SUBNORMALITY AND 2-HYPONORMALITY FOR TOEPLITZ OPERATORS

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In this article we provide an example of a Toeplitz operator which is 2–hyponormal but not subnormal, and we consider 2-hyponormal Toeplitz operators with finite rank self-commutators.

The present article concerns the gap between subnormality and 2-hyponormality for Toeplitz operators. We begin with a brief survey of research related to P.R. Halmos's Problem 5 (cf. [Ha1],[Ha2]):

(Prob 5) Is every subnormal Toeplitz operator either normal or analytic ?

As we know, (Prob 5) was answered in the negative by C. Cowen and J. Long [CoL]. Directly connected with it is the following problem:

(0.1) Which Toeplitz operators are subnormal ?

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \cdots\}$. An element $f \in L^2(\mathbb{T})$ is said to be analytic if $f \in H^2(\mathbb{T})$, and co-analytic if $f \in L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. If P denotes the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$ the operators T_φ and H_φ on $H^2(\mathbb{T})$ defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi(g) := (I-P)(\varphi g) \qquad (g \in H^2(\mathbb{T}))$$

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are called the *Toeplitz operator* and the *Hankel operator*, respectively, with symbol φ.

(Prob 5) has been answered in the affirmative for *trigonometric* Toeplitz operators [ItW], and for *quasinormal* Toeplitz operators [AIW]. In 1976, M. B. Abrahamse [Abr] gave a general sufficient condition for the answer to (Prob 5) to be affirmative.

Theorem 1 ([Abr]). *If*

(i) T_φ *is hyponormal;*
(ii) φ *or* $\bar{\varphi}$ *is of bounded type (i.e.,* φ *or* $\bar{\varphi}$ *is a quotient of two analytic functions);*
(iii) *ker* $[T_\varphi^*, T_\varphi]$ *is invariant for* T_φ*,*

then T_φ *is normal or analytic.*

Since ker $[T^*, T]$ is invariant for every subnormal operator T, Theorem 1 answers (Prob 5) affirmatively when φ or $\bar{\varphi}$ is of bounded type. Also, in [Abr], Abrahamse proposed the following question, as a strategy to answer (Prob 5):

(Abr) Is the Bergman shift unitarily equivalent to a Toeplitz operator ?

To study this question, recall that given a bounded sequence of positive numbers α : $\alpha_0, \alpha_1, \cdots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bergman shift is a weighted shift W_α with weights $\alpha := \left\{\sqrt{\frac{n}{n+1}}\right\}_{n=1}^\infty$; it is well known that the Bergman shift is subnormal. In 1983, S. Sun [Sun] showed that if a Toeplitz operator T_φ is unitarily equivalent to a hyponormal weighted shift W_α with weight sequence α, then α must be of the form

$$\alpha = \left\{(1-\beta^{2n+2})^{\frac{1}{2}} ||T_\varphi||\right\}_{n=0}^\infty \tag{1.1}$$

for some β ($0 < \beta < 1$), thus answering (Abr) in the negative. Cowen and Long [CoL] showed that a unilateral weighted shift with weight sequence of the form (1.1) must be subnormal (see also [Fa]). Consequently, we have:

Theorem 2 ([Sun], [Cow2]). *Every hyponormal Toeplitz operator which is unitarily equivalent to a weighted shift must be subnormal.*

Finally, in 1984 Cowen and Long [CoL] constructed a symbol φ for which T_φ is unitarily equivalent to a weighted shift with weight sequence (1.1). This helped answer (Prob 5) in the negative.

Theorem 3 ([CoL],[Cow2]). *Let* $0 < \alpha < 1$ *and let* ψ *be a conformal map of the unit disk onto the interior of the ellipse with vertices* $\pm(1+\alpha)i$ *and passing through* $\pm(1-\alpha)$. *If* $\varphi = (1-\alpha^2)^{-1}(\psi + \alpha\bar{\psi})$, *then* T_φ *is a weighted shift with weight sequence* $\alpha_n = (1-\alpha^{2n+2})^{-\frac{1}{2}}$. *Therefore,* T_φ *is subnormal but neither normal nor analytic. In particular,* φ *is not of bounded type.*

On the other hand, the Bram–Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([Bra],[Con, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1). \tag{3.1}$$

Condition (3.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (3.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (3.1) for all k. If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B, and if we define T to be *k–hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{3.2}$$

is positive, or equivalently, the $(k+1) \times (k+1)$ operator matrix in (3.1) is positive (via the operator version of Choleski's Algorithm), then the Bram–Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k–hyponormal for every $k \geq 1$ ([CMX]). It is then natural to try to understand the gap between k–hyponormality and subnormality for Toeplitz operators. In [CuL1] and [CuL3], as a first inquiry in this line of thought the following question was raised.

Question A. *Is every 2–hyponormal Toeplitz operator subnormal?*

In [CuL1], the following was shown: *Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic. Hence, in particular, every 2–hyponormal trigonometric Toeplitz operator is subnormal.* We can extend this result.

Theorem 4 ([CuL3, Corollary 6])**.** *If T_φ is 2–hyponormal and if φ or $\bar{\varphi}$ is of bounded type then T_φ is normal or analytic, so that in particular T_φ is subnormal.*

Proof. It is known (cf. [CuL2]) that if $T \in \mathcal{L}(\mathcal{H})$ is 2–hyponormal then

$$T\big(\ker [T^*, T]\big) \subseteq \ker [T^*, T]. \tag{4.1}$$

Thus the result follows at once from Theorem 1. □

We now answer Question A in the negative: there is a gap between 2–hyponormality and subnormality for Toeplitz operators. We first establish a theorem which provides an example of a non-weighted shift which is 2–hyponormal but not subnormal.

Theorem 5. *For $0 < \alpha < 1$, let $T \equiv W_\beta$ be the weighted shift with weight sequence $\beta = \{\beta_n\}_{n=0}^\infty$ with*

$$\beta_n := \left(\sum_{j=0}^{n} \alpha^{2j} \right)^{\frac{1}{2}} .$$

If $S_\lambda := T + \lambda T^*$ ($\lambda \in \mathbb{C}$), then

(i) *S_λ is hyponormal if and only if $|\lambda| \le 1$.*
(ii) *S_λ is subnormal if and only if $\lambda = 0$ or $|\lambda| = \alpha^k$ for some $k = 0, 1, 2, \cdots$.*
(iii) *S_λ is 2–hyponormal if and only if $|\lambda| = 1$ or $|\lambda| \le \alpha$.*

Proof. The statements (i) and (ii) are known from [Cow1, Theorem 2.3]. Thus it suffices to focus on the assertion (iii). Since $T + \lambda T^*$ is unitarily equivalent to $e^{\frac{i\theta}{2}}(T + |\lambda|\, T^*)$ with $|\lambda| = \lambda e^{-i\theta}$ (cf. [Cow1, Lemma 2.1]), it follows that $T + \lambda T^*$ is 2–hyponormal if and only if $T + |\lambda|\, T^*$ is 2–hyponormal. Thus we may assume $\lambda \ge 0$. If $\lambda = 0, 1$, then evidently S_λ is 2–hyponormal because T is subnormal (cf. [CoL]). Thus, in view of (i), we assume $0 < \lambda < 1$. A straightforward calculation shows that

$$\begin{aligned} M_2(S_\lambda) &\equiv \begin{pmatrix} [S_\lambda^*, S_\lambda] & [S_\lambda^{*2}, S_\lambda] \\ [S_\lambda^*, S_\lambda^2] & [S_\lambda^{*2}, S_\lambda^2] \end{pmatrix} \\ &=: (1 - \lambda^2) \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}, \end{aligned} \tag{5.1}$$

where

$$\begin{cases} A &= [T^*, T] \\ B &= [T^{*2}, T] + \lambda\, [T^*, T^2] \\ C &= (1 + \lambda^2)[T^{*2}, T^2] + \lambda \left([T^{*3}, T] + [T^*, T^3] + T^*[T^*, T]T^* + T[T^*, T]T \right). \end{cases} \tag{5.2}$$

By (3.2), S_λ is 2–hyponormal if and only if $M_2(S_\lambda) \ge 0$. Recall now Smul'jan's Theorem ([Smu], [CuF, Proposition 2.2]), which states that if

$$M := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}, \tag{5.3}$$

if $A \ge 0$, and if $B = A^{\frac{1}{2}} V$ for some $V : \mathcal{H}_2 \to \mathcal{H}_1 \ominus N(A)$, then $M \ge 0$ if and only if $C \ge V^* V$. Now apply this result to (5.1). We first argue that in (5.2),

$$B := [T^{*2}, T] + \lambda\, [T^*, T^2] = (1 + \alpha^2)\, [T^*, T]\, (\frac{\lambda}{\alpha^2} T + T^*). \tag{5.4}$$

To see this, observe that $[T^*, T^2]$ and $[T^*, T]T$ are both unilateral weighted shifts, so it suffices to check the $(n+1, n)$–entries:

$$([T^*, T^2] e_n,\, e_{n+1}) = \beta_n(\beta_{n+1}^2 - \beta_{n-1}^2) = \beta_n(\alpha^{2n+2} + \alpha^{2n})$$

and

$$([T^*,T]Te_n,\, e_{n+1}) = \beta_n(\beta_{n+1}^2 - \beta_n^2) = \beta_n \alpha^{2n+2},$$

which implies that

$$\begin{aligned}
(\lambda[T^*,T^2]e_n,\, e_{n+1}) &= \lambda\,\beta_n(\alpha^{2n+2}+\alpha^{2n}) \\
&= \lambda\,\beta_n(1+\frac{1}{\alpha^2})\alpha^{2n+2} \\
&= (\lambda(1+\frac{1}{\alpha^2})[T^*,T]Te_n,\, e_{n+1}),
\end{aligned}$$

giving $\lambda[T^*,T^2] = (1+\alpha^2)\frac{\lambda}{\alpha^2}[T^*,T]T$, and similarly $[T^{*2},T] = (1+\alpha^2)[T^*,T]T^*$, which proves (5.4). Since, in (5.1), $A \geq 0$ and $N(A) = \{0\}$, it follows from (5.4) that we may take

$$V := (1+\alpha^2)[T^*,T]^{\frac{1}{2}}\,(\frac{\lambda}{\alpha^2}T + T^*).$$

Then a straightforward calculation shows that

$$\begin{aligned}
C - V^*V &= (1+\lambda^2)[T^{*2},T^2] + \lambda\big([T^{*3},T] + [T^*,T^3] + T^*[T^*,T]T^* + T[T^*,T]T\big) \\
&\quad - (1+\alpha^2)^2\,(\frac{\lambda}{\alpha^2}T^* + T)\,[T^*,T]\,(\frac{\lambda}{\alpha^2}T + T^*) \\
&= (1+\lambda^2)[T^{*2},T^2] - \frac{(1+\alpha^2)^2}{\alpha^4}\big(\lambda^2 T^*[T^*,T]T + \alpha^4 T[T^*,T]T^*\big) + \lambda\,(Q+Q^*),
\end{aligned}$$

where

$$Q := [T^*,T^3] - (\alpha^2 + 1 + \frac{1}{\alpha^2})\,T[T^*,T]T.$$

Observe that both $[T^*,T^3]$ and $T[T^*,T]T$ are unilateral weighted shifts of multiplicity 2. Thus to determine Q, it suffices to check the $(n+2,n)$–entries. Now

$$\begin{aligned}
(Qe_n,\, e_{n+2}) &= ([T^*,T^3]e_n,\, e_{n+2}) - (\alpha^2+1+\frac{1}{\alpha^2})(T[T^*,T]Te_n,\, e_{n+2}) \\
&= \beta_n\beta_{n+1}(\beta_{n+2}^2 - \beta_{n-1}^2) - (\alpha^2+1+\frac{1}{\alpha^2})\beta_n\beta_{n+1}(\beta_{n+1}^2 - \beta_n^2) \\
&= \beta_n\beta_{n+1}\left(\alpha^{2n+4} + \alpha^{2n+2} + \alpha^{2n} - (\alpha^2+1+\frac{1}{\alpha^2})\alpha^{2n+2}\right) \\
&= 0,
\end{aligned}$$

which implies $Q \equiv 0$. Therefore,

$$C - V^*V = (1+\lambda^2)\,[T^{*2},T^2] - \left(1+\frac{1}{\alpha^2}\right)^2\big(\lambda^2 T^*[T^*,T]T + \alpha^4 T[T^*,T]T^*\big)\,.$$

Observe that $[T^{*2}, T^2]$, $T^*[T^*,T]T$, and $T[T^*,T]T^*$ are all diagonal. Thus to determine $C - V^*V$, it suffices to check the (n,n)–entries. Now

$$\begin{aligned}
&\big((C - V^*V)e_n,\, e_n\big)\\
&= (1+\lambda^2)([T^{*2},T^2]e_n,\, e_n) - \left(1+\frac{1}{\alpha^2}\right)^2 \big(\lambda^2(T^*[T^*,T]Te_n,\, e_n) + \alpha^4(T[T^*,T]T^*e_n,\, e_n)\big)\\
&= (1+\lambda^2)(\beta_{n+1}^2\beta_n^2 - \beta_{n-1}^2\beta_{n-2}^2) - \left(1+\frac{1}{\alpha^2}\right)^2 \big(\lambda^2\beta_n^2(\beta_{n+1}^2 - \beta_n^2) + \alpha^4\beta_{n-1}^2(\beta_{n-1}^2 - \beta_{n-2}^2)\big)\\
&= (1+\lambda^2)\left((\alpha^{2n-2} + \alpha^{2n})\sum_{j=1}^{n-1}\alpha^{2j} + (\alpha^{2n} + \alpha^{2n+2})\sum_{j=0}^{n}\alpha^{2j}\right)\\
&\qquad - \left(1+\frac{1}{\alpha^2}\right)^2\left(\lambda^2\alpha^{2n+2}\sum_{j=0}^{n}\alpha^{2j} + \alpha^4\alpha^{2n-2}\sum_{j=0}^{n-1}\alpha^{2j}\right)\\
&= \left(1+\frac{1}{\alpha^2}\right)^2 \alpha^{4n+2} - (1+\lambda^2)\alpha^{4n-2}(1+\alpha^2)\\
&= (1+\alpha^2)(\alpha^2-\lambda^2)\alpha^{4n-2}\\
&= \frac{(1+\alpha^2)(\alpha^2-\lambda^2)}{\alpha^2}(\beta_n^2 - \beta_{n-1}^2)^2\\
&= \frac{(1+\alpha^2)(\alpha^2-\lambda^2)}{\alpha^2}\left([T^*,T]e_n,\, e_n\right)^2,
\end{aligned}$$

which implies

$$C - V^*V = \frac{(1+\alpha^2)(\alpha^2-\lambda^2)}{\alpha^2}[T^*,T]^2.$$

Therefore $C \geq V^*V$ if and only if $0 \leq \lambda \leq \alpha$, and hence S_λ is 2–hyponormal if and only if $0 \leq \lambda \leq \alpha$. This completes the proof. □

In the following theorem, the proofs of the statements (i) and (ii) are given in [Cow1, Theorem 2.4].

Theorem 6. *Let $0 < \alpha < 1$ and let ψ be the conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. Let $\varphi = \psi + \lambda\bar{\psi}$ and let T_φ be the corresponding Toeplitz operator on H^2. Then*

(i) *T_φ is hyponormal if and only if λ is in the closed unit disk $|\lambda| \leq 1$.*

(ii) *T_φ is subnormal if and only if $\lambda = \alpha$ or λ is in the circle $\left|\lambda - \frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}\right| = \frac{\alpha^k(1-\alpha^2)}{1-\alpha^{2k+2}}$ for $k = 0, 1, 2\cdots$.*

(iii) *T_φ is 2–hyponormal if and only if λ is in the unit circle $|\lambda| = 1$ or in the closed disk $\left|\lambda - \frac{\alpha}{1+\alpha^2}\right| \leq \frac{\alpha}{1+\alpha^2}$.*

Proof. It was shown in [CoL] that $T_{\psi+\alpha\bar{\psi}}$ is unitarily equivalent to $(1-\alpha^2)^{\frac{3}{2}}T$, where T is the weighted shift in Theorem 5. Thus T_ψ is unitarily equivalent to $(1-\alpha^2)^{\frac{1}{2}}(T - \alpha T^*)$, so T_φ is unitarily equivalent to

$$(1-\alpha^2)^{\frac{1}{2}}(1-\lambda\,\alpha)(T+\frac{\lambda-\alpha}{1-\lambda\,\alpha}T^*) \quad \text{(cf. [Cow1, Theorem 2.4])},$$

applying Theorem 5 with $\frac{\lambda-\alpha}{1-\lambda\,\alpha}$ in place of λ. Now, for $k=0,1,2,\cdots$,

$$\begin{aligned}\left|\frac{\lambda-\alpha}{1-\lambda\,\alpha}\right| \le \alpha^k &\Longleftrightarrow |\lambda-\alpha|^2 \le \alpha^{2k}|1-\lambda\,\alpha|^2\\ &\Longleftrightarrow |\lambda|^2-\frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}(\lambda+\bar{\lambda})+\frac{\alpha^2-\alpha^{2k}}{1-\alpha^{2k+2}}\le 0\\ &\Longleftrightarrow \left|\lambda-\frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}\right| \le \frac{\alpha^k(1-\alpha^2)}{1-\alpha^{2k+2}}.\end{aligned}$$

If $k=0$ then $\left|\frac{\lambda-\alpha}{1-\lambda\,\alpha}\right|\le 1 \Longleftrightarrow |\lambda|\le 1$. If $k=1$ then $\left|\frac{\lambda-\alpha}{1-\lambda\,\alpha}\right|\le\alpha \Longleftrightarrow \left|\lambda-\frac{\alpha}{1+\alpha^2}\right|\le\frac{\alpha}{1+\alpha^2}$. This completes the proof. □

A moment's reflection reveals that the circles in Theorem 6 (ii) form a nested sequence in the sense of their convex hulls.

Corollary 7. *In Theorem 6, if λ lies in the open annulus between the two circles*

$$\left|\lambda-\frac{\alpha}{1+\alpha^2}\right|=\frac{\alpha}{1+\alpha^2} \quad \text{and} \quad \left|\lambda-\frac{\alpha(1-\alpha^4)}{1-\alpha^6}\right|=\frac{\alpha^2(1-\alpha^2)}{1-\alpha^6}$$

then the corresponding Toeplitz operator T_φ is 2–hyponormal but not subnormal.

Now we would like to pose the following conjecture.

Conjecture A. *In Theorem 6, we have:*

(i) *T_φ is quadratically hyponormal if and only if T_φ is 2–hyponormal;*

(ii) *T_φ is k-hyponormal if and only if λ is in the circle $\left|\lambda-\frac{\alpha(1-\alpha^{2j})}{1-\alpha^{2j+2}}\right|=\frac{\alpha^j(1-\alpha^2)}{1-\alpha^{2j+2}}$ for $j=0,\cdots,k-2$ or in the closed disk $\left|\lambda-\frac{\alpha(1-\alpha^{2(k-1)})}{1-\alpha^{2k}}\right|\le\frac{\alpha^{k-1}(1-\alpha^2)}{1-\alpha^{2k}}$.*

In a recent preprint, C. Gu [Gu] has announced a proof of part (ii) of Conjecture A, in the context of extending our results to $k\ge 3$. His techniques are different from ours, and seem to exploit a special case of Smul'jan's Theorem (see (5.3) above). However, we have been unable to verify his proof; concretely, we do not follow his beginning argument in the proof of [Gu, Theorem 3.3], using [Gu, Lemma 2.6].

In spite of Theorem 6, it seems to be interesting to consider the following problem:

Which 2–hyponormal Toeplitz operators are subnormal ?

The first inquiry involves the self-commutator. Subnormal operators with finite rank self-commutators have been studied by many authors ([Ale], [McCY], [Mor], [OTT], [Xi1],

[Xi2]). In 1975, I. Amemiya, T. Ito and T. Wong [AIW] showed that if T_φ is a subnormal Toeplitz operator with rank–one self-commutator then φ is a linear function of a Blaschke product of degree 1. More generally, B. Morrel [Mor] showed that a pure subnormal operator with rank-one self-commutator is unitarily equivalent to a linear function of the unilateral shift. Very recently, in [CuL2], it was shown that every pure 2–hyponormal operator with rank-one self-commutator is a linear function of the unilateral shift. On the other hand, J. McCarthy and L. Yang [McCY] have classified all rationally cyclic subnormal operators with finite rank self-commutators. However it is still open which are the pure subnormal operators with finite rank self–commutator. Related to this, in [CuL3] we formulated the following:

Question B. *If T_φ is a 2–hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that T_φ is analytic ?*

In fact, it remains still open even whether subnormal Toeplitz operators with finite rank self–commutator are normal. As a strategy to answer to Question B we would like to pose the following conjecture.

Conjecture B. *If T_φ is a 2–hyponormal Toeplitz operator with finite rank self-commutator then*

$$\dim \left(cl\, H_{\overline{\varphi}}(ker\,[T_\varphi^*, T_\varphi])\right)^\perp \le \dim \left(cl\, H_{\varphi}(ker\,[T_\varphi^*, T_\varphi])\right)^\perp .$$

If $\ker [T_\varphi^*, T_\varphi] = q\, H^2$ for some inner function q, then Conjecture B is true: indeed, if k is a function in $\mathcal{E}(\varphi)$ then

$$H_\varphi(\ker [T_\varphi^*, T_\varphi]) = H_{k\,\overline{\varphi}}(qH^2) = H_{\overline{\varphi}} T_k(qH^2) \subseteq H_{\overline{\varphi}}(qH^2) = H_{\overline{\varphi}}(\ker [T_\varphi^*, T_\varphi]),$$

and hence $\left(\mathrm{cl}\, H_{\overline{\varphi}}(\ker [T_\varphi^*, T_\varphi])\right)^\perp \subseteq \left(\mathrm{cl}\, H_{\varphi}(\ker [T_\varphi^*, T_\varphi])\right)^\perp$, which implies that Conjecture A is true.

We now have:

Theorem 8. *Suppose that Conjecture B is true. If T_φ is a 2–hyponormal Toeplitz operator with finite rank self-commutator then T_φ is normal or analytic.*

Proof. If φ or $\overline{\varphi}$ is of bounded type then the result follows at once from Theorem 4 with no restriction on the self-commutator. Thus we suppose that φ and $\overline{\varphi}$ both are not of bounded type. Suppose φ is not analytic and T_φ is a hyponormal operator with finite rank self-commutator. It suffices to show that T_φ is normal. By an argument of [Abr, Lemmas 3 and 4] we have that $\ker H_\varphi = \ker H_{\overline{\varphi}} = \{0\}$, and $\mathrm{ran}\, H_\varphi$ and $\mathrm{ran}\, H_{\overline{\varphi}}$ are both dense in H^2. Observe (cf. [Abr, Lemma 1]) that for $h \in H^2$,

$$(8.1) \qquad (h, [T_\varphi^*, T_\varphi]h) = ||H_{\overline{\varphi}} h||^2 - ||H_\varphi h||^2.$$

Since T_φ is hyponormal it follows that

$$(8.2) \qquad ||H_\varphi h|| \le ||H_{\overline{\varphi}} h|| \quad (h \in H^2).$$

Define an operator S on $\operatorname{ran} H_{\bar{\varphi}}$ by

$$S(H_{\overline{\varphi}}h) = H_\varphi h.$$

Then S is well defined and $||S|| \le 1$ by (8.2), so S has an extension to H^2 since $\operatorname{ran} H_{\overline{\varphi}}$ is dense in H^2. In [Cow3, Proof of Theorem 1] it was shown that

(i) S is a contraction on H^2;
(ii) S is a co-analytic Toeplitz operator, say $S := T_{\bar{k}}$ with $k \in H^\infty$;
(iii) $T_{\bar{k}} H_{\overline{\varphi}} = H_\varphi$;
(iv) $\tilde{k} \in \mathcal{E}(\varphi)$, where $\tilde{k} = \overline{k(\bar{z})}$.

Since $[T_\varphi^*, T_\varphi]$ is of finite rank, we have that $\ker [T_\varphi^*, T_\varphi]$ has finite co–dimension. Also by (8.1) we have that

$$||H_{\overline{\varphi}}h|| = ||H_\varphi h|| \quad \text{for all } h \in \ker [T_\varphi^*, T_\varphi].$$

Thus the restriction of $T_{\bar{k}}$ to $\operatorname{cl} H_{\overline{\varphi}}(\ker [T_\varphi^*, T_\varphi])$ is an isometry. Now put

$$M := \operatorname{cl} H_{\overline{\varphi}}(\ker [T_\varphi^*, T_\varphi]) \quad \text{and} \quad N := \operatorname{cl} H_\varphi(\ker [T_\varphi^*, T_\varphi]).$$

Since both $H_{\overline{\varphi}}$ and H_φ are one-one and have dense ranges it follows that $M^\perp$ and $N^\perp$ are finite-dimensional. If Conjecture B is true then we have

$$\dim M^\perp \le \dim N^\perp.$$

Since $T_{\bar{k}}|_M$ is an isometry, we can see that $T_k T_{\bar{k}} = I + K$, where K is a finite rank operator. Then by Douglas's Theorem, which states that if $T_{\varphi_1} T_{\varphi_2} - T_{\varphi_3}$ is compact then $\varphi_1\varphi_2 = \varphi_3$, we have that $|k|^2 = 1$, so that T_k is an isometry because T_k is analytic. Write $T_{\bar{k}}$ as the following 2×2 operator matrix:

$$T_{\bar{k}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} M \\ M^\perp \end{pmatrix} \longrightarrow \begin{pmatrix} N \\ N^\perp \end{pmatrix},$$

where A is an isometry. Since $T_{\bar{k}}$ is a contraction, it follows that $C = 0$. Also since T_k is an isometry, we have

$$T_{\bar{k}} T_k = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A^* & 0 \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} AA^* + BB^* & BD^* \\ DB^* & DD^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so

$$\begin{cases} AA^* + BB^* = 1 \\ BD^* = 0 \\ DD^* = 1. \end{cases}$$

Since $D^* : N^\perp \to M^\perp$ is an isometry, it follows that $\dim M^\perp = \dim N^\perp$, so that D^* is unitary. Thus we must have that $B = 0$ and in turn, $AA^* = 1$. This forces A to be unitary, so that T_k is unitary. Therefore k is a constant of modulus 1, and hence so is $\tilde{k}$. But since $\tilde{k} \in \mathcal{E}(\varphi)$, it follows that φ is of the form $\varphi = \bar{f} + e^{i\theta} f$ for some $f \in H^\infty$ and $\theta \in [0, 2\pi)$, which implies that T_φ is normal. This completes the proof. □

The core of the proof of Theorem 8 is that if φ is not of bounded type for which T_φ is a 2–hyponormal Toeplitz operator with nonzero finite rank self-commutator then T_φ is analytic. One might expect that this is true with "hyponormal" in place of " 2–hyponormal," but this is not the case. To see this let ψ be the conformal map in Theorem 3. Then $\varphi \equiv \bar\psi + z\psi$ is not of bounded type because, in Theorem 3, $\bar\psi + \alpha\psi$ is not of bounded type, so that $\{0\} \neq \ker H_{\bar\psi+\alpha\psi} = \ker H_{\bar\psi} = \ker H_\varphi$, and hence φ is not of bounded type by [Abr, Lemma 3]. Observe that $\varphi - z\bar\varphi \in H^\infty$ which, by Cowen's Theorem, says that T_φ is hyponormal. But since

$$\begin{aligned}[T_\varphi^*, T_\varphi] &= (T_\psi + T_{\bar z}T_{\bar\psi})(T_{\bar\psi} + T_z T_\psi) - (T_{\bar\psi} + T_z T_\psi)(T_\psi + T_{\bar z}T_{\bar\psi})\\ &= T_\psi T_{\bar\psi} - T_\psi T_z T_{\bar z} T_{\bar\psi} = T_\psi(1 - T_z T_{\bar z})T_{\bar\psi},\end{aligned}$$

it follows that rank $[T_\varphi^*, T_\varphi] = 1$.

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