EXTENSIONS AND EXTREMALITY OF RECURSIVELY GENERATED WEIGHTED SHIFTS

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Abstract. Given an n-step extension $\alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$ of a recursively generated weight sequence $(0 < \alpha_0 < \cdots < \alpha_k)$, and if W_α denotes the associated unilateral weighted shift, we prove that

$$W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } ([\frac{k+1}{2}]+1)\text{-hyponormal} & (n=1) \\ W_\alpha \text{ is } ([\frac{k+1}{2}]+2)\text{-hyponormal} & (n>1). \end{cases}$$

In particular, the subnormality of an extension of a recursively generated weighted shift is independent of its length if the length is bigger than 1. As a consequence we see that if $\alpha(x)$ is a canonical rank-one perturbation of the recursive weight sequence α, then subnormality and k-hyponormality for $W_{\alpha(x)}$ eventually coincide. We then examine a converse - an "extremality" problem: Let $\alpha(x)$ be a canonical rank-one perturbation of a weight sequence α and assume that $(k+1)$-hyponormality and k-hyponormality for $W_{\alpha(x)}$ coincide. We then show that $\alpha(x)$ is recursively generated, i.e., $W_{\alpha(x)}$ is recursive subnormal.

Introduction

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$ and *hyponormal* if $T^*T \geq TT^*$. Given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+)$. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bram-Halmos criterion for subnormality states that an operator T is *subnormal* if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([**Br**],[**Con**, III.1.9]). Using Choleski's algorithm for operator matrices, it is easy to see that this is equivalent to the following positivity test:

$$(0.1) \qquad \begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1).$$

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Condition (0.1) provides a measure of the gap between hyponormality and subnormality, and k-hyponormality has been introduced and studied in an attempt to bridge the gap between subnormality and hyponormality ([**At**],[**Cu1**],[**Cu2**],[**CF1**], [**CF2**],[**CF3**],[**CL1**],[**CMX**], [**McCP**]). In fact, the positivity condition (0.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (0.1) for all k. If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B, and if we define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{0.2}$$

is positive, or equivalently, the $(k+1)\times(k+1)$ operator matrix in (0.1) is positive, then the Bram-Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([**CMX**]).

If W_α is the weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, then the *moments* of W_α are usually defined by $\beta_0 := 1$, $\beta_{n+1} := \alpha_n \beta_n$ $(n \geq 0)$ [**Shi**]; however, we reserve this term for the sequence $\gamma_n := \beta_n^2$ $(n \geq 0)$. A criterion for k-hyponormality can be given in terms of moments ([**Cu1**, Theorem 4]): if we build a $(k+1)\times(k+1)$ Hankel matrix $A(n;k)$ by

$$A(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0), \tag{0.3}$$

then

$$W_\alpha \text{ is } k\text{-hyponormal} \iff A(n;k) \geq 0 \quad (n \geq 0). \tag{0.4}$$

In [**Sta**], J. Stampfli showed that given $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}$ with $0 < a < b < c$, there always exists a subnormal completion of α, but that for $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}$ $(a < b < c < d)$ such a subnormal completion may not exist.

There are instances where k-hyponormality implies subnormality for weighted shifts. For example, in [**CF3**], it was shown that if $\alpha(x) : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ $(a < b < c)$ then $W_{\alpha(x)}$ is 2-hyponormal if and only if it is subnormal: more concretely, $W_{\alpha(x)}$ is 2-hyponormal if and only if

$$\sqrt{x} \leq H_2(\sqrt{a}, \sqrt{b}, \sqrt{c}) := \sqrt{\frac{ab(c-b)}{(b-a)^2 + b(c-b)}},$$

in which case $W_{\alpha(x)}$ is subnormal. In this paper we extend the above result to weight sequences of the form $\alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$ with $0 < \alpha_0 < \cdots < \alpha_k$. Our main results are as follows.

Extensions of Recursively Generated Weighted Shifts. *If $\alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$ then*

$$W_\alpha \textit{ is subnormal} \iff \begin{cases} W_\alpha \textit{ is } ([\frac{k+1}{2}]+1)\textit{-hyponormal} & (n = 1) \\ W_\alpha \textit{ is } ([\frac{k+1}{2}]+2)\textit{-hyponormal} & (n > 1). \end{cases}$$

In particular, the above theorem shows that the subnormality of an extension of the recursive shift is independent of its length if the length is bigger than 1.

Canonical Rank–One Perturbations. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty = (\alpha_0, \cdots, \alpha_k)^\wedge$. If $W_{\alpha'}$ is a perturbation of W_α at the j-th weight then*

$$W_{\alpha'} \textit{ is subnormal} \iff \begin{cases} W_{\alpha'} \textit{ is } ([\frac{k+1}{2}]+1)\textit{-hyponormal} & (j = 0) \\ W_{\alpha'} \textit{ is } ([\frac{k+1}{2}]+2)\textit{-hyponormal} & (j \geq 1). \end{cases}$$

Extremality Criterion. *Let $\alpha(x)$ be a canonical rank-one perturbation of a weight sequence α. If $(k+1)$-hyponormality and k-hyponormality for $W_{\alpha(x)}$ coincide, then $\alpha(x)$ is recursively generated, i.e., $W_{\alpha(x)}$ is recursive subnormal.*

1. Extensions of Recursively Generated Shifts

C. Berger's characterization of subnormality for unilateral weighted shifts (cf. [**Hal**], [**Con**, III.8.16]) states that W_α is subnormal if and only if there exists a Borel probability measure μ supports in $[0, ||W_\alpha||^2]$, with $||W_\alpha||^2 \in \operatorname{supp}\mu$, such that

$$\gamma_n = \int t^n d\mu(t) \quad \text{for all } n \geq 0.$$

Given an initial segment of weights $\alpha : \alpha_0, \cdots \alpha_r$, the sequence $\hat{\alpha} \in \ell^\infty(\mathbb{Z}_+)$ such that $\hat{\alpha}_i = \alpha_i$ $(i = 0, \cdots, r)$ is said to be *recursively generated* by α if there exist $r \geq 1$ and $\varphi_0, \cdots, \varphi_{r-1} \in \mathbb{R}$ such that

$$\gamma_{n+r} = \varphi_0\gamma_n + \cdots + \varphi_{r-1}\gamma_{n+r-1} \quad \text{for all } n \geq 0,$$

where $\gamma_0 := 1$, $\gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2$ $(n \geq 1)$. In this case the weighted shift $W_{\hat{\alpha}}$ with a weight sequence $\hat{\alpha}$ is said to be *recursively generated* (or simply *recursive*). If

$$g(t) := t^r - \left(\varphi_{r-1}t^{r-1} + \cdots + \varphi_0\right),$$

then g has r distinct real roots $0 \leq s_0 < \cdots < s_{r-1}$ ([**CF2**, Theorem 3.9]). Let

$$V := \begin{pmatrix} 1 & 1 & \dots & 1 \\ s_0 & s_1 & \dots & s_{r-1} \\ \vdots & \vdots & & \vdots \\ s_0^{r-1} & s_1^{r-1} & \dots & s_{r-1}^{r-1} \end{pmatrix}$$

and let

$$\begin{pmatrix} \rho_0 \\ \vdots \\ \rho_{r-1} \end{pmatrix} := V^{-1} \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{r-1} \end{pmatrix}.$$

If $W_{\hat{\alpha}}$ is a recursively generated subnormal shift then the Berger measure of $W_{\hat{\alpha}}$ is of the form

$$\mu := \rho_0\delta_{s_0} + \cdots + \rho_{r-1}\delta_{r-1}.$$

Given an initial segment of weights

$$\alpha : \ \alpha_0, \cdots, \alpha_{2k} \quad (k \geq 0),$$

suppose $\hat{\alpha} \equiv (\alpha_0, \cdots, \alpha_{2k})^\wedge$, i.e., $\hat{\alpha}$ is recursively generated by α. Write

$$\mathbf{v}_n := \begin{pmatrix} \gamma_n \\ \vdots \\ \gamma_{n+k} \end{pmatrix} \qquad (0 \leq n \leq k+1).$$

Then $\{\mathbf{v}_0, \cdots, \mathbf{v}_{k+1}\}$ is linearly dependent in $\mathbb{R}^{k+1}$. Now the *rank* of α is defined by the smallest integer i $(1 \leq i \leq k+1)$ such that $\mathbf{v}_i$ is a linear combination of $\mathbf{v}_0, \cdots, \mathbf{v}_{i-1}$. Since $\{\mathbf{v}_0, \cdots, \mathbf{v}_{i-1}\}$ is linearly independent, there exists a unique i-tuple $\varphi \equiv (\varphi_0, \cdots, \varphi_{i-1}) \in \mathbb{R}^i$ such that $\mathbf{v}_i = \varphi_0\mathbf{v}_0 + \cdots + \varphi_{i-1}\mathbf{v}_{i-1}$, or equivalently,

$$\gamma_j = \varphi_{i-1}\gamma_{j-1} + \cdots + \varphi_0\gamma_{j-i} \quad (i \leq j \leq k+i),$$

which says that $(\alpha_0, \cdots, \alpha_{k+i})$ is recursively generated by $(\alpha_0, \cdots, \alpha_i)$. In this case, W_α is said to be *i-recursive* (cf. [**CF3**, Definition 5.14]).

We begin with:

Lemma 1.1 ([CF2, Propositions 2.3, 2.6, and 2.7]). *Let $A, B \in M_n(\mathbb{C})$, $\tilde{A}, \tilde{B} \in M_{n+1}(\mathbb{C})$ $(n \geq 1)$ be such that*

$$\tilde{A} = \begin{pmatrix} A & * \\ * & * \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} * & * \\ * & B \end{pmatrix}.$$

Then we have:

(i) *If $A \geq 0$ and if $\tilde{A}$ is a flat extension of A (i.e., $\operatorname{rank}(\tilde{A}) = \operatorname{rank}(A)$) then $\tilde{A} \geq 0$;*
(ii) *If $A \geq 0$ and $\tilde{A} \geq 0$ then $\det(A) = 0$ implies $\det(\tilde{A}) = 0$;*
(iii) *If $B \geq 0$ and $\tilde{B} \geq 0$ then $\det(B) = 0$ implies $\det(\tilde{B}) = 0$.*

Lemma 1.2. *If $\alpha \equiv (\alpha_0, \cdots, \alpha_k)^\wedge$ then*

$$W_\alpha \text{ is subnormal} \iff W_\alpha \text{ is } ([\frac{k}{2}] + 1)\text{-hyponormal.} \tag{1.2.1}$$

In the cases where W_α is subnormal and $i := \operatorname{rank}(\alpha)$, we have $\alpha = (\alpha_0, \cdots, \alpha_{2i-2})^\wedge$.

Proof. We only need to establish the sufficiency condition in (1.2.1). Let $i := \operatorname{rank}(\alpha)$. Since W_α is i-recursive, [**CF3**, Proposition 5.15] implies that the subnormality of W_α follows after we verify that $A(0, i-1) \geq 0$ and $A(1, i-1) \geq 0$. Now observe that $i - 1 \leq [\frac{k}{2}] + 1$ and

$$A(j, [\frac{k}{2}] + 1) = \begin{pmatrix} A(j, i-1) & * \\ * & * \end{pmatrix} \quad (j = 0, 1),$$

so the positivity of $A(0, i-1)$ and $A(1, i-1)$ is a consequence of the positivity of the $([\frac{k}{2}] + 1)$-hyponormality of W_α. For the second assertion, observe that $\det A(n, i) = 0$ for all $n \geq 0$. By assumption $A(n, i+1) \geq 0$, so by Lemma 1.1 (ii) we have $\det A(n, i+1) = 0$, which says that $(\alpha_0, \cdots, \alpha_{2i-1}) \subset (\alpha_0, \cdots, \alpha_{2i-2})^\wedge$. By iteration we obtain $(\alpha_0, \cdots, \alpha_k) \subset (\alpha_0, \cdots, \alpha_{2i-2})^\wedge$, and therefore $(\alpha_0, \cdots, \alpha_k)^\wedge = (\alpha_0, \cdots, \alpha_{2i-2})^\wedge$. This proves the lemma. $\square$

In what follows, and for notational convenience, we shall set $x_{-j} := \alpha_j$ $(0 \leq j \leq k)$.

Theorem 1.3 (Subnormality Criterion). *If $\alpha : x_n, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$ then*

$$W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } ([\frac{k+1}{2}] + 1)\text{-hyponormal} & (n = 1) \\ W_\alpha \text{ is } ([\frac{k+1}{2}] + 2)\text{-hyponormal} & (n > 1). \end{cases} \tag{1.3.1}$$

Furthermore, in the cases where the above equivalence holds, if $\operatorname{rank}(\alpha_0, \cdots, \alpha_k) = i$ then

$$W_\alpha \text{ is subnormal} \iff \begin{cases} W_\alpha \text{ is } i\text{-hyponormal} & (n = 1) \\ W_\alpha \text{ is } (i+1)\text{-hyponormal} & (n > 1). \end{cases} \tag{1.3.2}$$

In fact,

$$\begin{cases} x_1 = H_i(x_0, \cdots, x_{2-2i}) \\ x_2 = H_i(x_1, \cdots, x_{3-2i}) \\ \qquad \cdots\cdots\cdots \\ x_{n-1} = H_i(x_{n-2}, \cdots, x_{n-2i}) \\ x_n \leq H_i(x_{n-1}, \cdots, x_{n-2i+1}), \end{cases}$$

where H_i is the modulus of i-hyponormality (cf.[**CF3**, Proposition 3.4 and (3.4)]), *i.e.,*

$$H_i(\alpha) := \sup\{x > 0 : \ W_{x\alpha} \text{ is } i\text{-hyponormal}\}.$$

Therefore, $W_\alpha = W_{x_n(x_{n-1},\cdots,x_{n-2i+1})^\wedge}$.

Proof. Consider the $(k+1)\times(l+1)$ "Hankel" matrix $A(n;k,l)$ by (cf. [**CL1**])

$$A(n;k,l) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+l} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+1+l} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+k+l} \end{pmatrix} \quad (n \ge 0).$$

Case 1 ($\alpha : x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$): Let $\hat{A}(n;k,l)$ and $A(n;k,l)$ denote the Hankel matrices corresponding to the weight sequences $(\alpha_0,\cdots,\alpha_k)^\wedge$ and α, respectively. Suppose W_α is $([\frac{k+1}{2}]+1)$-hyponormal. Then by Lemma 1.2, $W_{(\alpha_0,\cdots,\alpha_k)^\wedge}$ is subnormal. Observe that

$$A(n+1;m,m) = x_1^2\,\hat{A}(n;m,m) \quad \text{for all } n \ge 0 \text{ and all } m \ge 0.$$

Thus it suffices to show that $A(0;m,m)\ge 0$ for all $m \ge [\frac{k+1}{2}]+2$. Also observe that if $\tilde{B}$ denotes the $(k-1)\times k$ matrix obtained by eliminating the first row of a $k\times k$ matrix B then

$$\tilde{A}(0;m,m) = x_1^2\,\hat{A}(0;m-1,m) \quad \text{for all } m \ge [\frac{k+1}{2}]+2.$$

Therefore, for every $m \ge [\frac{k+1}{2}]+2$, $A(0;m,m)$ is a flat extension of $A(0;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1)$. This implies $A(0;m,m)\ge 0$ for all $m\ge[\frac{k+1}{2}]+2$ and therefore W_α is subnormal.

Case 2 ($\alpha : x_n,\cdots,x_1,(\alpha_0,\cdots,\alpha_k)^\wedge)$): As in Case 1, let $\hat{A}(n;k,l)$ and $A(n;k,l)$ denote the Hankel matrices corresponding to the weight sequences $(\alpha_0,\cdots,\alpha_k)^\wedge$ and α, respectively. Observe that $\det\hat{A}(n;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1)=0$ for all $n\ge 0$. Suppose W_α is $([\frac{k+1}{2}]+2)$-hyponormal. Observe that

$$A(n+1;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1) = x_1^2\cdots x_n^2\hat{A}(1;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1),$$

so that

$$\det A(n+1;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1) = 0. \tag{1.3.3}$$

Also observe that

$$A(n-1;[\frac{k+1}{2}]+2,[\frac{k+1}{2}]+2) = \begin{pmatrix} x_2^2\cdots x_n^2 & * \\ * & A(n+1;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1) \end{pmatrix}.$$

Since W_α is $([\frac{k+1}{2}]+1)$-hyponormal, it follows from Lemma 1.1 (iii) and (1.3.3) that $\det A(n-1;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1)=0$. Note that

$$A(n-1;[\frac{k+1}{2}]+1,[\frac{k+1}{2}]+1) = x_1^2\cdots x_n^2 \begin{pmatrix} \frac{1}{x_1^2} & \hat{\gamma}_0 & \cdots & \hat{\gamma}_{[\frac{k+1}{2}]+1} \\ \hat{\gamma}_0 & \hat{\gamma}_1 & \cdots & \hat{\gamma}_{[\frac{k+1}{2}]+2} \\ \vdots & \vdots & & \vdots \\ \hat{\gamma}_{[\frac{k+1}{2}]+1} & \hat{\gamma}_{[\frac{k+1}{2}]+2} & \cdots & \hat{\gamma}_{2[\frac{k+1}{2}]+2} \end{pmatrix},$$

where $\hat{\gamma}_j$ denotes the moments corresponding to the weight sequence $(\alpha_0,\cdots,\alpha_k)^\wedge$. Therefore x_1 is determined uniquely by $\{\alpha_0,\cdots,\alpha_k\}$ such that $(x_1,\alpha_0,\cdots,\alpha_{k-1})^\wedge = x_1,(\alpha_0,\cdots,\alpha_k)^\wedge$: more precisely, if $i:=\operatorname{rank}(\alpha)$ and $\varphi_0,\cdots,\varphi_{i-1}$ denote the coefficients of recursion in $(\alpha_0,\cdots,\alpha_k)^\wedge$ then

$$x_1 = H_i[(\alpha_0,\cdots,\alpha_k)^\wedge] = \left[\frac{\varphi_0}{\hat{\gamma}_{i-1}-\varphi_{i-1}\hat{\gamma}_{i-2}-\cdots-\varphi_1\hat{\gamma}_0}\right]^{\frac{1}{2}}$$

(cf. [**CF3**,(3.4)]). Continuing this process we can see that $x_1, \cdots, x_{n-1}$ are determined uniquely by a telescoping method such that

$$(x_{n-1}, \cdots, x_{n-1-k})^\wedge = x_{n-1}, \cdots, x_1, (\alpha_0, \cdots, \alpha_k)^\wedge$$

and $W_{(x_{n-1},\cdots,x_{n-1-k})^\wedge}$ is subnormal. Therefore, after $(n-1)$ steps, Case 2 reduces to Case 1. This completes the proof of the first assertion. For the second assertion, note that if $\text{rank}(\alpha_0, \cdots, \alpha_k) = i$ then

$$\det \hat{A}(n; i, i) = 0.$$

Now applying the above argument with i in place of $[\frac{k+1}{2}]+1$ gives that $x_1, \cdots, x_{n-1}$ are determined uniquely by $\alpha_0, \cdots, \alpha_{2i-2}$ such that $W_{(x_{n-1},\cdots,x_{n-2i-1})^\wedge}$ is subnormal. Thus the second assertion immediately follows. Finally, observe that the preceding argument also establish the remaining assertions. □

Remark 1.4. (a) From Theorem 1.3 we note that the subnormality of an extension of a recursive shift is independent of its length if the length is bigger than 1.

(b) In Theorem 1.3, "$[\frac{k+1}{2}]$" can not be relaxed to "$[\frac{k}{2}]$". For example consider the following weight sequences:

(i) $\alpha : \sqrt{\frac{1}{2}}, (\sqrt{\frac{3}{2}}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$ with $\varphi_0 = 0$;

(ii) $\alpha' : \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge$.

Observe that α equals α'. Then a straightforward calculation shows that W_α (and hence $W_{\alpha'}$) is 2-hyponormal but not 3-hyponormal (and hence, not subnormal). Note that $k = 3$ and $n = 1$ in (i) and $k = 2$ and $n = 2$ in (ii).

(c) Note that the second assertion of Theorem 1.3 does *not* imply that if $\text{rank}(\alpha_0, \cdots, \alpha_k) = i$ then (1.3.2) holds in general. Theorem 1.3 says only that when W_α is $([\frac{k+1}{2}]+1)$-hyponormal $(n = 1)$, i-hyponormality and subnormality coincide, and that when W_α is $([\frac{k+1}{2}]+2)$-hyponormal $(n > 1)$, $(i+1)$-hyponormality and subnormality coincide. For example consider the weight sequence

$$\hat{\alpha} \equiv (\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}, 2)^\wedge \quad \text{with } \varphi_0 = 0 \text{ (here } \varphi_1 = 0 \text{ also)}.$$

Since $(\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}) \subset (\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}})^\wedge$, we can see that $\text{rank}(\alpha) = 2$. Put

$$\beta \equiv 1, (\sqrt{2}, \sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}}, 2)^\wedge.$$

If (1.3.2) held true without assuming (1.3.1), then 2-hyponormality would imply subnormality for W_β. However, a straightforward calculation shows that W_β is 2-hyponormal but not 3-hyponormal (and hence not subnormal): in fact, $\det A(n, 2) = 0$ for all $n \geq 0$ except for $n = 2$ and $\det A(2, 2) = 160 > 0$, while since

$$\varphi_3 = -\frac{\alpha_3^2\alpha_4^2(\alpha_5^2 - \alpha_4^2)}{\alpha_4^2 - \alpha_3^2} = -102 \quad \text{and} \quad \varphi_4 = \frac{\alpha_4^2(\alpha_5^2 - \alpha_3^2)}{\alpha_4^2 - \alpha_3^2} = 34$$

(so that $\alpha_6 = \sqrt{\varphi_4 - \frac{\varphi_3}{\alpha_5^2}} = \sqrt{\frac{17}{2}}$), we have that

$$\det A(1,3) = \det \begin{pmatrix} 1 & 2 & 6 & 20 \\ 2 & 6 & 20 & 68 \\ 6 & 20 & 68 & 272 \\ 20 & 68 & 272 & 2312 \end{pmatrix} = -3200 < 0.$$

(d) On the other hand, Theorem 1.3 does show that if $\alpha \equiv (\alpha_0, \cdots, \alpha_k)$ is such that rank$(\alpha) = i$ and $W_{\hat{\alpha}}$ is subnormal with associated Berger measure μ, then $W_{\hat{\alpha}}$ has an n-step $(i+1)$-hyponormal extension $W_{x_n,\cdots,x_1,\hat{\alpha}}$ $(n \geq 2)$ if and only if $\frac{1}{t^n} \in L^1(\mu)$,

$$x_{j+1} = \left[\frac{\varphi_0}{\gamma_{i-1}^{(j)} - \varphi_{i-1}\gamma_{i-2}^{(j)} - \cdots - \varphi_1 \gamma_0^{(j)}} \right]^{\frac{1}{2}} \qquad (0 \leq j \leq n-2),$$

and

$$x_n \leq \left[\frac{\varphi_0}{\gamma_{i-1}^{(n-1)} - \varphi_{i-1}\gamma_{i-2}^{(n-1)} - \cdots - \varphi_1 \gamma_0^{(n-1)}} \right]^{\frac{1}{2}},$$

where $\varphi_0, \cdots, \varphi_{i-1}$ denote the coefficients of recursion in $(\alpha_0, \cdots, \alpha_{2i-2})^\wedge$ and $\gamma_m^{(j)}$ $(0 \leq m \leq i-1)$ are the moments corresponding to the weight sequence $(x_j, \cdots, x_1, \alpha_0, \cdots, \alpha_{k-j})^\wedge$ with $\gamma_m^{(0)} = \gamma_m$.

We now observe that the determination of k-hyponormality and subnormality for canonical rank-one perturbations of recursive shifts falls within the scope of the theory of extensions.

Corollary 1.5. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty} = (\alpha_0, \cdots, \alpha_k)^\wedge$. If $W_{\alpha'}$ is a perturbation of W_α at the j-th weight then*

$$W_{\alpha'} \text{ is subnormal } \Longleftrightarrow \begin{cases} W_{\alpha'} \text{ is } ([\frac{k+1}{2}]+1)\text{-hyponormal} & (j=0) \\ W_{\alpha'} \text{ is } ([\frac{k+1}{2}]+2)\text{-hyponormal} & (j \geq 1). \end{cases}$$

Proof. Observe that if $j = 0$ then $\alpha' = x, (\alpha_1, \cdots, \alpha_{k+1})^\wedge$ and if instead $j \geq 1$ then $\alpha' = \alpha_0, \cdots, \alpha_{j-1}, x, (\alpha_{j+1}, \cdots, \alpha_{j+k+1})^\wedge$. Thus the result immediately follows from Theorem 1.3. □

2. Extremality of Recursively Generated Shifts

In Corollary 1.5, we showed that if $\alpha(x)$ is a canonical rank-one perturbation of a recursive weight sequence then subnormality and k-hyponormality for the corresponding shift eventually coincide. In this section we consider a converse.

Problem 2.1 (Extremality Problem). *Let $\alpha(x)$ be a canonical rank-one perturbation of a weight sequence α. If there exists $k \geq 1$ such that $(k+1)$-hyponormality and k-hyponormality for the corresponding shift $W_{\alpha(x)}$ coincide, does it follow that $\alpha(x)$ is recursively generated?*

In [**CF3**], the following extremality criterion was established.

Lemma 2.2 (Extremality Criterion)[CF3; Theorem 5.12, Proposition 5.13]. *Let α be a weight sequence and let $k \geq 1$.*

(i) *If W_α is k-extremal (i.e., $\det A(j,k) = 0$ for all $j \geq 0$) then W_α is recursive subnormal.*
(ii) *If W_α is k-hyponormal and if $\det A(i_0, j_0) = 0$ for some $i_0 \geq 0$ and some $j_0 < k$ then W_α is recursive subnormal.*

In particular, Lemma 2.2 (ii) shows that if W_α is subnormal and if $\det A(i_0, j_0) = 0$ for some $i \geq 0$ and some $j \geq 0$ then W_α is recursive subnormal.

We now answer Problem 2.1 affirmatively.

Theorem 2.3. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence and let $\alpha_j(x)$ be a canonical perturbation of α in the j-th weight. Write*

$$\mathfrak{H}_k := \{x \in \mathbb{R}^+ : \ W_{\alpha_j(x)} \text{ is } k\text{-hyponormal}\}.$$

If $\mathfrak{H}_k = \mathfrak{H}_{k+1}$ for some $k \geq 1$, and if $x \in \mathfrak{H}_k$, then $\alpha_j(x)$ is recursively generated, i.e., $W_{\alpha_j(x)}$ is recursive subnormal.

Proof. Suppose $\mathfrak{H}_k = \mathfrak{H}_{k+1}$ and let $H_k := \sup_x \mathfrak{H}_k$. To avoid triviality we assume $\alpha_{j-1} < x < \alpha_{j+1}$.

Case 1 ($j = 0$): In this case, clearly H_k^2 is the nonzero root of the equation $\det A(0,k) = 0$ and for $x \in (0, H_k]$, $W_{\alpha_0(x)}$ is k-hyponormal. By assumption $H_k = H_{k+1}$, so $W_{\alpha_0(H_{k+1})}$ is $(k+1)$-hyponormal. The result now follows from Lemma 2.2 (ii).

Case 2 ($j \geq 1$): Let $A_x(n,k)$ denote the Hankel matrix corresponding to $\alpha_j(x)$. Since $W_{\alpha_j(x)}$ is $(k+1)$-hyponormal for $x \in \mathfrak{H}_k$, we have that $A_x(n, k+1) \geq 0$ for all $n \geq 0$ and all $x \in \mathfrak{H}_k$. Observe that if $n \geq j+1$ then

$$A_x(n,k) = \alpha_0^2 \cdots \alpha_{j-1}^2 \, x^2 \begin{pmatrix} \tilde{\gamma}_{n-j-1} & \cdots & \tilde{\gamma}_{n-j-1+k} \\ \vdots & & \vdots \\ \tilde{\gamma}_{n-j-1+k} & \cdots & \tilde{\gamma}_{n-j-1+2k} \end{pmatrix},$$

where $\tilde{\gamma}_*$ denotes the moments corresponding to the subsequence $\alpha_{j+1}, \alpha_{j+2}, \cdots$. Therefore for $n \geq j+1$, the positivity of $A_x(n,k)$ is independent of the values of $x > 0$. This gives

$$W_{\alpha_j(x)} \text{ is } k\text{-hyponormal} \Longleftrightarrow A_x(n,k) \geq 0 \text{ for all } n \leq j.$$

Write

$$\mathfrak{H}_k^{(i)} := \Big\{ x : \ \det A_x(i,k) \geq 0 \ \text{ and } \ \alpha_{j-1} < x < \alpha_{j+1} \Big\} \quad (0 \leq i \leq j)$$

and

$$H_k^{(i)} = \sup_x \mathfrak{H}_k^{(i)} \quad (0 \leq i \leq j).$$

Since $\det A_x(i,k)$ is a polynomial in x we have $\det A_{H_k^{(i)}}(i,k) = 0$. Observe that

$$\bigcap_{i=0}^{j} \mathfrak{H}_k^{(i)} = \mathfrak{H}_k \quad \text{and} \quad \max_{0 \leq i \leq j} H_k^{(i)} = H_k.$$

Since $\mathfrak{H}_k$ is a closed interval, by [**CL2**, Theorem 2.11], it follows that $H_k \in \mathfrak{H}_k$, say $H_k = H_k^{(p)}$ for some $0 \leq p \leq j$. Then $\det A_{H_k^{(p)}}(p,k) = 0$ and $W_{\alpha(H_k^{(p)})}$ is $(k+1)$-hyponormal. Therefore it follows from Lemma 2.2 (ii) that W_α is recursive subnormal. This completes the proof. □

We conclude this section with two corollaries of independent interest.

Corollary 2.4. *With the notations in Theorem 2.3, if $j \geq 1$ and $\mathfrak{H}_k = \mathfrak{H}_{k+1}$ for some k, then $\mathfrak{H}_k$ is a singleton set.*

Proof. By [**CL2**, Theorem 2.2],

$$\mathfrak{H}_\infty := \{x \in \mathbb{R}^+ : W_{\alpha_j(x)} \text{ is subnormal}\}$$

is a singleton set. By Theorem 2.3, we have that $\mathfrak{H}_k = \mathfrak{H}_\infty$. □

Corollary 2.5. *If W_α is a nonrecursive shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ and if $\alpha(x)$ is a canonical rank-one perturbation of α, then for every $k \geq 1$ there always exists a gap between k-hyponormality and $(k+1)$-hyponormality for $W_{\alpha(x)}$. More concretely, if we let*

$$\mathfrak{H}_k := \{x : W_{\alpha(x)} \text{ is } k\text{-hyponormal}\},$$

then $\{\mathfrak{H}_k\}_{k=1}^{\infty}$ is a strictly decreasing nested sequence of closed intervals in $(0, \infty)$ except when the perturbation occurs in the first weight. In that case, the intervals are of the form $(0, H_k]$.

Proof. Straightforward from Theorem 2.3. □

3. Some Revealing Examples

We now illustrate our results with two examples. Consider $\alpha(y,x) : \sqrt{y}, \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$, where $a < b < c$. Without loss of generality, we assume $a = 1$. Observe that

$$H_2(1, \sqrt{b}, \sqrt{c}) = \sqrt{\frac{bc - b^2}{1 + bc - 2b}} \quad \text{and} \quad \left(H_2(\sqrt{x}, 1, \sqrt{b})\right)^2 = \frac{x(b-1)}{(x-1)^2 + (b-1)} := f(x).$$

Thus $W_{\alpha(y,x)}$ is 2-hyponormal if and only if $0 < x \leq \frac{bc-b^2}{1+bc-2b}$ and $0 < y \leq f(x)$. To completely describe the region $\mathcal{R} := \{(x,y) : W_{\alpha(y,x)} \text{ is 2-hyponormal}\}$, we study the graph of f. Observe that

$$f'(x) = \frac{(b-1)(b-x^2)}{(b-2x+x^2)^2} > 0 \quad \text{and} \quad f''(x) = \frac{2(b-1)(2b-3bx+x^3)}{(b-2x+x^2)^3}.$$

Note that $b - 2x + x^2 = (b-1) + (1-x)^2 > 0$ and $f'(\sqrt{b}) = 0$. To consider the sign of f'', we let $g(x) := 2b - 3bx + x^3$. Then $g'(\sqrt{b}) = 0$, $g(0) = 2b > 0$, $g(1) = -b + 1 < 0$, and $g''(x) > 0$ $(x > 0)$. Hence there exists $x_0 \in (0,1)$ such that $f''(x_0) = 0$, $f''(x) > 0$ on $0 < x < x_0$, and $f''(x) < 0$ on $x_0 < x \leq 1$. We investigate which of the two values x_0 or $\widetilde{H} := H_2(1, \sqrt{b}, \sqrt{c})^2$ is bigger. By a simple calculation, we have

$$g(\widetilde{H}) = \frac{(-1+b)b \cdot g_1(b,c)}{(1-2b+bc)^3},$$

where

$$g_1(b,c) = -(2 - 10b + 17b^2 - 11b^3 + b^4 + 3bc - 9b^2c + 9b^3c - 3b^3c^2 + b^2c^3).$$

For notational convenience we let $b := 1 + h$, $c := 1 + h + k$. Then

$$g_1(b,c) = 2h^5 + (3h^3 + 3h^4)k + (-1 - 2h - h^2)k^3.$$

If h is sufficiently small (i.e., b is sufficiently close to 1), then $g_1 < 0$, i.e., $\widetilde{H} > x_0$. If k is sufficiently small (i.e., b is sufficiently close to c), then $g_1 > 0$, i.e., $\widetilde{H} < x_0$. Thus, if $\widetilde{H} \leq x_0$, then f is concave up on $x \leq \widetilde{H}$. If $\widetilde{H} > x_0$, then $(x_0, f(x_0))$ is an inflection point. Thus, f is concave up on $0 < x < x_0$ and concave down on $x_0 < x \leq \widetilde{H}$. Moreover, $W_{\alpha(y,x)}$ is 2-hyponormal if and only if $(x,y) \in \{(x,y) | 0 \leq y \leq f(x),\ 0 < x \leq \widetilde{H}\}$, and $W_{\alpha(y,x)}$ is k-hyponormal $(k \geq 3)$ if and only if $x = \widetilde{H}$ and $0 \leq y \leq f(\widetilde{H})$.

Example 3.1 ($b = 2$, $c = 3$)**.**

$$f(x) = \frac{x}{1 + (1-x)^2}.$$

The graph of $\mathcal{R}$ is given in Figure 1; notice that f is concave up in this case.

Figure 1

Example 3.2 **($b = \frac{11}{10}$, $c = 10$).**

$$f(x) = \frac{x}{11 - 20x + 10x^2}.$$

The graph of $\mathcal{R}$ is given in Figure 2; in this case, f has an inflection point at $x_0 \approx 0.85821$.

Figure 2

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