For the reverse inclusion, let \( 0 \neq g \in \ker \Delta^* \). Then it follows that \( g(z) = [g](z)1 = (|g|^i[g]^e)(z)1 = [g]^i(z)\overline{([g]^e(z)1)} \in [g]^i H^2 \).
Thus we have

$$\Omega H^2_{D'} = \ker \Delta^* \subseteq \bigvee \{ [g]^* H^2 : g \in \ker \Delta^* \} = \Delta_c H^2_{D'}. $$

Therefore, by the Beurling-Lax-Halmos Theorem, \( \Omega = \Delta_c \) and \( D' = D'' \), which gives (a). Note that \( \Delta^* \Delta_c = 0 \). We thus have

$$\left[ \begin{array}{c} \Delta^* \\ \Delta_c \end{array} \right] = \left[ \begin{array}{cc} I_D & 0 \\ 0 & I_{D''} \end{array} \right],$$

which implies that \( [\Delta, \Delta_c] \) is an inner function with values in \( B(D \oplus D', E) \), which gives (b). For (c), we first note that \( \Delta^* H^2_D \) and \( \ker \Delta^* \) are orthogonal and

$$\Delta^* H^2_D \oplus \ker \Delta^* \subseteq \ker H_{\Delta^*}.$$

For the reverse inclusion, suppose that \( f \in H^2_E \) and \( f \notin \Delta^* H^2_D \oplus \ker \Delta^* \equiv M \). Write

$$f_1 := P_M f \quad \text{and} \quad f_2 := f - f_1 \neq 0.$$

Since \( f_2 \in H^2_E \ominus M = \mathcal{H}(\Delta) \cap (H^2_E \ominus \ker \Delta^*) \), it follows from Corollary 3.11 that \( \Delta^* f_2 \in L^2_E \ominus \Delta^* H^2_E \ominus \ker \Delta^* f_2 \neq 0 \). We thus have \( H_{\Delta^*} f = J(\Delta^* f_2) \), and hence, \( ||H_{\Delta^*} f|| = ||\Delta^* f_2|| \neq 0 \), which implies that \( f \notin \ker H_{\Delta^*} \). We thus have that

$$\ker H_{\Delta^*} = \Delta^* H^2_D \oplus \ker \Delta^*.$$

Thus it follows from (a) that

$$\ker H_{\Delta^*} = \Delta^* H^2_D \oplus \Delta_c H^2_{D'} = [\Delta, \Delta_c] H^2_{D \oplus D'},$$

which gives (c). This completes the proof. \( \Box \)

For a subset \( F \) of \( H^2_E \), let \( E^*_F \) denote the smallest \( S^*_E \)-invariant subspace containing \( F \), i.e.,

$$E^*_F = \bigvee \{ S^*_E F : n \geq 0 \}.$$

Then by the Beurling-Lax-Halmos Theorem, \( E^*_F = \mathcal{H}(\Delta) \) for an inner function \( \Delta \) with values in \( B(D, E) \). In general, if \( \dim E = 1 \), then every \( S^*_E \)-invariant subspace \( M \) admits a generating vector, i.e., \( M = E^*_F \) for some \( f \in H^2 \). However, if \( \dim E \geq 2 \), then this is not such a case. For example, if \( M = \mathcal{H}(\Delta) \) with \( \Delta = [z^0] \), then \( M \) does not admit a generating vector, i.e., \( M \neq E^*_F \) for any vector \( f \in H^2_E \).

For a function \( f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n \in L^2_E \) (where \( \hat{f}(\cdot) \) denotes the Fourier coefficient of \( f \)), write

$$\overline{f}(z) := \sum_{n=-\infty}^{\infty} \overline{f(n)} z^{-n}, \quad \text{for} \quad n \geq 0,$$

where \( \overline{f(n)} \) is defined by

$$\overline{f(n)} := \sum_{n \geq 1} \alpha_n e_n \quad \text{if} \quad \hat{f}(n) = \sum_{n \geq 1} \alpha_n e_n \quad (\alpha_n \in \mathbb{C}),$$

where \( \{e_n\}_{n \geq 1} \) is an orthonormal basis for \( E \).

If \( \Phi \in H^2_{\mathbb{C}}(B(D, E)) \) and \( \{d_k\}_{k \geq 1} \) is an orthonormal basis for \( D \), write

$$\phi_k := \Phi d_k \in H^2_E \cong H^2_{\mathbb{C}}(B(D', E)).$$
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We then define
\[ \{ \Phi \} := \{ \phi_k \}_{k \geq 1} \subseteq H^2_E. \]
Hence, $\{ \Phi \}$ may be regarded as the set of “column” vectors (in $H^2_E$) of $\Phi$, in which we may think of $\Phi$ as an infinite matrix-valued function.

We then have:

**Lemma 3.17.** For $\Phi \in H^2_s(B(D, E))$, we have
\[ E^*_\Phi = \text{cl ran } H_{z\tilde{\Phi}}. \tag{23} \]

**Remark.** By definition, $\{ \Phi \}$ depends on the orthonormal basis of $D$. However, Lemma 3.17 shows that $E^*_\Phi$ is independent of a particular choice of the orthonormal basis of $D$ because the right-hand side of (23) is independent of the orthonormal basis of $D$.

**Proof.** We first claim that if $g \in L^2_E$, then
\[ JS^*_E P^- g = P^- \tilde{g}. \tag{24} \]
Indeed, if $g \in L^2_E$, then we may write $g(z) = \sum_{k=-\infty}^{\infty} \hat{g}(k)z^k$. Thus we have
\[
JS^*_E P^- g(z) = JS^*_E \sum_{k=-\infty}^{-1} \hat{g}(k)z^{-k} = J\left(\sum_{k=-\infty}^{-1} \hat{g}(k)z^{-k-1}\right) = \sum_{n=-\infty}^{-1} \hat{g}(k)z^n = P^- \tilde{g}(z),
\]
which proves (24). We next claim that if $f \in H^2_E$, then
\[ E^*_f = \text{cl ran } H_f. \tag{25} \]

To see this, observe that for each $k = 1, 2, \ldots$,
\[ S^*_E f(z) = S^*_E \left( \sum_{j=0}^{\infty} \tilde{f}(k+j)z^{j+1} \right) = S^*_E \left( P^- (z^{-1} f(z)) \right). \]
Thus, for each $k = 1, 2, \ldots$,
\[
S^*_E f = S^*_E P^- (z^{-1} f) = JJS^*_E P^- (z^{-1} f) = JP^- (z^{-1} \tilde{f}) \quad (JJ = I) = H_{\tilde{f}} z^k.
\]
which proves (25). Let $\{d_k\}_{k \geq 1}$ be an orthonormal basis for $D$, and let $\phi_k := \Phi d_k$. Since by (25), $E^*_\phi_k = \text{cl ran } H_{\tilde{\phi}_k}$ for each $k = 1, 2, 3, \ldots$, it follows that
\[ E^*_\Phi = \bigvee \text{ran } H_{\tilde{\phi}_k} = \text{cl ran } H_{\tilde{\Phi}}, \]
which gives the result.

We introduce:
In particular, if \( \text{nc} (F) = \sup \dim \{ g(\zeta) : g \in H_E^2 \} \) for some two-sided inner function \( \Delta^1 \), then the degree of cyclicity, denoted by \( \text{dc}(F) \), of \( F \) is defined by the number (cf. [VN])

\[
\text{dc}(F) := r - \text{nc}(F).
\]

In particular, if \( E^*_E = \mathcal{H}(\Delta) \), then \( \Delta \) is two-sided inner if and only if \( \text{nc}(F) = r \).

We then have:

**Lemma 3.19.** Let \( \Phi \) be a strong \( L^2 \)-function with values in \( \mathcal{B}(D, E) \). In view of the Beurling-Lax-Halmos Theorem and Corollary 3.15, we may write

\[
E^*_E = \mathcal{H}(\Delta) \quad \text{and} \quad \ker H^*_\Phi = \Theta H^2_E,
\]

for some inner functions \( \Delta \) and \( \Theta \) with values in \( \mathcal{B}(E', E) \) and \( \mathcal{B}(E', E') \), respectively. Then

\[
\Delta = \Theta \Delta^1
\]

for some two-sided inner function \( \Delta^1 \) with values in \( \mathcal{B}(E', E') \). Hence, in particular,

\[
\ker H^*_\Phi = \Theta H^2_E \iff \text{nc}\{\Phi \} = \dim E'.
\]

**Proof.** Suppose that \( \ker H^*_\Phi = \Theta H^2_E \) for some inner function \( \Theta \) with values in \( \mathcal{B}(E', E') \) and \( E^*_E = \mathcal{H}(\Delta) \) for some inner function \( \Delta \) with values in \( \mathcal{B}(E', E') \). Then it follows from Lemma 3.17 that

\[
\mathcal{H}(\Delta) = E^*_E \cap \ker \Theta = \mathcal{H}(\Delta) \cap \ker \Theta.
\]

It thus follows from Lemma 3.14 that

\[
\Delta H^2_E = \ker H^*_\Phi
\]

which implies that \( \Theta \) is a left inner divisor of \( \Delta \). Thus we can write

\[
\Delta = \Theta \Delta^1
\]

for some inner function \( \Delta^1 \in H^\infty(\mathcal{B}(E', E')) \). By the same argument as above, we also have \( z\Theta H^2_E \subseteq \Delta H^2_{E'} \), so that we may write \( z\Theta = \Delta \Delta_2 \) for some inner function \( \Delta_2 \in \)
A canonical decomposition of strong $L^2$-functions $H^\infty(\mathcal{B}(E',E''))$. Therefore by (28), we have $zI_{E'} = \Delta_1\Delta_2$, and hence by Lemma 3.5, $\Delta_1$ is two-sided inner. This proves (26) and in turn (27). This completes the proof. □

**Remark 3.20.** From Lemma 3.19, we obtain several useful observations as follows.

1. Lemma 3.19 shows that if $\Phi$ is a strong $L^2$-function with value in $\mathcal{B}(D,E)$, then the following are equivalent:
   - (a) $E^*_{\Phi_+} = H^2_E$;
   - (b) $\text{nc}\{\Phi_+\} = 0$;
   - (c) $\ker H^*_\Phi = \{0\}$.

2. Lemma 3.19 together with Lemma 3.16(c) shows that if $\Delta$ is an inner function with values in $\mathcal{B}(D,E)$ and if $\Delta_c$ is the complementary factor, with values in $\mathcal{B}(D',E)$, of $\Delta$, then $\text{nc}\{\Delta\} = \dim D + \dim D'$.

3. From Corollary 2 of [Ni1, p.47], (5) and Lemma 3.19, we can see that if $\Phi$ is an $n \times m$ matrix $L^2$-function, i.e., $\Phi \in L^2_{M_n \times m}$, then the following are equivalent:
   - (a) $\Phi$ is of bounded type (i.e., $\Phi$ is of entrywise bounded type);
   - (b) $\ker H^*_\Phi = \Theta H^2_C$, for some two-sided inner matrix function $\Theta$;
   - (c) $\text{nc}\{\Phi_+\} = n$.

   The equivalence (a)$\Rightarrow$(b) was known from [GHR] for the cases of $\Phi \in L^\infty_{M_n}$. It was also known ([Ab, Lemma 4]) that if $\phi \in L^\infty$, then $\phi$ is of bounded type $\iff \ker H^*_\phi \neq \{0\}$. (29)

   Thus the equivalence (a)$\iff$(b) shows that (29) still holds for $L^2$-functions.

4. The above remark (3) together with Lemma 3.16 shows that if $\Delta$ is an $n \times r$ inner matrix function then the following are equivalent:
   - (a) $\Delta^*$ is of bounded type, or equivalently, $\tilde{\Delta}$ is of bounded type;
   - (b) $[\Delta, \Delta_c]$ is two-sided inner,

   where $\Delta_c$ is the complementary factor of $\Delta$.

On the other hand, in view of Remark 3.20(4), we may ask a more general complementation: If $\Delta$ is an $n \times r$ inner matrix function, which condition on $\Delta$ allows us to complement $\Delta$ to an $n \times (r + q)$ inner matrix function $[\Delta, \Omega]$ by aid of an $n \times q$ inner matrix function $\Omega$?

We can give an answer, to this question, whose proof uses the complementary factor.

**Proposition 3.21.** If $\Delta$ is an $n \times r$ inner matrix function, then $[\Delta, \Omega]$ is inner for some $n \times q$ ($q \geq 1$) inner matrix function $\Omega$ if and only if $q \leq \text{nc}\{\Delta\} - r$.

In particular, $\Delta$ is complemented to a two-sided inner function if and only if $\text{nc}\{\Delta\} = n$.

**Proof.** Suppose that $[\Delta, \Omega]$ is an inner matrix function for some $n \times q$ ($q \geq 1$) inner matrix function $\Omega$. Then

$$I_{r+q} = [\Delta, \Omega]^*[\Delta, \Omega] = \begin{bmatrix} I_r & \Delta^*\Omega \\ \Omega^*\Delta & I_q \end{bmatrix},$$

which implies that $\Omega H^2_C \subseteq \ker \Delta^*$. Since by Lemma 3.16, $ker \Delta^* = \Delta_c H^2_C$, it follows that $\Omega H^2_C \subseteq \Delta_c H^2_C$, so that $\Delta_c$ is a left inner divisor of $\Omega$. Thus we can write

$$\Omega = \Delta_c \Omega_1$$

for some $p \times q$ inner matrix function $\Omega_1$. 
Thus we have $q \leq p$. But since by Remark 3.20(2), $nc(\Delta) = r + p$, it follows that $q \leq nc(\Delta) - r$. For the converse, suppose that $q \leq nc(\Delta) - r$. Then it follows from Remark 3.20(2) that the complementary factor $\Delta_c$ of $\Delta$ is in $H^\infty_{M_{n,s}}$ for some $p \geq q$. Thus if we take $\Omega := \Delta_c|_{C_q}$, then $[\Delta, \Omega]$ is inner.

### 3.4. Strong $L^2$-functions of bounded type

In this section we introduce the notion of “bounded type” for strong $L^2$-functions. Recall that a matrix-valued function of bounded type was defined by a matrix whose entries are of bounded type (see p. 3). But this definition is not appropriate for operator-valued functions, in particular, strong $L^2$-functions even though “entry” may be properly interpreted. Thus we need an idea of defining “bounded type” strong $L^2$-functions, which is equivalent to the condition that each entry is of bounded type when the function is matrix-valued. Indeed, we get an inspiration from the equivalence (a)$\iff$(b) in Remark 3.20(3).

**Definition 3.22.** A strong $L^2$-function $\Phi$ with values in $B(D, E)$ is said to be of bounded type if $\ker H^*_\Phi = \Theta H^2_E$ for some two-sided inner function $\Theta$ with values in $B(E)$.

On the other hand, in [FB], it was shown that if $\Phi$ belongs to $L^\infty(B(D, E))$, then $\Phi$ admits a Douglas-Shapiro-Shields factorization (see p. 19) if and only if $E^*_\Phi \subseteq H(\Theta)$ for a two-sided inner function $\Theta$. Thus by Lemma 3.19, we can see that if $\Phi \in L^\infty(B(D, E))$, then $\hat{\Phi}$ is of bounded type $\iff$ $\Phi$ admits a Douglas-Shapiro-Shields factorization. (30)

We can prove more:

**Lemma 3.23.** Let $\Phi$ be a strong $L^2$-function with values in $B(D, E)$. Then the following are equivalent:

(a) $\hat{\Phi}$ is of bounded type;
(b) $E^*_\Phi = H(\Delta)$ for some two-sided inner function $\Delta$ with values in $B(E)$;
(c) $E^*_\Phi \subseteq H(\Theta)$ for some two-sided inner function $\Theta$ with values in $B(E)$;
(d) $\{\Phi_+\} \subseteq H(\Theta)$ for some two-sided inner function $\Theta$ with values in $B(E)$;
(e) For $\{\varphi_{k_1}, \varphi_{k_2}, \cdots\} \subseteq \{\Phi\}$, write $\Psi \equiv [\varphi_{k_1}, \varphi_{k_2}, \cdots]$. Then $\Psi$ is of bounded type.

**Proof.** (a) $\Rightarrow$ (b): Suppose that $\hat{\Phi}$ is of bounded type. Then $\ker H^*_\Phi = \Theta H^2_E$ for some two-sided inner function $\Theta$ with values in $B(E)$. It thus follows from Lemma 3.19 that $E^*_\Phi = H(\Delta)$ for some two-sided inner function $\Delta$ with values in $B(E)$.

(b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d): Clear.

(d) $\Rightarrow$ (e): Suppose that $\{\varphi_{k_1}, \varphi_{k_2}, \cdots\} \subseteq \{\Phi\}$ and $\{\Phi_+\} \subseteq H(\Theta)$ for some two-sided inner function $\Theta \in H^\infty(B(E))$. Write $\Psi \equiv [\varphi_{k_1}, \varphi_{k_2}, \cdots]$. Then $\{\Phi_+\} \subseteq H(\Theta)$, so that $E^*_\Psi \subseteq H(\Theta)$. Suppose that $E^*_\Psi = H(\Delta)$ for some inner function $\Delta$ with values in $B(D', E)$. Thus $\Theta H^2_E \subseteq \Delta H^2_{D'}$, so that by Lemma 3.5, $\Delta$ is two-sided inner. Thus, by Lemma 3.19, $\ker H^*_\Psi = \Omega H^2_{D'}$ for some two-sided inner function $\Omega$ with values in $B(E)$, so that $\Psi$ is of bounded type.

(e) $\Rightarrow$ (a): Clear.

□
Corollary 3.24. Let $\Delta$ be an inner function with values in $\mathcal{B}(D, E)$. Then

$\Delta$ is of bounded type $\iff [\Delta, \Delta_c]$ is two-sided inner,

where $\Delta_c$ is the complementary factor of $\Delta$. Hence, in particular, if $\Delta$ is a two-sided inner function with values in $\mathcal{B}(E)$, then $\Delta$ is of bounded type.

Proof. The first assertion follows from Lemma 3.16. The second assertion follows from the first assertion together with the observation that if $\Delta$ is two-sided inner then $[\Delta, \Delta_c] = \Delta$.

Corollary 3.25. Let $\Delta$ be an inner function with values in $\mathcal{B}(D, E)$. Then $[\Delta, \Omega]$ is two-sided inner for some inner function $\Omega$ with values in $\mathcal{B}(D', E)$ if and only if $\hat{\Delta}$ is of bounded type.

Proof. Suppose that $[\Delta, \Omega]$ is two-sided inner for some inner function $\Omega$ with values in $\mathcal{B}(D', E)$. Then $\Delta^* \Omega = 0$, so that $\Omega H_{D'}^2 \subseteq \ker \Delta^* = \Delta_c H_{D''}^2$. Thus $\Delta_c$ is a left inner divisor of $\Omega$, and hence $[\Delta, \Delta_c]$ is a left inner divisor of $[\Delta, \Omega]$. Therefore by Lemma 3.5, $[\Delta, \Delta_c]$ is two-sided inner, so that by Corollary 3.24, $\hat{\Delta}$ is of bounded type. The converse comes at once from Corollary 3.24 with $\Omega = \Delta_c$.

A $\mathcal{B}(D, E)$-valued function $\Phi$ is said to be meromorphic of bounded type in $D_e$ if it can be represented by

$$\Psi = \frac{G}{\theta},$$

where $G$ is a strong $H^2$-function in $D_e$, with values in $\mathcal{B}(D, E)$ and $\theta$ is a scalar inner function in $D_e$. (cf. [Fu2]). A function $\Phi \in L^2_s(\mathcal{B}(D, E))$ is said to have a meromorphic pseudocontinuation $\hat{\Phi}$ of bounded type in $D_e$ if $\Phi$ is meromorphic of bounded type in $D_e$ and $\Phi$ is the nontangential SOT limit of $\hat{\Phi}$, that is, for all $x \in D$,

$$\Phi(x) = \lim_{r \to 1^+} \hat{\Phi}(rz)x \equiv \hat{\Phi}(z)x \quad \text{for almost all } z \in \mathbb{T}.$$

Note that for almost all $z \in \mathbb{T}$,

$$\hat{\Phi}(z)x = \lim_{r \to 1^+} \Phi(rz)x = \lim_{r \to 1^+} \Phi_D^*(r^{-1}z)x = \check{\Phi}_D^*(z)x \quad (x \in D).$$

We then have:

Lemma 3.26. Let $\Phi$ be a strong $L^2$-function with values in $\mathcal{B}(D, E)$. If $\Phi$ has a meromorphic pseudocontinuation of bounded type in $D_e$, then $\hat{\Phi}$ is of bounded type.
where the last equality follows from the fact that $z$ is a strong $H^2$-function in $D$.

Thus it follows from Lemma 3.5 that $\tilde{\Phi}$ is of bounded type.

3.14, we have that $D$ is a strong $H^2$-function in $D$. Then for all $x \in D$,

$$\Phi(z)x = \tilde{\Phi}(z)x = \frac{G(z)x}{\delta(z)} = \delta(z)G^*_D(z)x \quad \text{for almost all } z \in \mathbb{T}.$$  

Thus for all $x \in D$, $p \in \mathcal{P}_E$, and $n = 1, 2, 3, \cdots$,

$$\int_T \langle \Phi(z)x, z^n\delta_D(z)p(z) \rangle_E dm(z) = \int_T \langle G^*_D(z)x, z^n p(z) \rangle_E dm(z) = \langle x, z^nG_D(z)p(z) \rangle_{L^2_E} = 0,$$

where the last equality follows from the fact that $z^nG_D(z)p(z) \in zH^2_E$. Thus by Lemma 3.14, we have that

$$\delta_D H^2_E = \text{cl } \delta_D \mathcal{P}_E \subseteq \ker H^*_E.$$

It thus follows from Lemma 3.5 that $\tilde{\Phi}$ is of bounded type.

\[ \square \]

The following lemma was proved in [Fu1] under the setting of $H^\infty_E(D,E)$.

**Lemma 3.27.** Let $\Phi \in L^\infty(B(D,E))$. Then the following are equivalent:

(a) $\Phi$ has a meromorphic pseudocontinuation of bounded type in $D$;

(b) $\Delta H^2_E \subseteq \ker H^*_E$ for some scalar inner function $\theta$;

(c) $\Phi = \theta A^*$ (a.e. on $\mathbb{T}$) for a scalar inner function $\theta$ and some $A \in H^\infty(B(E,D))$.

**Proof.** First of all, recall that $L^\infty(B(D,E)) \subseteq L^2(B(D,E))$.

(a) $\Rightarrow$ (b): This follows from the proof of Lemma 3.26.

(b) $\Rightarrow$ (c): Suppose that $\Delta H^2_E \subseteq \ker H^*_E$ for some scalar inner function $\theta$. Then it follows from Lemma 3.14 that for all $x \in D$, $f \in H^2_E$, and $n = 1, 2, 3, \cdots$,

$$0 = \int_T \langle \Phi(z)x, z^n\theta(z)f(z) \rangle_E dm(z) = \int_T \langle \overline{\theta}(z)\Phi(z)x, z^n f(z) \rangle_E dm(z).$$

Thus, again by Lemma 3.14, $H^2_E = 0$, and hence $H^*_E = 0$. Put $A := \theta \Phi^*$. Then by Lemma 3.6 and Lemma 3.12, $A$ belongs to $H^\infty(B(E,D))$ and $\Phi = \theta A^*$.

(c) $\Rightarrow$ (a): Suppose that $\Phi = \theta A^*$ (a.e. on $\mathbb{T}$) for a scalar inner function $\theta$ and some $A \in H^\infty(B(E,D))$. Thus it follows from Lemma 3.3(b) that $A$ is a strong $H^2$-function. Let

$$\Phi(z) := \frac{A^*(1/z)}{\theta(1/z)} \quad (z \in D).$$

Then $\Phi$ is meromorphic of bounded type in $D$ and for all $x \in D$,

$$\Phi(z)x = \frac{A^*(z)x}{\theta(z)} = \theta(z)A^*(z)x = \Phi(z)x \quad \text{for almost all } z \in \mathbb{T},$$

which implies that $\Phi$ has a meromorphic pseudocontinuation of bounded type in $D$.  \[ \square \]
An examination of the proof of Lemma 3.27 shows that Lemma 3.27 still holds for every function $\Phi \in L^2_{B(D,E)}$.

**Corollary 3.28.** If $\Phi \in L^2_{B(D,E)}$, then Lemma 3.27 holds with $A \in H^2_{B(E,D)}$ in place of $A \in H^\infty(B(E,D))$.

We now have:

**Proposition 3.29.** Let $D$ and $E$ be separable complex Hilbert spaces and let $\{d_j\}$ and $\{e_i\}$ be orthonormal bases of $D$ and $E$, respectively. If $\Phi \in L^2_{B(D,E)}$ has a meromorphic pseudocontinuation of bounded type in $D_e$, then $\hat{\phi}_{ij}(z) \equiv \langle \Phi(z)d_j, e_i \rangle_E$ is of bounded type for each $i,j$.

**Proof.** Let $\Phi \in L^2_{B(D,E)}$. Suppose that $\Phi$ has a meromorphic pseudocontinuation of bounded type in $D_e$. Then by Corollary 3.28, $\Phi = \theta A^*$ (a.e. on $\mathbb{T}$) for a scalar inner function $\theta$ and some $A \in H^2_{B(E,D)}$. Write

$$\phi_{ij}(z) := \langle \Phi(z)d_j, e_i \rangle_E \quad \text{and} \quad a_{ij}(z) := \langle \tilde{A}(z)d_j, e_i \rangle_E.$$ 

Then for each $i,j$,

$$\int_\mathbb{T} |\phi_{ij}(z)|^2dm(z) = \int_\mathbb{T} |\langle \Phi(z)d_j, e_i \rangle_E|^2dm(z) \leq \int_\mathbb{T} ||\Phi(z)||^2_{B(D,E)}dm(z) < \infty,$$

which implies $\phi_{ij} \in L^2$. Similarly, $a_{ij} \in L^2$ and for $n = 1,2,3,\ldots$,

$$\tilde{a}_{ij}(-n) = \int_\mathbb{T} z^n \langle \tilde{A}(z)d_j, e_i \rangle_E dm(z) = \langle d_j, z^{-n}\tilde{A}(z)e_i \rangle_{L^2_E} = 0,$$

which implies $a_{ij} \in H^2$. Note that

$$\tilde{\phi}_{ij}(z) = \tilde{\theta}(z)\langle \tilde{A}(z)d_j, e_i \rangle_E = \tilde{\theta}(z)a_{ij}(z),$$

which implies that $\tilde{\phi}_{ij}$ is of bounded type for each $i,j$. \(\square\)

**Example 3.30.** Let $\{\alpha_n\}$ be a sequence of distinct points in $\mathbb{D}$ such that $\sum_{n=1}^{\infty} (1 - |\alpha_n|) = \infty$ and put $\Delta := \text{diag}(b_{\alpha_n})$, where $b_{\alpha_n}(z) := \frac{z - \overline{\alpha_n}}{1 - \overline{\alpha_n}z}$. Then $\Delta$ is two-sided inner, and hence by Corollary 3.24, $\tilde{\Delta}$ is of bounded type. On the other hand, by Lemma 3.16, $\ker H^2_{\Delta^*} = \Delta H^2_{\Delta^*}$. Thus if $\Delta$ had a meromorphic pseudocontinuation of bounded type in $D_e$, then by Lemma 3.27, we would have $\theta H^2_{\Delta^*} \subseteq H^2_{\Delta^*}$ for a scalar inner function $\theta$, so that $\theta(\alpha_n) = 0$ for each $n = 1,2,\ldots$, and hence $\theta = 0$, a contradiction. Therefore, $\Delta$ cannot have a meromorphic pseudocontinuation of bounded type in $D_e$.

For the matrix-valued case, the function having a meromorphic pseudocontinuation of bounded type in $D_e$ is collapsed to the function whose flip is of bounded type.

**Corollary 3.31.** For $\Phi \equiv [\phi_{ij}] \in L^2_{\mathcal{M}_m}$, the following are equivalent:

(a) $\Phi$ has a meromorphic pseudocontinuation of bounded type in $D_e$;

(b) $\tilde{\Phi}$ is of bounded type;
Proof. (a) ⇒ (b): This follows from Lemma 3.26.

(b) ⇒ (a): Suppose that $\Phi$ is of bounded type. Then $\ker H_\Phi^* = \Theta H_{\mathbb{C}^n}^2$ for some two-sided inner function $\Theta \in H_{\mathbb{C}^n}^\infty$. Thus by the Complementing Lemma (cf. p. 19), there exists a function $G$ in $H_{\mathbb{C}^n}^\infty$ such that $G\Theta = \Theta G = \theta I_n$, and hence, $\Theta H_{\mathbb{C}^n}^2 = \Theta GH_{\mathbb{C}^n}^2 \subseteq \ker H_\Phi^*$. It thus follows from Corollary 3.28 that $\Phi$ has a meromorphic pseudocontinuation of bounded type in $D$.

(a) ⇔ (c): This follows from Lemma 3.27 and Proposition 3.29. □

However, by contrast to the matrix-valued case, it may happen that an $L_\infty$ function $\Phi$ is not of bounded type in the sense of Definition 3.22 even though each entry $\phi_{ij}$ of $\Phi$ is of bounded type.

Example 3.32. Let $\{\alpha_j\}$ be a sequence of distinct points in $(0, 1)$ satisfying $\sum_{j=1}^\infty (1 - \alpha_j) < \infty$. For each $j \in \mathbb{Z}_+$, choose a sequence $\{\alpha_{ij}\}$ of distinct points on the circle $C_j := \{z \in \mathbb{C} : |z| = \alpha_j\}$. Let

$$B_{ij} := \frac{b_{\alpha_{ij}}}{(i + j)!} \quad (i, j \in \mathbb{Z}_+),$$

where $b_\alpha(z) := \frac{z - \alpha}{1 - w_2}$, and let

$$\Phi := [B_{ij}] = \left[ \begin{array}{ccc} \frac{\delta_{02}}{2} & \frac{\delta_{03}}{3!} & \cdots \\ \frac{\delta_{12}}{2} & \frac{\delta_{13}}{3!} & \cdots \\ \frac{\delta_{22}}{2} & \frac{\delta_{23}}{3!} & \cdots \end{array} \right].$$

Observe that

$$\sum_{i,j} |B_{ij}(z)|^2 = \sum_i \frac{i}{(1 + i)!^2} \leq \sum_i \frac{1}{(1 + i)^2} < \infty,$$

which implies that $\Phi \in L_\infty(\mathcal{B}(\mathbb{E}_2))$. For a function $f \in H_{\mathbb{C}^2}^2$, we write $f = (f_1, f_2, f_3, \cdots)^t$ ($f_n \in H^2$). Thus if $f = (f_1, f_2, f_3, \cdots)^t \in \ker H_\Phi$, then $\sum_j \frac{\delta_{ij}}{(i + j)!} f_j \in H^2$ for each $i \in \mathbb{Z}_+$, which forces that $f_j(\alpha_{ij}) = 0$ for each $i, j$. Thus $f_j = 0$ for each $j$ (by the Identity Theorem). Therefore we can conclude that $\ker H_\Phi^* = \{0\}$, so that $\Phi$ is not of bounded type. But we note that every entry of $\Phi$ is of bounded type.

4. Proof of Theorem A

In this section, we give a proof of Theorem A. To understand a canonical decomposition of strong $L^2$-functions, we first consider an example of a matrix-valued $L^2$-function that does not admit a Douglas-Shapiro-Shields factorization. Suppose that $\theta_1$ and $\theta_2$ are coprime inner functions. Consider

$$\Phi := \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{bmatrix} \equiv [\phi_1, \phi_2, \phi_3] \in H_{\mathbb{C}^3}^\infty,$$
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where $a \in H^\infty$ is such that $\pi$ is not of bounded type. Then a direct calculation shows that

$$\ker H_{\Phi^*} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & 0 \end{bmatrix} H^2_{C_2} \equiv \Delta H^2_{C_2}.$$ 

Since $\Delta$ is not two-sided inner, it follows from Lemma 3.13 that $\Phi$ does not admit a Douglas-Shapiro-Shields factorization. For a decomposition of $\Phi$, suppose that

$$\Phi = \Omega A^*,$$

(31)

where $\Omega, A \in H^2_{M_{3\times k}} (k = 1, 2, 3)$, $\Omega$ is an inner function, and $\Omega$ and $A$ are right coprime. We then have

$$\Phi^* \Omega = A \in H^2_{M_{3\times k}}.$$

(32)

But since $a$ is not of bounded type, it follows from (32) that the 3rd row vector of $\Omega$ is zero. Thus by (31), we must have $a = 0$, a contradiction. Therefore we could not get any decomposition of the form $\Phi = \Omega A^*$ with a $3 \times k$ inner matrix function $\Omega$ for each $k = 1, 2, 3$. To get another idea, we note that $\ker \Delta^* = [0 0 1]^t H^2 \equiv \Delta H^2_{C_2}$. Then by a direct manipulation, we can get

$$\Phi = \begin{bmatrix} f \\ g \\ 0 \end{bmatrix} \equiv [\phi_1, \phi_2, \phi_3] \in H^\infty_{M_3},$$

where $f$ and $g$ are given in Example 3.4, $\theta$ is inner, and $a \in H^\infty$ is such that $\theta$ and $a$ are coprime. It then follows from Lemma 3.16 that

$$\ker H_{\Phi} = \begin{bmatrix} f \\ g \end{bmatrix} H^2.$$ 

We thus have that

$$\ker H_{\Phi^*} = \ker H_{\Phi} \bigoplus \ker H_{\pi a} = \begin{bmatrix} f \\ g \\ 0 \end{bmatrix} H^2_{C_2} \equiv \Delta H^2_{C_2}.$$

By the same argument as the preceding example, we see that $\Phi$ does not admit a Douglas-Shapiro-Shields factorization. Observe that

$$\Phi = \begin{bmatrix} f \\ g \\ 0 \end{bmatrix} \equiv [\phi_1, \phi_2, \phi_3] \in H^\infty_{M_3},$$

where $f$ and $g$ are given in Example 3.4, $\theta$ is inner, and $a \in H^\infty$ is such that $\theta$ and $a$ are coprime. It then follows from Lemma 3.16 that

$$\ker H_{\Phi} = \begin{bmatrix} f \\ g \end{bmatrix} H^2.$$ 

We thus have that

$$\ker H_{\Phi^*} = \ker H_{\Phi} \bigoplus \ker H_{\pi a} = \begin{bmatrix} f \\ g \\ 0 \end{bmatrix} H^2_{C_2} \equiv \Delta H^2_{C_2}.$$

By the same argument as the preceding example, we see that $\Phi$ does not admit a Douglas-Shapiro-Shields factorization. Observe that

$$\Phi = \begin{bmatrix} f \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & 0 \end{bmatrix} H^2_{C_2} \equiv \Delta H^2_{C_2}.$$ 

(33)

Since $\theta$ and $\pi$ are coprime, it follows that $\Delta$ and $A$ are right coprime. Note that $\Delta$ is not two-sided inner and $\ker \Delta^* = \{0\}.$

The above examples (33) and (34) show that the decomposition of a matrix-valued $H^2$-functions $\Phi$ satisfying $\ker H^*_\Phi = \Delta H^2_{C_2}$ is affected by the kernel of $\Delta^*$ and in turn, the complementary factor $\Delta_c$ of $\Delta$. Indeed, if we regard $\Delta^*$ as an operator acting from $L^2_E$,
and hence \( \ker \Delta^* \subseteq L^2_E \), then \( B \) in the canonical decomposition (2) satisfies the inclusion \( \{ B \} \subseteq \ker \Delta^* \). Theorem A gives a canonical decomposition of strong \( L^2 \)-functions which realizes the idea inside those examples.

We are ready for a proof of Theorem A.

**Proof of Theorem A.** If \( \ker H^*_\Phi = \{ 0 \} \), take \( E' := \{ 0 \} \) and \( B := \Phi \). Then \( \tilde{\Delta} \) and \( \tilde{A} \) are zero operator with codomain \( \{ 0 \} \). Thus \( \Phi = \Delta A^* + B \), where \( \Delta \) and \( A \) are right coprime. It also follows from Lemma 3.19 that \( \text{nc} \{ \Phi_+ \} = 0 \), which gives the inequality (iv). If instead \( \ker H^*_\Phi \neq \{ 0 \} \), then in view of Corollary 3.15, we may suppose \( \ker H^*_\Phi = \Delta H^2_E \) for some nonzero inner function \( \Delta \) with values in \( \mathcal{B}(E', E') \). Put \( A := \Phi \). Then it thus follows from Lemma 3.6 and Lemma 3.9 that \( \tilde{\Delta} = \tilde{\Delta} \Phi \) is a strong \( L^2 \)-function with values in \( \mathcal{B}(D, E') \). Since \( \ker H^*_\Phi = \Delta H^2_E \), it follows that for all \( p \in \mathcal{P}_D \) and \( h \in H^2_E \),

\[
0 = \langle H^*_\Phi p, \Delta h \rangle_{L^2_E} = \int_T \langle \Phi(z)p(z), \overline{\Delta}(\overline{z})h(z) \rangle_{E} dm(z)
\]

which implies \( H^*_\Delta = 0 \). Thus by Lemma 3.12, \( \tilde{A} \) belongs to \( H^2_\mathcal{B}(D, E') \). Put \( B := \Phi - \Delta A^* \). Then by Lemma 3.9, \( B \) is a strong \( L^2 \)-function with values in \( \mathcal{B}(D, E') \). Observe that

\[
\Phi = \Delta A^* + B \quad \text{and} \quad \Delta^* B = 0.
\]

For the first assertion, we need to show that \( \Delta \) and \( A \) are right coprime. To see this, we suppose that \( \Omega \) is a common left inner divisor, with values in \( \mathcal{B}(E'', E') \), of \( \tilde{\Delta} \) and \( \tilde{A} \). Then we may write

\[
\tilde{\Delta} = \Omega \tilde{\Delta}_1 \quad \text{and} \quad \tilde{A} = \Omega \tilde{A}_1,
\]

where \( \tilde{\Delta}_1 \in H^\infty(\mathcal{B}(E, E'')) \) and \( \tilde{A}_1 \in H^2_\mathcal{B}(D, E'') \). Thus we have

\[
\Delta = \Delta_1 \tilde{\Omega} \quad \text{and} \quad A = A_1 \tilde{\Omega}.
\]

(35)

Since \( \Omega \) is inner, it follows that \( I_{E''} = \overline{\Omega} \tilde{\Omega} = \tilde{\Omega} \Omega^* \). Thus by (35), \( \Delta_1 = \Delta \tilde{\Omega}^* \), and hence, by Lemma 3.6, \( \Delta_1 \) is inner. We now claim that

\[
\Delta_1 H^2_{E''} = \ker H^*_\Phi = \Delta H^2_{E'}.
\]

(36)

Since \( \Omega \) is an inner function with values in \( \mathcal{B}(E'', E') \), we know that \( \tilde{\Omega} \in H^\infty(\mathcal{B}(E', E'')) \) by Lemma 3.6. Thus it follows from Corollary 3.10 and (35) that

\[
\Delta H^2_{E'} = \Delta \tilde{\Omega} H^2_{E'} \subseteq \Delta_1 H^2_{E''}.
\]
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For the reverse inclusion, by (35), we may write $\Phi = \Delta_1 A_1^* + B$. Since $0 = \Delta^* B = \Omega^* \Delta_1 B$, it follows that $\Delta_1^* B = 0$. Therefore for all $f \in H_{E''}^2$, $x \in D$ and $n = 1, 2, \cdots$, we have

$$
\int_T \langle \Phi(z)x, z^n \Delta_1(z)f(z) \rangle_{E'} dm(z) = \int_T \langle (\Delta_1(z)A_1^*(z) + B(z))x, z^n \Delta_1(z)f(z) \rangle_{E'} dm(z)
$$

$$
= \int_T \langle A_1^*(z)x, z^n f(z) \rangle_{E''} dm(z)
$$

$$
= \langle A_1^*(z)x, z^n f(z) \rangle_{L^2_{E''}}
$$

$$
= 0,
$$

where the last equality follows from the fact that $A_1^*(z)x = \tilde{A}_1(z)x \in L^2_{E''} \ominus zH^2_{E''}$. Thus by Lemma 3.14, we have

$$
\Delta_1 H_{E''}^2 \subseteq \ker H^*_\Phi = \Delta H_{E''}^2,
$$

which proves (36). Thus it follows from the Beurling-Lax-Halmos Theorem and (35) that $\tilde{\Omega}$ is a unitary operator, and so is $\Omega$. Therefore $A$ and $\Delta$ are right coprime. The assertion (iv) on the nc-number comes from Lemma 3.19. This proves the first assertion (2).

Suppose dim $E' < \infty$. For the uniqueness of the expression (2), we suppose that $\Phi = \Delta_1 A_1^* + B_1 = \Delta_2 A_2^* + B_2$ are two canonical decompositions of $\Phi$. We want to show that $\Delta_1 = \Delta_2$, which gives

$$
A_1^* = \Delta_1^*(\Delta_1 A_1^* + B_1) = \Delta_2^*(\Delta_2 A_2^* + B_2) = A_2^*
$$

and in turn, $B_1 = B_2$, which implies that the representation (2) is unique. To prove $\Delta_1 = \Delta_2$, it suffices to show that if $\Phi = \Delta A^* + B$ is a canonical decomposition of $\Phi$, then

$$
\ker H^*_\Phi = \Delta H_{E''}^2.
$$

(37)

If $E' = \{0\}$, then $\text{nc}\{\Phi_+\} = 0$. Thus it follows from Remark 3.20(1) that

$$
\ker H^*_\Phi = \{0\} = \Delta H_{E''}^2,
$$

which proves (37). If instead $E' \neq \{0\}$, then we suppose $r := \dim E' < \infty$. Thus, we may assume that $E' \equiv \mathbb{C}^r$, so that $\Delta$ is an inner function with values in $\mathcal{B}(\mathbb{C}^r, E)$. Suppose that $\Phi = \Delta A^* + B$ is a canonical decomposition of $\Phi$ in $L^2_{E''}(\mathcal{B}(D, E))$. Then

(i) $\tilde{A}$ belongs to $H^2_{E''}(\mathcal{B}(D, \mathbb{C}^r))$ such that $\Delta$ and $A$ are right coprime;

(ii) $B$ is a strong $L^2$-function with values in $\mathcal{B}(D, E)$ such that $\Delta^* B = 0$;

(iii) $\text{nc}\{\Phi_+\} \leq r$.

We first claim that

$$
\Delta H_{E''}^2 \subseteq \ker H^*_\Phi.
$$

(38)

Observe that for each $g \in H_{E''}^2$, $x \in D$ and $k = 1, 2, 3, \cdots$,

$$
\int_T \langle \Phi(z)x, z^k \Delta(z)g(z) \rangle_{E} dm(z) = \int_T \langle A^*(z)x, z^k g(z) \rangle_{E'} dm(z)
$$

$$
= \langle \tilde{A}(z)x, z^k g(z) \rangle_{L^2_{E'}}
$$

$$
= 0.
$$

It thus follows from Lemma 3.14 that $\Delta H_{E''}^2 \subseteq \ker H^*_\Phi$, which proves (38). In view of Corollary 3.15, we may assume that $\ker H^*_\Phi = \Theta H_{E''}^2$, for some inner function $\Theta$ with values in $\mathcal{B}(E'', E)$. Then by Lemma 3.19,

$$
p = \dim E'' = \text{nc}\{\Phi_+\} \leq r.
$$

(39)
Thus we may assume $E'' \equiv \mathbb{C}^p$. Since
\[ \Delta H_{2r}^p \subseteq \ker H_\Phi^* = \Theta H_{2r}^p, \tag{40} \]
it follows that $\Theta$ is left inner divisor of $\Delta$, i.e., there exists a $p \times r$ inner matrix function $\Delta_1$ such that $\Delta = \Theta \Delta_1$. Since $\Delta_1$ is inner, it follows that $r \leq p$. But since by (39), $p \leq r$, we must have $r = p$, which implies that $\Delta_1$ is two-sided inner. Thus we have
\[ \Theta^* \Phi = \Delta_1 A^* + \Delta_1 \Delta^* B = \Delta_1 A^*. \tag{41} \]
Since $\ker H_\Phi^* = \Theta H_{2r}^p$, it follows from Lemma 3.14 and (41) that for all $f \in H_{2r}^p$, $x \in D$ and $n = 1, 2, \ldots$,
\[ \int_T \langle \Delta_1(z) A^* (z) x, z^n f(z) \rangle_{c^2} \, dm(z) = \int_T \langle \Phi(z) x, z^n \Theta(z) f(z) \rangle_{E} \, dm(z) = 0. \tag{42} \]
Write $\Psi := \Delta_1 A^*$. Then by Lemma 3.9, $\Psi \in L_2^2(\mathcal{B}(D, C^r))$. Thus by Lemma 3.12, Lemma 3.14 and (42), we have $\Psi \in H_{2r}^2(\mathcal{B}(D, C^r))$. Since $A = \tilde{\Delta}_1 \Psi$, it follows that $\tilde{\Delta}_1$ is a common left inner divisor of $\Delta$ and $\tilde{\Delta}$. But since $\Delta$ and $A$ are right coprime, it follows that $\tilde{\Delta}_1$ is a unitary matrix, and so is $\Delta_1$, which proves (37). This proves the uniqueness of the expression (2) when $\dim E' < \infty$.

For the last assertion, we suppose that $\tilde{\Delta}$ is of bounded type. Then by Corollary 3.24, $[\Delta, \Delta_c]$ is two-sided inner, where $\Delta_c$ is the complementary factor of $\Delta$, with values in $\mathcal{B}(D', E)$. We thus have
\[ I = [\Delta, \Delta_c] [\Delta, \Delta_c]^* = \Delta \Delta^* + \Delta_c \Delta_c^*, \]
so that
\[ B = \Phi - \Delta A^* = (I - \Delta \Delta^*) \Phi = \Delta_c \Delta_c^* \Phi. \]
This proves the last assertion.

This completes the proof. \hfill \Box

\textbf{Remark 4.1.} If $\tilde{\Delta}$ is of bounded type and $\dim E' < \infty$ in Theorem A, then the dimension of the domain $D'$ of $\Delta_c$ can be computed by the degrees of non-cyclicity of $\Phi_+$ and $\Delta$: i.e., $\dim D' = \text{nc}\{\Delta\} - \text{nc}\{\Phi_+\}$. Indeed, this follows from the facts that $\text{nc}\{\Phi_+\} = \dim E' < \infty$ (by Lemma 3.19) and $\text{nc}\{\Delta\} = \dim E' + \dim D'$ (by Remark 3.20(2)).

The following corollary is an extension of Lemma 3.13 (the Douglas-Shapiro-Shields factorization) to strong $L^2$-functions.

\textbf{Corollary 4.2.} If $\Phi$ is a strong $L^2$-function with values in $\mathcal{B}(D, E)$, then the following are equivalent:

(a) The flip $\tilde{\Phi}$ of $\Phi$ is of bounded type;
(b) $\Phi = \Delta A^*$ ($\Delta$ is two-sided inner) is a canonical decomposition of $\Phi$.

\textbf{Proof.} The implication (a) $\Rightarrow$ (b) follows from the proof of Theorem A. For the implication (b) $\Rightarrow$ (a), suppose $\Phi = \Delta A^*$ ($\Delta$ is two-sided inner) is a canonical decomposition of $\Phi$. By Corollary 3.15, there exists an inner function $\Theta$ with values in $\mathcal{B}(D', E)$ such that $\ker H_\Phi^* = \Theta H_{2r}^p$. Then it follows from Lemma 3.14 that $\Delta H_{2r}^p \subseteq \ker H_\Phi^* = \Theta H_{2r}^p$. Since $\Delta$ is two-sided inner, we have that by Lemma 3.5, $\Theta$ is two-sided inner, and hence the flip $\tilde{\Phi}$ of $\Phi$ is of bounded type. This completes the proof. \hfill \Box
Remark 4.3. (a) If \( \dim E' = \infty \) (even though \( \dim D < \infty \)), the canonical decomposition (2) may not be unique even if \( \Phi \) is of bounded type. To see this, let \( \Phi \) be an inner function with values in \( B(C^2, \ell^2) \) defined by

\[
\Phi := \begin{bmatrix}
\theta_1 & 0 \\
0 & 0 \\
0 & \theta_2 \\
0 & 0 \\
0 & 0 \\
\vdots & \vdots
\end{bmatrix},
\]

where \( \theta_1 \) and \( \theta_2 \) are scalar inner functions. Then by Lemma 3.16, we have

\[
\ker H^*_\Phi = \ker H_{\Phi^*} = \Phi H^2_{\ell^2} \oplus \ker \Phi^* = \text{diag}(\theta_1, 1, \theta_2, 1, 1, \cdots) H^2_{\ell^2} \equiv \Theta H^2_{\ell^2},
\]

which implies that \( \Phi \) is of bounded type since \( \Theta \) is two-sided inner (see Definition 3.22). Let

\[
A := \Phi^* \Theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
\end{bmatrix} \quad \text{and} \quad B := 0.
\]

Then \( A \) belongs to \( H^2_s(B(C^2, \ell^2)) \) and \( \Delta H^2_{\ell^2} \bigcup \tilde{\Delta} H^2_{\ell^2} = H^2_{\ell^2} \), which implies that \( \Theta \) and \( A \) are right coprime. Clearly, \( \Theta^* B = 0 \) and \( \text{nc} \{\Phi_+\} \leq \dim \ell^2 = \infty \). Therefore,

\[
\Phi = \Theta A^\star
\]

is the BLH-canonical decomposition of \( \Phi \). On the other hand, to get another canonical decomposition of \( \Phi \), let

\[
\Delta := \begin{bmatrix}
\theta_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \theta_2 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}.
\]

Then \( \Delta \) is an inner function. If we define

\[
A_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
\end{bmatrix} \quad \text{and} \quad B := 0,
\]

then \( A_1 \) belongs to \( H^2_s(B(C^2, \ell^2)) \) such that \( \Delta \) and \( A_1 \) are right coprime, \( \Delta^\star B = 0 \) and \( \text{nc} \{\Phi_+\} \leq \dim \ell^2 = \infty \). Therefore \( \Phi = \Delta A_1^\star \) is also a canonical decomposition of \( \Phi \). In this case, ker \( H^2_\Phi \neq \Delta H^2_\tilde{\Delta} \). Therefore, the canonical decomposition of \( \Phi \) is not unique.

(b) Let \( \Delta \) be an inner matrix function with values in \( B(E', E) \). Then Theorem A says that if \( \dim E' < \infty \), the expression (2) satisfying the conditions (i) - (iv) in Theorem A gives ker \( H^2_\Phi = \Delta H^2_\tilde{\Delta} \). We note that the condition (iv) on nc-number cannot be dropped from the assumptions of Theorem A. To see this, let

\[
\Delta := \frac{1}{\sqrt{2}} \begin{bmatrix} z \\
1
\end{bmatrix}, \quad A := \begin{bmatrix} \sqrt{2} \\
0
\end{bmatrix} \quad \text{and} \quad B := 0.
\]
If $Φ := ΔA^* + B = \begin{bmatrix} z & 0 \\ 1 & 0 \end{bmatrix}$, then $Φ$ satisfies the conditions (i), (ii), and (iii), but $\ker H_Φ^* = zH^2 \oplus H^2 \neq ΔH^2$. Note that by Lemma 3.19, $nc\{Φ_+\} = 2$, which does not satisfy the condition on nc-number, say $nc\{Φ_+\} \leq 1$.

5. The Beurling degree: Proof of Theorem B

In this section we give a proof of Theorem B. To do so, we first consider the question: If $Δ$ is an inner function with values in $B(E',E)$, does there exist a strong $L^2$-function $Φ$ with values in $B(D,E)$ satisfying the equation

$$\ker H_Φ^* = ΔH^2_{E'}? \quad (43)$$

To closely understand an answer to the question (43), we examine a question whether there exists an inner function $Ω$ satisfying $\ker H_Ω^* = ΔH^2_{E'}$ if $Δ$ is an inner function with values in $B(E',E)$. In fact, the answer to this question is negative. Indeed, if $\ker H_Ω^* = ΔH^2_{E'}$ for some inner function $Ω$, then by Lemma 3.16, we have $[Ω, Ω_c] = Δ$, and hence $Δ_c = 0$. Conversely, if $Δ_c = 0$ then by again Lemma 3.16, we should have $\ker H_Δ^* = ΔH^2_{E'}$. Consequently, $\ker H_Ω^* = ΔH^2_{E'}$ for some inner function $Ω$ if and only if $Δ_c = 0$. Thus if

$$Δ := \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

then there exists no inner function $Ω$ such that $\ker H_Ω^* = ΔH^2_{E'}$. On the other hand, we note that the solution $Φ$ is not unique although there exists an inner function $Φ$ satisfying the equation (43). For example, if $Δ := \text{diag}(z,1,1)$, then the following $Φ$ are all such solutions:

$$Φ = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} z & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Δ.$$

The following lemma gives an affirmative answer to the question (43): we can always find a strong $L^2$-function $Φ$ with values in $B(D,E)$ satisfying the equation $\ker H_Φ^* = ΔH^2_{E'}$.

**Lemma 5.1.** Let $Δ$ be an inner function with values in $B(E',E)$. Then there exists a function $Φ$ in $H^2_s(B(D,E))$, with either $D = E'$ or $D = C \oplus E'$, satisfying

$$\ker H_Φ^* = ΔH^2_{E'}.$$

**Proof.** If $\ker Δ^* = \{0\}$, take $Φ = Δ$. Then it follows from Lemma 3.16 that

$$\ker H_Φ^* = \ker H_Δ^* = ΔH^2_{E'}.$$

If instead $\ker Δ^* \neq \{0\}$, let $Δ_c$ be the complementary factor of $Δ$ with values in $B(E'',E)$ for some nonzero Hilbert space $E''$. Choose a cyclic vector $g ∈ H^2_{E''}$ of $S^2_{E''}$, and define

$$Φ := [[zΔ_c g], Δ],$$

where $[zΔ_c g](z) : C \to E$ is given by $[zΔ_c g](z) = αzΔ_c(z)g(z)$. Then it follows from Lemma 3.7 and Lemma 3.9 that $Φ$ belongs to $H^2_s(B(D,E))$, where $D = C \oplus E'$. For each
x \equiv \alpha \oplus x_0 \in D, f \in H^2_{E_r}, and n = 1, 2, 3, \ldots, we have

\begin{align*}
\int_T \langle \Phi(z)x, z^n \Delta(z)f(z) \rangle_{E} \, dm(z) &= \int_T \langle \alpha z \Delta_c(z)g(z) + \Delta(z)x_0, z^n \Delta(z)f(z) \rangle_{E} \, dm(z) \\
&= \int_T \langle x_0, z^n f(z) \rangle_{E} \, dm(z) \quad \text{(since } \Delta^* \Delta_c = 0) \\
&= 0.
\end{align*}

It thus follows from Lemma 3.14 that

\[ \Delta H^2_{E^r} \subseteq \ker H^*_{\Phi}. \quad (44) \]

For the reverse inclusion, suppose \( h \in \ker H_{\Phi}^* \). Then by Lemma 3.14, we have that for each \( x \equiv \alpha \oplus x_0 \in D \) and \( n = 1, 2, 3, \ldots, \)

\[ \int_T \langle \Delta(z)x_0, z^n h(z) \rangle_{E} \, dm(z) = 0. \]

Take \( \alpha = 0 \). Then we have

\[ \int_T \langle \Delta(z)x_0, z^n h(z) \rangle_{E} \, dm(z) = 0, \]

which implies, by Lemma 3.14, that \( h \in \ker H_{\Delta^*} \). It thus follows from Lemma 3.16 that

\[ \ker H^*_{\Phi} \subseteq \ker H_{\Delta^*} = \Delta H^2_{E^r} \ominus \Delta_c H^2_{E^r} \quad (45) \]

Assume to the contrary that \( \ker H^*_{\Phi} \neq \Delta H^2_{E^r} \). Then by (44) and (45), there exists a nonzero function \( f \in H^2_{E^r} \) such that \( \Delta_c f \in \ker H^*_{\Phi} \). It thus follows from Lemma 3.14 that for each \( x \equiv \alpha \oplus x_0 \in D \) and \( n = 1, 2, 3, \ldots, \)

\[ 0 = \int_T \langle \Phi(z)x, z^n \Delta_c(z)f(z) \rangle_{E} \, dm(z) \\
= \int_T \langle \alpha z \Delta_c(z)g(z) + \Delta(z)x_0, z^n \Delta_c(z)f(z) \rangle_{E} \, dm(z) \\
= \int_T \langle z g(z) \alpha, z^n f(z) \rangle_{E^r} \, dm(z) \quad \text{(since } \Delta^* \Delta_c = 0), \]

which implies that \( f \in \ker H^*_{\Phi_{[g]}} \). Since \( g \) is a cyclic vector of \( S_{E^r} \), it thus follows from Lemma 3.17 that

\[ f \in (\cl \ran H_{\Phi_{[g]}})^{\perp} = (E^*_{g})^{\perp} = \{0\}, \]

which is a contradiction. This completes the proof. \( \square \)

If \( \Delta \) is an \( n \times r \) inner matrix function, then we can find a solution \( \Phi \in H^\infty_{M_{n \times m}} \) (with \( m \leq r + 1 \)) of the equation \( \ker H^*_{\Phi} = \Delta H^2_{E^r} \).

**Corollary 5.2.** For a given \( n \times r \) inner matrix function \( \Delta \), there exists at least a solution \( \Phi \in H^\infty_{M_{n \times m}} \) (with \( m \leq r + 1 \)) of the equation \( \ker H^*_{\Phi} = \Delta H^2_{E^r} \).

**Proof.** If \( \ker \Delta^* = \{0\} \), then this is obvious. Let \( \ker \Delta^* \neq \{0\} \) and \( \Delta_c \in H^\infty_{M_{p \times n}} \) be the complementary factor of \( \Delta \). Then by Lemma 3.16, \( 1 \leq p \leq n - r \). For \( j = 1, 2, \ldots , p \), put

\[ g_j := e^j \frac{1}{\overline{\rho}_j}, \]
where $\alpha_j$ are distinct points in the interval $[2,3]$. Then it is known that (cf. [Ni1, P. 55])

$$g := \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_p \end{bmatrix} \in H_{C^p}^\infty$$

is a cyclic vector of $S_{C^p}^*$. Put $\Phi := [(z\Delta g), \Delta]$. Then by Lemma 3.7, we have $\Phi \in H_{M_n \times (r+1)}^\infty$. The same argument as the proof of Lemma 5.1 gives the result. □

**Corollary 5.3.** If $\Delta$ is an inner function with values in $B(E', E)$, then there exists a function $\Phi \in L^2(B(D, E))$ (with $D = E'$ or $D = \mathbb{C} \oplus E'$) such that $\Phi \equiv \Delta A^* + B$ is the BLH-canonical decomposition of $\Phi$.

**Proof.** By Lemma 5.1, there exists a function $\Phi \in H^2_s(B(D, E))$ such that $\text{ker } H_\Phi^* = \Delta H^2_{C^r}$. If we put $A := \Phi^* \Delta$ and $B := \Phi - \Delta A^*$, then by the proof of the first assertion of Theorem A, $\Phi = \Delta A^* + B$ is the BLH-canonical decomposition of $\Phi$. □

**Remark 5.4.** In view of Corollary 5.2, it is reasonable to ask whether such a solution $\Phi \in L^2(B_{M_n \times m})$ of the equation $\text{ker } H_\Phi^* = \Delta H^2_{C^r}$ (where $\Delta$ is an inner matrix function) exists for each $m = 1, 2, \ldots$ even though it exists for some $m$. For example, let

$$\Delta := \frac{1}{\sqrt{2}} \begin{bmatrix} z \\ 1 \end{bmatrix}. \quad (46)$$

Then, by Corollary 5.2, there exists a solution $\Phi \in L^2_{M_2 \times m}$ ($m = 1$ or $2$) of the equation $\text{ker } H_\Phi^* = \Delta H^2$. For $m = 2$, let

$$\Phi := \begin{bmatrix} z & za \\ 1 & -a \end{bmatrix} \in H_{M_2}^\infty, \quad (47)$$

where $a \in H^\infty$ is such that $\pi$ is not of bounded type. Then a direct calculation shows that $\text{ker } H_\Phi^* = \text{ker } H_{\Phi^*} = \Delta H^2$. We may then ask how about the case $m = 1$. In this case, the answer is affirmative. To see this, let

$$\Psi := \begin{bmatrix} z + za \\ 1 - a \end{bmatrix} \in H_{M_2 \times 1}^\infty,$$

where $a \in H^\infty$ is such that $\pi$ is not of bounded type. Then a direct calculation shows that $\text{ker } H_{\Phi^*} = \Delta H^2$. Therefore, if $\Delta$ is given by (46), then we may assert that there exists a solution $\Phi \in L^2_{M_n \times m}$ of the equation $\text{ker } H_\Phi^* = \Delta H^2$ for each $m = 1, 2$. However, this assertion is not true in general, i.e., a solution exists for some $m$, but may not exist for another $m_0 < m$. To see this, let

$$\Delta := \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{bmatrix} \in H_{M_3 \times 3}^\infty.$$
Then $\Delta$ is inner. We will show that there exists no solution $\Phi \in L^2_{M_{4 \times 1}}$ (i.e., the case $m = 1$) of the equation $\ker H_{\Phi}^* = \Delta H_{\Phi}^2$. Assume to the contrary that $\Phi \in L^2_{M_{4 \times 1}}$ is a solution of the equation $\ker H_{\Phi}^* = \Delta H_{\Phi}^2$. By Theorem A, $\Phi$ can be written as

$$\Phi = \Delta A^* + B,$$

where $A \in H^2_{M_{4 \times 1}}$ is such that $\Delta$ and $A$ are right coprime. But since $\tilde{\Delta} H_{\Phi}^2 = zH^2 \oplus zH^2 \oplus H^2$, it follows that

$$\tilde{\Delta} H_{\Phi}^2 \cap \Delta H^2 \neq H^2_{\phi},$$

which implies that $\Delta$ and $A$ are not right coprime, a contradiction. Therefore we cannot find any solution $\Phi$, in $L^2_{M_{4 \times 1}}$ (the case $m = 1$), of the equation $\ker H_{\Phi}^* = \Delta H_{\Phi}^2$. However, if $m = 2$, then we can find a solution $\Phi \in L^2_{M_{4 \times 2}}$. Indeed, let

$$\Phi := \begin{bmatrix} z & 0 \\ 0 & z \\ 0 & 0 \\ a & 0 \end{bmatrix},$$

where $a \in H^\infty$ is such that $\Phi$ is not of bounded type. A straightforward calculation shows that $\ker H_{\Phi}^* = zH^2 \oplus zH^2 \oplus H^2 \oplus \{0\} = \Delta H_{\Phi}^2$. Thus we obtain a solution for $m = 2$ although there exists no solution for $m = 1$.

Let $\Delta$ be an inner function with values in $\mathcal{B}(E', E)$. In view of Remark 5.4, we may ask how to determine a possible dimension of $D$ for which there exists a solution $\Phi \in L^2(B(D,E))$ of the equation $\ker H_{\Phi}^* = \Delta H_{\Phi}^2$. In fact, if we have a solution $\Phi \in L^2(B(D,E))$ of the equation $\ker H_{\Phi}^* = \Delta H_{\Phi}^2$, then a solution $\Psi \in L^2(D', E)$ also exists if $D'$ is a separable complex Hilbert space containing $D$: indeed, if $0$ denotes the zero operator in $\mathcal{B}(D' \oplus D, E)$ and $\Psi := [\Phi, 0]$, then it follows from Lemma 3.14 that $\ker H_{\Phi}^* = \ker H_{\Psi}^*$. Thus we would like to ask what is the infimum of $\dim D$ such that there exists a solution $\Phi \in L^2(B(D,E))$ of the equation $\ker H_{\Phi}^* = \Delta H_{\Phi}^2$.

To answer this question, we introduce a notion of the “Beurling degree” for an inner function, by employing a canonical decomposition of strong $L^2$-functions induced by the given inner function.

**Definition 5.5.** Let $\Delta$ be an inner function with values in $\mathcal{B}(E', E)$. Then the **Beurling degree** of $\Delta$, denoted by $\deg_B(\Delta)$, is defined by the infimum of the dimension of nonzero space $D$ for which there exists a pair $(A, B)$ such that $\Phi = \Delta A^* + B$ is a canonical decomposition of $\Phi$ in $L^2(B(D,E))$: i.e.,

$$\deg_B(\Delta) := \inf \dim D \in \mathbb{Z}_+ \cup \{\infty\}.$$  

**Note.** By Corollary 5.3, $\deg_B(\Delta)$ is well-defined: indeed, $1 \leq \deg_B(\Delta) \leq 1 + \dim E'$. In particular, if $E' = \{0\}$, then $\deg_B(\Delta) = 1$. Also if $\Delta$ is a unitary operator then clearly, $\deg_B(\Delta) = 1$.

Given a backward shift-invariant subspace $\mathcal{H}(\Delta)$, where $\Delta$ is an inner function with values in $\mathcal{B}(E', E)$, we may ask what is the smallest number $m$ of vectors in $F$ satisfying $\mathcal{H}(\Delta) = E_F^m$. We recall that the smallest number $m$ of vectors in $F$ such that $\mathcal{H}(\Delta) = E_F^m$...
is exactly the spectral multiplicity $\mu_T$ of the truncated backward shift $T \equiv S^+_E|_{\mathcal{H}(\Delta)}$.

Now it is somewhat surprising that if $\Delta$ is an inner function with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$ (in particular, $\Delta$ is an inner matrix function), then the spectral multiplicity of the model operator, i.e., truncated backward shift on $\mathcal{H}(\Delta)$ coincides with the Beurling degree of the inner function $\Delta -$ this is the second object of this paper.

We are ready for:

**Proof of Theorem B.** We first claim that

$$\deg_B(\Delta) = \inf \{ \dim D : \ker H^*_\Phi = \Delta H^2_{E'} \text{ for some } \Phi \in L^2_*(\mathcal{B}(D, E)) \text{ with } D \neq \{0\} \}. \tag{48}$$

To see this, let $\Delta$ be an inner function with values in $\mathcal{B}(E', E)$, with $\dim E' < \infty$. Suppose that $\Phi = \Delta A^* + B$ is a canonical decomposition of $\Phi$ in $L^2_*(\mathcal{B}(D, E))$. Then by the uniqueness of $\Delta$ in Theorem A, we have

$$\ker H^*_\Phi = \Delta H^2_{E'}, \tag{49}$$

which implies

$$\deg_B(\Delta) \geq \inf \{ \dim D : \ker H^*_\Phi = \Delta H^2_{E'} \text{ for some } \Phi \in L^2_*(\mathcal{B}(D, E)) \text{ with } D \neq \{0\} \}. \tag{50}$$

For the reverse inequality of (50), suppose $\Phi \in L^2_*(\mathcal{B}(D, E))$ satisfies $\ker H^*_\Phi = \Delta H^2_{E'}$. Then by the same argument as in the proof of the first assertion of Theorem A,

$$\Phi = \Delta A^* + B \quad (A := \Phi^* \Delta \text{ and } B := \Phi - \Delta A^*)$$

is a canonical decomposition of $\Phi$, and hence we have the reverse inequality of (50). This proves the claim (48). We will next show that

$$\deg_B(\Delta) \leq \mu_T. \tag{51}$$

If $\mu_T = \infty$, then (51) is trivial. Suppose $p \equiv \mu_T < \infty$. Then there exists a subset $G = \{g_1, g_2, \ldots, g_p\} \subseteq H^2_{E'}$ such that $E^*_G = \mathcal{H}(\Delta)$. Put

$$\Psi := z[G].$$

Then by Lemma 3.7, $\Psi \in H^2_*(\mathcal{B}(\mathbb{C}^p, E))$. It thus follows from Lemma 3.17 that

$$\mathcal{H}(\Delta) = E^*_G = \text{cl ran } H^*_\Psi = \text{cl ran } H^*_\Psi,$$

which implies $H^*_\Psi = \Delta H^2_{E'}$. Thus by (48), $\deg_B(\Delta) \leq p = \mu_T$, which proves (51). For the reverse inequality of (51), suppose that $r \equiv \dim E' < \infty$. Write $m_0 \equiv \deg_B(\Delta)$. Then it follows from Lemma 5.1 and (48) that $m_0 \leq r + 1 < \infty$ and there exists a function $\Phi \in L^2_*(\mathcal{B}(\mathbb{C}^{m_0}, E))$ such that

$$\ker H^*_\Phi = \Delta H^2_{E'}. \tag{52}$$

Now let

$$G := \Phi_\Psi - \Phi(0).$$

Thus we may write $G = zF$ for some $F \in H^2_*(\mathcal{B}(\mathbb{C}^{m_0}, E))$. Then by Lemma 3.17, we have that

$$E^*_F = \text{cl ran } H^*_\Psi = (\ker H^*_\Phi)^\perp = \mathcal{H}(\Delta),$$

which implies $\mu_T \leq m_0 = \deg_B(\Delta)$. This completes the proof of Theorem B. \qed

**Remark 5.6.** We give some interesting remarks on Theorem B.
A canonical decomposition of strong $L^2$-functions

(a) From a careful analysis of the proof of Theorem B, we can see that (51) holds in general without the assumption “dim $E' < \infty$”; more concretely, given an inner function $\Delta$ with values in $\mathcal{B}(E', E)$, if $T := S^*_E|_{\mathcal{H}(\Delta)}$, then

$$\text{deg}_B(\Delta) \leq \mu_T.$$ 

(b) Suppose $\Delta$ is an inner function with values in $\mathcal{B}(E', E)$, with dim $E' < \infty$. If $\Phi = \Delta A + B$ is a canonical decomposition of $\Phi$ in $L^2_s(\mathcal{B}(D, E))$. Then by Theorem A, we have that

$$\ker H^*_\Phi = \Delta H^2_{E'},$$

It thus follows from the proof of Theorem B that

$$E^*_{\{F\}} = H(\Delta),$$

where $F$ is defined by $F(z) := \overline{z}(\Phi(z) - \hat{\Phi}(0))$. This gives an answer to the problem of describing the set $\{F\}$ in $H^2_E$ such that $H(\Delta) = E^*_{\{F\}}$, given an inner function $\Delta$ with values in $\mathcal{B}(E', E)$, with dim $E' < \infty$ (cf. p.2).

(c) From Remark 5.4 and (48), we see that if

$$\Delta := \begin{bmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\text{deg}_B(\Delta) = 2$. Let $T := S^*_C|_{\mathcal{H}(\Delta)}$. Observe that $\mathcal{H}(\Delta) = \mathcal{H}(z) \oplus \mathcal{H}(z) \oplus \{0\} \oplus H^2$. Since $\mathcal{H}(z) \oplus \mathcal{H}(z)$ has no cyclic vector, we must have $\mu_T \neq 1$. In fact, if we put $f = (1 0 0 a)^t$ (a is not of bounded type) and $g = (0 1 0 0)^t$, then $E^*_{\{f,g\}} = \mathcal{H}(\Delta)$, which implies $\mu_T = 2$. This illustrates Theorem B.

It was known (cf. [Ni1, p. 41]) that if $\Delta$ is an $n \times r$ inner matrix function and $T := S^*_C|_{\mathcal{H}(\Delta)}$, then $\mu_T \leq r + 1$. Its proof uses the Wold-Kolmogorov Lemma and the Hahn-Banach Theorem. As a corollary of Theorem B, we can say more:

**Corollary 5.7.** Let $\Delta$ be an $n \times r$ inner matrix function and let $T := S^*_C|_{\mathcal{H}(\Delta)}$. Then $\mu_T \leq r + 1$. Further if ker $\Delta^* = \{0\}$ (e.g. $\Delta$ is square-inner), then $\mu_T \leq r$.

**Proof.** This follows from Lemma 3.16, Corollary 5.2, (48) and Theorem B.

6. The case of the Beurling degree 1

In this section, we consider the question (4): For which inner function $\Delta$ with values in $\mathcal{B}(E', E)$, does it follow that

$$\text{deg}_B(\Delta) = 1?$$

and then prove Theorem C. If dim $E' < \infty$, then in the viewpoint of Theorem B, the question (4) is equivalent to the following: if $T$ is the truncated backward shift $S^*_E|_{\mathcal{H}(\Delta)}$, then $\mu_T = 1$, i.e., $T$ has a cyclic vector? To answer the question (4), in section 6.1, we extend the notion of the characteristic scalar inner function to operator-valued inner functions having a meromorphic pseudocontinuation of
bounded type in $D_e$. In Section 6.2, we give an answer to the question (4) and in turn a proof of Theorem C.

6.1. Characteristic scalar inner functions

We first consider the characteristic scalar inner functions of operator-valued inner functions, by using the results of Section 3. The characteristic scalar inner function of a two-sided inner matrix function has been studied in [Hel], [SFBK], [CHL3], etc.

Let $\Delta \in H^\infty (\mathcal{B}(D, E))$ have a meromorphic pseudocontinuation of bounded type in $D_e$. Then by Lemma 3.27, there exists a scalar inner function $\delta$ such that $\delta H^2_E \subseteq \ker H^*$. Put $G := \delta \Delta^* \in H^\infty (\mathcal{B}(E, D))$. If further $\Delta$ is inner then $G \Delta = \delta I_D$, so that

$$\text{g.c.d. } \{ \delta : G \Delta = \delta I_D \text{ for some } G \in H^\infty (\mathcal{B}(E, D)) \}$$

always exists. Thus we may introduce:

**Definition 6.1.** Let $\Delta$ be an inner function with values in $\mathcal{B}(D, E)$. If $\Delta$ has a meromorphic pseudocontinuation of bounded type in $D_e$, define

$$m_{\Delta} := \text{g.c.d. } \{ \delta : G \Delta = \delta I_D \text{ for some } G \in H^\infty (\mathcal{B}(E, D)) \},$$

where $\delta$ is a scalar inner function. The inner function $m_{\Delta}$ is called the characteristic scalar inner function of $\Delta$.

The notion of $m_{\Delta}$ arises in the Sz.-Nagy–Foiaş theory of contraction operators $T$ of class $C_0$ (cf. [Ber], [SFBK], [CHL3]).

If $\Delta$ is an $n \times n$ square inner matrix function then we may write $\Delta \equiv [\theta_{ij} b_{ij}]$, where $\theta_{ij}$ is inner and $\theta_{ij}$ and $b_{ij} \in H^\infty$ are coprime for each $i, j = 1, 2, \ldots, n$. In Lemma 4.12 of [CHL3], it was shown that

$$m_{\Delta} = \text{l.c.m. } \{ \theta_{ij} : i, j = 1, 2, \ldots, n \}.$$ 

In this section, we examine the cases of general inner function that has a meromorphic pseudocontinuation of bounded type in $D_e$.

On the other hand, if $\Phi \in H^\infty (\mathcal{B}(D, E))$ has a meromorphic pseudocontinuation of bounded type in $D_e$, then by Lemma 3.27, $\delta H^2_E \subseteq \ker H^\Phi$. for some scalar inner function $\delta$. Thus we may also define

$$\omega_{\Phi} := \text{g.c.d. } \{ \delta : \delta H^2_E \subseteq \ker H^\Phi \text{ for some scalar inner function } \delta \}.$$ 

If $\Delta$ is an inner function with values in $\mathcal{B}(D, E)$ and has a meromorphic pseudocontinuation of bounded type in $D_e$, then $m_{\Delta}$ is an inner divisor of $\omega_{\Delta}$. If further $\Delta$ is two-sided inner, then

$$\delta H^2_E \subseteq \ker H^\Delta \iff G \equiv \delta \Delta^* \in H^\infty (\mathcal{B}(E)) \iff G \Delta = \Delta G = \delta I_E,$$ 

which implies $m_{\Delta} = \omega_{\Delta}$. We would like to remark that

$$\text{g.c.d. } \{ \delta : G \Delta = \delta I_D \text{ for some } G \in H^\infty (\mathcal{B}(E, D)) \}$$

may exist for some inner function $\Delta$ that has no meromorphic pseudocontinuation of bounded type in $D_e$. To see this, let

$$\Delta := \begin{bmatrix} f \\ g \end{bmatrix}.$$
where $f$ and $g$ are given in Example 3.4. Then $\Delta^*$ is not of bounded type, so that by Corollary 3.31, $\Delta$ has no meromorphic pseudocontinuation of bounded type in $D_e$. On the other hand, since $\Delta$ is inner, by the Complementing Lemma (cf. p. 19), there exists a function $G \in H^\infty_{H_1, z_2}$ such that $G\Delta$ is a scalar inner function, so that (54) exists.

The following lemma shows how to determine $\omega_\Phi$ more easily.

**Lemma 6.2.** Let $D$ and $E$ be separable complex Hilbert spaces and let $\{d_j\}$ and $\{e_i\}$ be orthonormal bases of $D$ and $E$, respectively. Suppose $\Phi \in H^\infty (B(D, E))$ has a meromorphic pseudocontinuation of bounded type in $D_e$. In view of Proposition 3.29, we may write

$$\phi_{ij} = \langle \Phi d_j, e_i \rangle_E = \theta_{ij} \pi_{ij},$$

where $\theta_{ij}$ is inner and $\theta_{ij}$ and $a_{ij} \in H^\infty$ are coprime. Then we have

$$\omega_\Phi = \text{l.c.m.} \{ \theta_{ij} : i, j = 1, 2, \cdots \}.$$

**Proof.** Let $\Phi \in H^\infty (B(D, E))$ have a meromorphic pseudocontinuation of bounded type in $D_e$. By Lemma 3.27, we may write $\Phi = \theta A^*$ (a.e. on $\mathbb{T}$) for some $A \in H^\infty (B(E, D))$ and a scalar inner function $\theta$. Also by an analysis of the proof of Proposition 3.29, we can see that $\theta_0 \equiv \text{l.c.m.} \{ \theta_{ij} : i, j = 1, 2, \cdots \}$ is an inner divisor of $\theta$. Thus by Lemma 3.27, $\theta_0$ is an inner divisor of $\omega_\Phi$. Since $\Phi \in H^\infty (B(D, E))$, it follows that for all $f \in H^2_E$ and $j, n \geq 1$,

$$\langle \Phi(z) d_j, z^n \theta_0(z) f(z) \rangle_E \in L^2.$$  \hfill (55)

On the other hand, for all $f \in H^2_E$,

$$f(z) = \sum_{i \geq 1} \langle f(z), e_i \rangle e_i \equiv \sum_{i \geq 1} f_i(z) e_i$$ \quad \text{for almost all } z \in \mathbb{T} \quad (f_i \in H^2). \hfill (56)

Since $\theta_0 \equiv \text{l.c.m.} \{ \theta_{ij} : i, j = 1, 2, \cdots \}$, it follows from (55) and (56) that for all $j, n \geq 1$,

$$\int_{\mathbb{T}} \langle \Phi(z) d_j, z^n \theta_0(z) f(z) \rangle_E dm(z) = \int_{\mathbb{T}} z^n \sum_{i \geq 1} \overline{f_i(z)} \theta_{ij}(z) \pi_{ij}(z) dm(z) = 0,$$

where the last equality follows from the fact that $z^n \sum_{i \geq 1} \overline{f_i(z)} \theta_{ij}(z) \pi_{ij}(z) \in L^2 \otimes H^2$. Since $\{d_j\}$ is an orthonormal basis for $D$, it follows from the Fatou’s Lemma that for all $x \in D$ and $n = 1, 2, 3, \cdots$,

$$\int_{\mathbb{T}} \langle \Phi(z) x, z^n \theta_0(z) f(z) \rangle_E dm(z) = 0.$$

Thus by Lemma 3.14, $\theta_0 H^2_E \subseteq \ker H^\infty$, so that $\omega_\Phi$ is an inner divisor of $\theta_0$, and therefore $\theta_0 = \omega_\Phi$. This complete the proof. \hfill $\square$

**Corollary 6.3.** Let $\Delta$ be a two-sided inner function and have a meromorphic pseudocontinuation of bounded type in $D_e$. Thus, in view of Proposition 3.29, we may write $\Delta = [\theta_{ij} b_{ij}]$, where $\theta_{ij}$ is an inner function and $\theta_{ij}$ and $b_{ij} \in H^\infty$ are coprime for each $i, j = 1, 2, \cdots$. Then

$$\omega_\Delta = m_\Delta = \text{l.c.m.} \{ \theta_{ij} : i, j = 1, 2, \cdots \}.$$

**Proof.** Immediate from Lemma 6.2. \hfill $\square$
Remark 6.4. If $\Delta$ is not two-sided inner then Corollary 6.3 may fail. To see this, let

$$\Delta := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ z \end{bmatrix}.$$ 

Then by Corollary 3.31, $\Delta$ has a meromorphic pseudocontinuation of bounded type in $D_e$. It thus follows from Lemma 6.2 that $\omega_\Delta = z$. On the other hand, let $G := \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$. Then $G\Delta = 1$, so that $m_{[\Delta, \Delta_c]} = 1 \neq z = \omega_\Delta$. Note that by Corollary 6.3 $[\Delta, \Delta_c] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix}$ and $m_{[\Delta, \Delta_c]} = \omega_{[\Delta, \Delta_c]} = z$.

The following lemma shows that Remark 6.4 is not accidental.

Lemma 6.5. Let $\Delta$ be an inner function and have a meromorphic pseudocontinuation of bounded type in $D_e$. Then $m_{[\Delta, \Delta_c]} = \omega_{[\Delta, \Delta_c]} = \omega_\Delta$ and $\Delta_c$ has a meromorphic pseudocontinuation of bounded type in $D_e$: in this case, $\omega_{\Delta_c}$ is an inner divisor of $\omega_\Delta$.

Proof. Suppose that $\Delta$ is an inner function with values in $B(D, E)$ and has a meromorphic pseudocontinuation of bounded type in $D_e$. Then it follows from Corollary 3.24 and Lemma 3.26 that $[\Delta, \Delta_c]$ is two-sided inner. On the other hand, it follows from Lemma 3.16 that 

$$\ker H_{[\Delta, \Delta_c]}^* = \ker [\Delta, \Delta_c]H_{D_e}^2 = \ker H_{[\Delta, \Delta_c]}^*.$$ 

Thus by Lemma 3.27, $[\Delta, \Delta_c]$ has a meromorphic pseudocontinuation of bounded type in $D_e$ and $m_{[\Delta, \Delta_c]} = \omega_{[\Delta, \Delta_c]} = \omega_\Delta$. This proves the first assertion. Since $[\Delta, \Delta_c]$ has a meromorphic pseudocontinuation of bounded type in $D_e$, it follows from Lemma 3.27 that $\Delta_c$ has a meromorphic pseudocontinuation of bounded type in $D_e$. Since $[(\Delta_c)_e, \Delta_c]$ is a left inner divisor of $[\Delta, \Delta_c]$, it follows from Lemma 3.5 that $\omega_{\Delta_c} = \omega_{[(\Delta_c)_e, \Delta_c]}$ is an inner divisor of $\omega_\Delta = \omega_{[\Delta, \Delta_c]}$. This proves the second assertion.

6.2. Proof of Theorem C

In this section we give an answer to the question (4) and a proof of Theorem C. This is accomplished by several lemmas.

Lemma 6.6. Let $\Phi \in H^\infty(B(D, E))$ have a meromorphic pseudocontinuation of bounded type in $D_e$. Then for each cyclic vector $g$ of $S_D^*$,

$$\ker H_{[\Phi g]}^* = \ker \Phi^*,$$

where $[z \Phi g]^*$ denotes the flip of $[z \Phi g]$.

Proof. Let $\Phi \in H^\infty(B(D, E))$ have a meromorphic pseudocontinuation of bounded type in $D_e$. Then by Lemma 3.27, there exists a scalar inner function $\delta$ such that $\delta H_E^2 \subseteq \ker H_{\Phi}^*$. We thus have

$$\delta \Phi^* h \in H_D^2$$

for any $h \in H_E^2$. (57)
Let $g$ be a cyclic vector of $S_D^*$ and $h \in \ker H_{[z\Phi g]}^*$. Then it follows from Lemma 3.14 that for all $n = 1, 2, 3 \ldots$,

$$0 = \int_T \langle z\Phi(z)g(z), z^n\delta(z)h(z) \rangle_E dm(z)$$
$$= \int_T \langle S_D^{*\ell(n-1)}g(z), \delta(z)\Phi^*(z)h(z) \rangle_D dm(z)$$
$$= \langle S_D^{*\ell(n-1)}g(z), \delta(z)\Phi^*(z)h(z) \rangle_{L_2^E},$$

which implies, by (57), that $\delta\Phi^*h = 0$, and hence $h \in \ker \Phi^*$. We thus have

$$\ker H_{[z\Phi g]}^* \subseteq \ker \Phi^*.$$

The reverse inclusion follows at once from Lemma 3.14. This completes the proof. ∎

**Lemma 6.7.** Let $\Phi \in H^\infty(B(D, E))$ have a meromorphic pseudocontinuation of bounded type in $D_c$. Then for each cyclic vector $g$ of $S_D^*$, we have:

(a) If $\ker \Phi^* = \{0\}$, then $\Phi g$ is a cyclic vector of $S_E^*$;

(b) If $\ker \Phi^* \neq \{0\}$, then $E_{\{\phi g\}}^* = \mathcal{H}((\Phi^i)_c)$, where $\Phi^i$ denotes the inner part of the inner-outer factorization of $\Phi$.

Hence, in particular, if $\ker \Phi^* \neq \{0\}$ and $T := S_E^*|_{\mathcal{H}((\Phi^i)_c)}$, then $T$ has a cyclic vector.

**Proof.** Suppose that $\ker \Phi^* = \{0\}$. By Lemma 6.6, $\ker H_{[z\Phi g]}^* = \{0\}$. It thus follows from Lemma 3.17 that

$$E_{\{\phi g\}}^* = \ell \text{ ran } H_{[z\Phi g]}^* = (\ker H_{[z\Phi g]}^*)^\perp = H_{[z\Phi g]}^*,$$

which implies that $\Phi g$ is a cyclic vector of $S_E^*$. This proves (a). For (b), write

$$\Phi = \Phi^i\Phi^e \quad (\text{inner-outer factorization}),$$

where $\Phi^i$ is an inner function with values in $B(E', E)$ and $\Phi^e$ is an outer function with values in $B(D, E')$. Now we will show that

$$\ker \Phi^* = \ker (\Phi^i)^*.$$

Since $\Phi^* = (\Phi^i)^* (\Phi^e)^*$, it follows that $\ker (\Phi^i)^* \subseteq \ker \Phi^*$. For the reverse inclusion, let $f \in \ker \Phi^*$. If $f \notin \ker (\Phi^i)^*$, then $0 \neq (\Phi^i)^* f \in \ker (\Phi^e)^*$. Thus for all $h \in H_D^*$,

$$((\Phi^i)^*f, \Phi^eh)_{L_2^E} = (\Phi^e f, h)_{L_2^E} = 0,$$

which implies that $(\Phi^i)^* f = 0$, a contradiction. This proves (58). It thus follows from Lemma 3.16, Lemma 3.17 and Lemma 6.6 that

$$E_{\{\phi g\}}^* = (\ker H_{[z\Phi g]}^*)^\perp = (\ker \Phi^*)^\perp = \mathcal{H}((\Phi^i)_c),$$

which proves (b). □

The following corollary is a matrix-valued version of Lemma 6.7.

**Corollary 6.8.** Let $\Delta$ be an $n \times r$ inner matrix function such that $\delta\Delta$ is of bounded type. If $g$ is a cyclic vector of $S_{C^n}^*$, then $E_{\{\Delta g\}}^* = \mathcal{H}(\Delta_c)$. In particular, if $n > r$ and $T := S_{C^n}^*|_{\mathcal{H}(\Delta_c)}$, then $T$ has a cyclic vector.

**Proof.** It follows from Corollary 3.31 and Lemma 6.7. □
In the setting of Corollary 6.8, we may ask whether
\[ \{ \Delta g : g \text{ is a cyclic vector of } S^*_{C^\infty} \} \]
is the set of all cyclic vectors of \( T := S^*_{C^\infty}|_{H(\Delta_c)} \)? The answer to this question is negative. For example, let \( \Delta = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \).

Then \( \Delta_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \) and \( H(\Delta_c) = H^2_{\mathbb{C}^3} \oplus \{0\} \).

Choose a cyclic vector \((f_1, f_2)^t\) of \( S^*_{C^\infty} \) such that \( f_1(0) \neq 0 \). Then \( f = (f_1, f_2, 0)^t \) is a cyclic vector of \( S^*_{C^\infty}|_{H(\Delta_c)} \). However, \( f \neq \Delta q \) for any cyclic vector \( q \) of \( S^*_{C^\infty} \).

The following lemma shows that the flip of the adjoint of an inner function may be an outer function.

**Lemma 6.9.** Let \( \Delta \) be an inner function with values in \( B(D, E) \), with its complementary factor \( \Delta_c \) with values in \( B(D', E) \). If \( \dim D' < \infty \), then \( \Delta_c \) is an outer function.

**Proof.** If \( D' = \{0\} \), then this is trivial. Suppose that \( D' = \mathbb{C}^p \) for some \( p \geq 1 \). Write
\[ \Delta_c \equiv (\Delta_c)^t(\Delta_c)^* \quad \text{(inner-outer factorization),} \tag{59} \]
where \((\Delta_c)^t \in H^\infty_{M_{p \times q}}\) and \((\Delta_c)^* \in H^\infty(B(E, \mathbb{C}^q))\) for some \( q \leq p \). It thus follows that
\[ q \equiv \text{Rank}(\Delta_c)^t \geq \text{Rank} \quad \text{max} \quad \text{Rank}(\Delta_c)(\Delta_c)(\Delta_c)^* = p, \]
which implies \( p = q \). Since \((\Delta_c)^t \in H^\infty_{M_p}\) is inner, by the Complementing Lemma (cf. p. 19), there exists a function \( G \in H^\infty_{M_p} \) and a scalar inner function \( \theta \) such that \( G(\Delta_c)^t = \theta I_p \).

Thus by (59), we have \( G\Delta_c = \theta I_p(\Delta_c)^* \), and hence we have
\[ \hat{\theta} I_E \Delta_c \hat{G} = \theta I_p G^\sim \Delta_c = (\Delta_c)^* \in H^\infty(B(\mathbb{C}^p, E)). \]

Thus by Corollary 3.10, we have
\[ \hat{\theta} I_E \Delta_c \hat{G} H^2_{\mathbb{C}^p} \subseteq H^2_E. \tag{60} \]

It thus follows from Lemma 3.16 and (60) that
\[ \Delta_c \hat{\theta} I_E \Delta_c \hat{G} H^2_{\mathbb{C}^p} = \hat{\theta} I_E \Delta_c \hat{G} H^2_{\mathbb{C}^p} \subseteq \ker \Delta^* = \Delta_c H^2_{\mathbb{C}^p}, \]
which implies \( \hat{\theta} I_p \hat{G} H^2_{\mathbb{C}^p} \subseteq H^2_{\mathbb{C}^p} \). We thus have \( \hat{\theta} I_p \hat{G} \in H^\infty_{M_p} \), so that \( \overline{\theta} I_p \hat{G} \in H^\infty_{M_p} \). Therefore we may write \( G = \theta I_p G_1 \) for some \( G_1 \in H^\infty_{M_p} \). It thus follows that
\[ \theta I_p = G(\Delta_c)^t = \theta I_p G_1(\Delta_c)^t, \]
which gives that \( G_1(\Delta_c)^t = I_p \). Therefore, by Corollary 3.10, we have
\[ H^2_{\mathbb{C}^p} = (\Delta_c)^t G_1 H^2_{\mathbb{C}^p} \subseteq (\Delta_c)^t H^2_{\mathbb{C}^p}, \tag{61} \]
which implies that \((\Delta_c)^t\) is a unitary matrix, and so is \((\Delta_c)^t\). Thus, \( \Delta_c \) is an outer function. This completes the proof. □
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**Corollary 6.10.** If $\Delta$ is an inner matrix function, then $\Delta'_c$ is an outer function.

*Proof. Immediate from Lemma 6.9. □*

**Remark 6.11.** Let $T := S^*_c \mid_{H(\Delta)}$ for some non-square inner matrix function $\Delta$. Then Corollary 6.8 shows that if $\Delta = \Omega_c$ for an inner matrix function $\Omega$ such that $\hat{\Omega}$ is of bounded type, then $\mu_T = 1$. However, the converse is not true in general, i.e., the condition “$\mu_T = 1$” does not guarantee that $\Delta = \Omega_c$ for any inner matrix function. Let $f := (a \mathbf{1})^t$ ($\mathbf{a}$ is not of bounded type). Then $E^*_f = H(\Delta)$, so that $\mu_T = 1$.

**Lemma 6.12.** Let $\Delta$ be an inner function and have a meromorphic pseudocontinuation of bounded type in $D_e$. If $\hat{\Delta}$ is an outer function, then $\Delta_c = \Delta$.

*Proof. Let $\Delta$ be an inner function with values in $B(D, E)$ and have a meromorphic pseudocontinuation of bounded type in $D_e$. Then by Lemma 3.26, $\hat{\Delta}$ is of bounded type. Suppose that $\hat{\Delta}$ is an outer function and $\ker \Delta^* = \{0\}$. Then by Lemma 3.16 and Corollary 3.24, $\Delta$ is two-sided inner, and so is $\Delta$. Thus $\Delta$ is a unitary operator □*

For iterated complementary factors of an inner function $\Delta$, we write

$\Delta_{cc} \equiv (\Delta_c)^c, \Delta_{ccc} \equiv (\Delta_{cc})^c, \ldots$, etc.

The following lemma is a key idea for an answer to the question (4).

**Lemma 6.13.** Let $\Delta$ be an inner function and have a meromorphic pseudocontinuation of bounded type in $D_e$. If $\hat{\Delta}$ is an outer function, then $\Delta_c = \Delta$.

*Proof. Let $\Delta$ be an inner function with values in $B(D, E)$ and have a meromorphic pseudocontinuation of bounded type in $D_e$. Also suppose $\hat{\Delta}$ is an outer function. If $\ker \Delta^* = \{0\}$, then the result follows at one from Lemma 6.12. Suppose that $\ker \Delta^* \neq \{0\}$. By Lemma 3.26, $\hat{\Delta}$ is of bounded type, so that by Corollary 3.24, $[\Delta, \Delta_c]$ is a two-sided inner function with values in $B(D \oplus D', E)$ for some nonzero Hilbert space $D'$. We now claim that

$\Delta = \Delta_{cc} \Omega$ (62)

for a two-sided inner function $\Omega$ with values in $B(D)$.

Since $\Delta^* \Delta = 0$, it follows that $\Delta H_{D'}^2 \subseteq \ker \Delta^*$, and therefore $\Delta H_{D'}^2 \subseteq \Delta_{cc} H_{D'}^2$ for some nonzero Hilbert space $D''$ by Lemma 3.16(a), which implies that $\Delta_{cc}$ is a left inner divisor of $\Delta$. Thus we may write

$\Delta = \Delta_{cc} \Omega$ (63)

for an inner function $\Omega$ with values in $B(D, D'')$. Assume to the contrary that $\Omega$ is not two-sided inner. Since $\Delta$ has a meromorphic pseudocontinuation of bounded type in $D_e$, it follows from Lemma 3.27 that

$\theta H_{D'}^2 \subseteq \ker H_{\Delta^*} = \ker H_{\Omega^* \Delta_{cc}^*}$

for some scalar inner function $\theta$. Thus $\Omega^* \Delta_{cc}^* \theta H_{D'}^2 \subseteq H_{D'}^2$. In particular, we have

$\Omega^* \theta H_{D''}^2 \subseteq \Omega^* \Delta_{cc}^* \theta \Delta_{cc} H_{D''}^2 \subseteq H_{D'}^2$. 
and hence $\theta H^2_{D''} \subseteq \ker H^\ast$, which implies, by Lemma 3.27, that $\Omega$ has a meromorphic pseudostretch of bounded type in $D_e$. Thus by Lemma 3.26, $\Omega$ is of bounded type. It thus follows from Corollary 3.24 that

$$[\Omega, \Omega_c]$$

is two-sided inner, where $\Omega_c$ is an inner function, with values in $B(D_1, D'')$ for some nonzero Hilbert space $D_1$. On the other hand, for all $f \in H^2_{D_1}$,

$$[\Delta, \Delta_c]^\ast \Delta_c \Omega_c f = \begin{bmatrix} \Omega^\ast \Omega_c f \\ \Delta^c_c \Delta \Omega_c f \end{bmatrix} = 0,$$

which implies that $D_1 = \{0\}$, a contradiction. This proves (62). Thus we may write

$$\tilde{\Delta} = \tilde{\Omega} \tilde{\Delta} \tilde{\Omega}$$

(64)

for a two-sided inner function $\tilde{\Omega}$ with values in $B(D)$. Since $\tilde{\Delta}$ is an outer function and $\tilde{\Omega}$ is two-sided inner, it follows from (64) that $\tilde{\Omega}$ is a unitary operator, and so is $\Omega$. This completes the proof. □

Lemma 6.13 may fail if the condition “$\Delta$ has a meromorphic pseudostretch of bounded type in $D_e$” is dropped. To see this, let

$$\Delta := \begin{bmatrix} f \\ g \\ 0 \end{bmatrix},$$

where $f$ and $g$ are given in Example 3.4. Then $\tilde{\Delta}$ is an outer function. Observe that

$$\Delta_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \Delta_{cc} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \Delta.$$ 

Note that $\tilde{\Delta}$ is not of bounded type. Thus, by Corollary 3.31, $\Delta$ has no meromorphic pseudostretch of bounded type in $D_e$.

We are ready for:

**Proof of Theorem C.** Suppose that $\Delta$ is an inner function with values in $B(D, E)$ and has a meromorphic pseudostretch of bounded type in $D_e$. Also suppose $\tilde{\Delta}$ is an outer function. Let $T := S_{D_1}^\ast |_{H(D)}$. If $\ker \Delta^\ast = \{0\}$, then by Lemma 6.12, $\Delta$ is a unitary operator, so that evidently, $\deg_B(\Delta) = 1$. If instead $\ker \Delta^\ast \neq \{0\}$, then by Lemma 6.5, $\Delta_c$ has a meromorphic pseudostretch of bounded type in $D_e$. Since $\tilde{\Delta}$ is an outer function it follows from Lemma 6.13 that $\Delta = \Delta_{cc}$. Applying Lemma 6.7 with $\Phi \equiv \Delta_c$, we can see that $T$ has a cyclic vector, i.e., $\mu_T = 1$. Thus by Remark 5.6 (a), $\deg_B(\Delta) \leq 1$. But since by definition, $\deg_B(\Delta) \geq 1$, it follows that $\deg_B(\Delta) = 1$. □

The following corollary is an immediate result of Theorem C.

**Corollary 6.14.** Let $\Delta$ be an inner matrix function whose flip $\Delta$ is of bounded type. If $\Delta^t$ is an outer function, then $\deg_B(\Delta) = 1$.

**Proof.** This follows from Theorem C and Corollary 3.31. □
If $\Delta$ is an inner matrix function then the converse of Lemma 6.13 is also true.

**Corollary 6.15.** Let $\Delta$ be an inner matrix function whose flip $\tilde{\Delta}$ is of bounded type. Then the following are equivalent:

(a) $\Delta'$ is an outer function;
(b) $\tilde{\Delta}$ is an outer function;
(c) $\Delta_{cc} = \Delta$;
(d) $\Delta = \Omega_c$ for some inner matrix function $\Omega$.

Hence, in particular, $\Delta_{ccc} = \Delta_c$.

**Proof.** The implication $(a) \Rightarrow (b)$ is clear and the implication $(b) \Rightarrow (c)$ follows from Corollary 3.31 and Lemma 6.13. Also the implication $(c) \Rightarrow (d)$ is clear and the implication $(d) \Rightarrow (a)$ follows from Corollary 6.10. The second assertion follows from the first assertion together with Corollary 6.10. □

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