A NEW CRITERION FOR k-HYPONORMALITY VIA WEAK SUBNORMALITY

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Abstract. In this article we obtain a criterion for k-hyponormality via weak subnormality. Using this criterion we recapture Spitkovskii's subnormality criterion and give a simple proof of the main result in [Gu], which describes a gap between k-hyponormality and $(k+1)$-hyponormality for Toeplitz operators. In addition, we notice that the minimal normal extension of a subnormal operator is exactly the inductive limit of its minimal partially normal extensions.

Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Thus an operator T is subnormal if and only if there exist operators A and B such that $\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ is normal, i.e.,

$$\text{(0.1)} \qquad \begin{cases} [T^*,T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*,B] + A^*A = 0. \end{cases}$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly subnormal* ([CuL2]) if there exist operators $A \in \mathcal{L}(\mathcal{H},\mathcal{H}')$ and $B \in \mathcal{L}(\mathcal{H}')$ such that the first two conditions in (0.1) hold: $[T^*,T] = AA^*$ and $A^*T = BT^*$, or equivalently, there is an extension $\widehat{T}$ of T such that

$$\text{(0.2)} \qquad \widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f \text{ for all } f \in \mathcal{H}.$$

The operator $\widehat{T}$ is said to be a *partially normal extension* (briefly, p.n.e.) of T. Note that the condition (0.2) implies $||\widehat{T}f|| = ||\widehat{T}^*f||$ for all $f \in \mathcal{H}$, and that if (0.2) holds for all $f \in \mathcal{H} \oplus \mathcal{H}'$, then $\widehat{T}$ becomes normal, so T is in that case subnormal. We also say that $\widehat{T} \in \mathcal{L}(\mathcal{K})$ is a *minimal partially normal extension* (briefly, m.p.n.e.) of

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a weakly subnormal operator T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a partially normal extension of T. It is known [CuL2, Lemma 2.5] that if $\widehat{T}$ is a partially normal extension of $T \in \mathcal{L}(\mathcal{H})$ on $\mathcal{K}$ then $\widehat{T}$ is minimal if and only if $\mathcal{K} = \bigvee\{\widehat{T}^{*k}h : h \in \mathcal{H},\ k = 0, 1\}$. Clearly, subnormal $\Longrightarrow$ weakly subnormal $\Longrightarrow$ hyponormal; however, the converses are not true in general (cf.[CuL2]).

On the other hand, the Bram–Halmos criterion for subnormality states that an operator T is subnormal if and only if $\sum_{i,j}(T^i x_j, T^j x_i) \geq 0$ for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([Bra],[Con, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1). \tag{0.3}$$

Condition (0.3) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (0.3) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (0.3) for all k. If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B, and if we define T to be *k–hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{0.4}$$

is positive, or equivalently, the $(k+1) \times (k+1)$ operator matrix in (0.3) is positive (via the operator version of Choleski's Algorithm), then the Bram–Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k–hyponormal for every $k \geq 1$ ([CMX]). The classes of k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([Cu1], [Cu2], [CuF1], [CuF2], [CuF3], [CuL1], [CuL2], [CuL3], [CMX], [DPY], [McCP]).

In this paper we obtain a new, different criterion for k-hyponormality via weak subnormality. Our criterion is sometimes more helpful because it avoids the potentially complicated verification of positivity needed for (0.4). Using this criterion we recapture Spitkovskii's subnormality criterion [Spi] and give a simple proof of the main result in [Gu], which describes a gap between k-hyponormality and $(k+1)$-hyponormality for Toeplitz operators.

1. A New Criterion for k-Hyponormality

Given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$.

In 1966, Stampfli [Sta] explicitly exhibited for a subnormal weighted shift A_0 its minimal normal extension

$$N := \begin{pmatrix} A_0 & B_1 & & 0 \\ & A_1 & B_2 & \\ & & A_2 & \ddots \\ 0 & & & \ddots \end{pmatrix}, \tag{0.5}$$

where A_n is a weighted shift with weights $\{a_0^{(n)}, a_1^{(n)}, \cdots\}$, $B_n := \mathrm{diag}\{b_0^{(n)}, b_1^{(n)}, \cdots\}$, and these entries satisfy:

(I) $(a_j^{(n)})^2 - (a_{j-1}^{(n)})^2 + (b_j^{(n)})^2 \geq 0$ $(b_j^{(0)} = 0$ for all $j)$;

(II) $b_j^{(n)} = 0 \Longrightarrow b_{j+1}^{(n)} = 0$;

(III) there exists a constant M such that $|a_j^{(n)}| \leq M$ and $|b_j^{(n)}| \leq M$ for $n = 0, 1, \cdots$ and $j = 0, 1, \cdots$, where

$$b_j^{(n+1)} := [(a_j^{(n)})^2 - (a_{j-1}^{(n)})^2 + (b_j^{(n)})^2]^{\frac{1}{2}} \quad \text{and} \quad a_j^{(n+1)} := a_j^{(n)} \frac{b_{j+1}^{(n+1)}}{b_j^{(n+1)}}$$

(if $b_{j_0}^{(n)} = 0$, then $a_{j_0}^{(n)}$ is taken to be 0).

On the other hand, in 1982, I. Spitkovskii [Spi] gave the following subnormality criterion for arbitrary operators. (In 1985, J. Ma and S. Zhou [MaZ] independently proved the same result.)

Theorem 1 ([Spi]). *Let $A_0 \in \mathcal{L}(\mathcal{H}_0)$, where $\mathcal{H}_0$ is a separable complex Hilbert space. Then A_0 is subnormal if and only if there exists $M > 0$ such that for every $n > 0$ the following properties hold:*

(I′) $D_n \geq 0$;

(II′) $A_{n-1}(\mathrm{Ker}\, D_{n-1}) \subseteq \mathrm{Ker}\, D_{n-1}$ $(n \geq 1)$;

(III′) *there exists a constant M such that $||A_n||, ||D_n|| \leq M$, where*

$$D_0 := [A_0^*, A_0], \quad D_{n+1} := D_n|_{\mathcal{H}_{n+1}} + [A_{n+1}^*, A_{n+1}], \quad \mathcal{H}_{n+1} := \overline{\mathrm{Ran}\,(D_n)},$$

and A_{n+1} denotes the bounded extension of $D_n^{\frac{1}{2}} A_n D_n^{-\frac{1}{2}}$ to $\overline{\mathrm{Ran}\,(D_n)} (= \mathcal{H}_{n+1})$ from $\mathrm{Ran}\,(D_n)$.

In this case, the minimal normal extension of A_0 is given by the operator N in (0.5) with $D_{n-1}^{\frac{1}{2}}$ in place of B_n.

In 1988, P. Fan [Fan] noticed that if A_0 is a weighted shift, then (I) and (I′), (II) and (II′), and (III) and (III′) are equivalent, respectively. Consequently, Theorem 1 is a natural generalization of Stampfli's subnormality criterion for weighted shifts.

We will now discuss analogues of the preceding results for k-hyponormal operators. It was shown in [CuL2] that every 2-hyponormal weighted shift is weakly subnormal, but whether the same implication holds for arbitrary operators was left open. Very recently, R. Curto, I.B. Jung and S.S. Park [CJP] showed that every 2-hyponormal operator is indeed weakly subnormal. This follows from a special case of a more general result:

Lemma 2 ([CJP, Theorems 2.7 and 3.2]).

(i) *If $T \in \mathcal{L}(\mathcal{H})$ is 2-hyponormal then $[T^*, T]^{\frac{1}{2}} T [T^*, T]^{-\frac{1}{2}}|_{\mathrm{Ran}[T^*, T]}$ is bounded;*

(ii) *$T \in \mathcal{L}(\mathcal{H})$ is $(k+1)$-hyponormal if and only if T is weakly subnormal and $\widehat{T} :=$ m.p.n.e.(T) is k-hyponormal.*

In particular, it was shown in [CuL2, Lemma 2.8] that if $A_0 \in \mathcal{L}(\mathcal{H}_0)$ is weakly subnormal then the minimal partially normal extension of A_0 can be obtained as

$$\widehat{A_0} = \begin{pmatrix} A_0 & D_0 \\ 0 & A_1 \end{pmatrix} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1,$$

where $\mathcal{H}_1 \equiv \overline{\mathrm{Ran}\,[A_0^*, A_0]}$ and D_0 is the restriction of $[A_0^*, A_0]^{\frac{1}{2}}$ to $\mathcal{H}_1$ satisfying $D_0 A_0 = A_1 D_0$. Note that

$$\bigvee\{\widehat{A_0}^{*k} h_0 : h_0 \in \mathcal{H}_0,\ k = 0, 1\} = \mathcal{H}_0 \oplus \mathcal{H}_1. \tag{2.1}$$

By using Lemma 2(ii) inductively, we can see that if A_0 is k-hyponormal then we can define

$$\widehat{A_0}^{(n)} := \widehat{\widehat{A_0}^{(n-1)}} \quad \text{for } n = 1, \cdots, k-1; \quad \widehat{A_0}^{(0)} := A_0.$$

Our criterion on k-hyponormality now follows:

Theorem 3. *An operator $A_0 \in \mathcal{L}(\mathcal{H}_0)$ is k-hyponormal if and only if the following three conditions hold for all n such that $0 \le n \le k-1$:*

(I_n) $D_n \ge 0$;
(II_n) $A_{n-1}(\mathrm{Ker}\, D_{n-1}) \subseteq \mathrm{Ker}\, D_{n-1}$ $(n \ge 1)$;
(III_n) $D_{n-1}^{\frac{1}{2}} A_{n-1} D_{n-1}^{-\frac{1}{2}}|_{\mathrm{Ran}\,(D_{n-1})}$ $(n \ge 1)$ *is bounded,*

where

$$D_0 := [A_0^*, A_0], \quad D_{n+1} := D_n|_{\mathcal{H}_{n+1}} + [A_{n+1}^*, A_{n+1}], \quad \mathcal{H}_{n+1} := \overline{\mathrm{Ran}\,(D_n)}$$

and A_{n+1} denotes the bounded extension of $D_n^{\frac{1}{2}} A_n D_n^{-\frac{1}{2}}$ to $\overline{\mathrm{Ran}\,(D_n)} (= \mathcal{H}_{n+1})$ from $\mathrm{Ran}\,(D_n)$.

Proof. Suppose A_0 is k-hyponormal. We now use induction on k. If $k = 2$ then A_0 is 2-hyponormal, and so $D_0 := [A_0^*, A_0] \ge 0$. By Lemma 2(i), $D_0^{\frac{1}{2}} A_0 D_0^{-\frac{1}{2}}|_{\mathrm{Ran}\,(D_0)}$ is bounded. Let A_1 be the bounded extension of $D_0^{\frac{1}{2}} A_0 D_0^{-\frac{1}{2}}$ from $\mathrm{Ran}\,(D_0)$ to $\mathcal{H}_1 := \overline{\mathrm{Ran}\,(D_0)}$ and $D_1 := D_0|_{\mathcal{H}_1} + [A_1^*, A_1]$. Writing $\widehat{A_0} := \begin{pmatrix} A_0 & D_0^{\frac{1}{2}} \\ 0 & A_1 \end{pmatrix}$, we have $\widehat{A_0} = \mathrm{m.p.n.e.}\,(A_0)$, which is hyponormal by Lemma 2(ii). Thus

$$[\widehat{A_0}^*, \widehat{A_0}] = \begin{pmatrix} 0 & 0 \\ 0 & D_0|_{\mathcal{H}_1} + [A_1^*, A_1] \end{pmatrix} \ge 0.$$

and hence $D_1 \ge 0$. Also by [CuL2, Lemma 2.2], $A_0(\mathrm{Ker}\, D_0) \subseteq \mathrm{Ker}\, D_0$ whenever A_0 is 2-hyponormal. Thus (I_n), (II_n), and (III_n) hold for $n = 0, 1$. Assume now that if A_0 is k-hyponormal then (I_n),(II_n) and (III_n) hold for all $0 \le n \le k-1$. Suppose A_0 is $(k+1)$-hyponormal. We must show that (I_n),(II_n) and (III_n) hold for $n = k$. Define

$$S := \begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{k-2}^{\frac{1}{2}} \\ 0 & & & & A_{k-1} \end{pmatrix} : \bigoplus_{i=0}^{k-1} \mathcal{H}_i \longrightarrow \bigoplus_{i=0}^{k-1} \mathcal{H}_i.$$

By our inductive assumption, $D_{k-1} \ge 0$. Writing $\widehat{T}^{(n)} := \mathrm{m.p.n.e.}(\widehat{T}^{(n-1)})$ when it exists, we can see by our assumption that $S = \widehat{A_0}^{(k-1)}$: indeed, if

$$S_l := \begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{l-2}^{\frac{1}{2}} \\ 0 & & & & A_{l-1} \end{pmatrix}$$

then since by assumption $[S_l^*, S_l] = 0 \oplus D_l$ and $A_l = D_{l-1}^{\frac{1}{2}} A_{l-1} D_{l-1}^{-\frac{1}{2}}|_{\mathrm{Ran}\,(D_{l-1})}$, it follows that S_l is the minimal partially normal extension of S_{l-1} $(1 \le l \le k-1)$. But since by our assumption A_0 is $(k+1)$-hyponormal, it follows from Lemma 2(ii) that S is 2-hyponormal. Thus by Lemma 2(i), $[S^*, S]^{\frac{1}{2}} S [S^*, S]^{-\frac{1}{2}}|_{\mathrm{Ran}\,([S^*,S])}$ is bounded, which says that $D_{k-1}^{\frac{1}{2}} A_{k-1} D_{k-1}^{-\frac{1}{2}}|_{\mathrm{Ran}\,(D_{k-1})}$ is bounded, proving (III_n) for $n = k$. Observe that A_k, $\mathcal{H}_k$ and D_k are well-defined. Writing $\widehat{S} := \begin{pmatrix} S & D_{k-1}^{\frac{1}{2}} \\ 0 & A_k \end{pmatrix}$, we can see that $\widehat{S} = \mathrm{m.p.n.e.}(S)$, which is hyponormal, again by Lemma 2(ii). Thus, since $[\widehat{S}^*, \widehat{S}] = \begin{pmatrix} 0 & 0 \\ 0 & D_k \end{pmatrix} \ge 0$, we have $D_k \ge 0$, proving (I_n) for $n = k$. On the other hand, since S is 2-hyponormal, it follows that $S(\mathrm{Ker}[S^*, S]) \subseteq \mathrm{Ker}[S^*, S]$. Since $[S^*, S] = \begin{pmatrix} 0 & 0 \\ 0 & D_{k-1} \end{pmatrix}$, we have $\mathrm{Ker}\,[S^*, S] = \bigoplus_{i=0}^{k-2} \mathcal{H}_i \bigoplus \mathrm{Ker}\,(D_{k-1})$. Thus, since

$$\begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{k-2}^{\frac{1}{2}} \\ 0 & & & & A_{k-1} \end{pmatrix} \begin{pmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{k-2} \\ \mathrm{Ker}\,(D_{k-1}) \end{pmatrix} \subseteq \begin{pmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{k-2} \\ \mathrm{Ker}\,(D_{k-1}) \end{pmatrix},$$

we must have that $A_{k-1}(\mathrm{Ker}\,(D_{k-1})) \subseteq \mathrm{Ker}\,(D_{k-1})$, proving (II_n) for $n = k$. This proves the necessity condition.

Toward sufficiency, suppose that conditions (I_n), (II_n) and (III_n) hold for all n such that $0 \le n \le k-1$. Define

$$S_n := \begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{n-2}^{\frac{1}{2}} \\ 0 & & & & A_{n-1} \end{pmatrix} \quad (1 \le n \le k-1).$$

Then S_{k-2} is weakly subnormal and $S_{k-1} = \mathrm{m.p.n.e.}\,(S_{k-2})$. Since, by assumption, $D_{k-1} \ge 0$, we have $[S_{k-1}^*, S_{k-1}] = \begin{pmatrix} 0 & 0 \\ 0 & D_{k-1} \end{pmatrix} \ge 0$. It thus follows from Lemma 2(ii) that S_{k-2} is 2-hyponormal. Note that $S_n = \mathrm{m.p.n.e.}\,(S_{n-1})$ for $n = 1, \cdots, k-1$ $(S_0 := A_0)$. Thus, again by Lemma 2(ii), S_{k-3} is 3-hyponormal. Now repeating this argument, we can conclude that $S_0 \equiv A_0$ is k-hyponormal. This completes the proof. □

We now present a strengthened version of Spitkovskii's subnormality criterion.

Corollary 4. *An operator $A_0 \in \mathcal{L}(\mathcal{H}_0)$ is subnormal if and only if the conditions (I_n), (II_n), and (III_n) hold for all $n \ge 0$. In this case, the minimal normal extension N of A_0 is given by*

$$N = \begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & \\ & & A_2 & \ddots \\ 0 & & & \ddots \end{pmatrix} : \bigoplus_{i=0}^{\infty} \mathcal{H}_i \;\to\; \bigoplus_{i=0}^{\infty} \mathcal{H}_i.$$

Proof. The first assertion follows from Theorem 3 together with the fact that A_0 is subnormal if and only if A_0 is k-hyponormal for all $k \geq 1$. Toward the second assertion, observe by a straightforward calculation that $N^*N = NN^*$, i.e., N is a (possibly unbounded) normal operator. For the boundedness of N, observe that

(i)
$$\begin{pmatrix} A_0 & D_0^{\frac{1}{2}} & & & 0 \\ & A_1 & D_1^{\frac{1}{2}} & & \\ & & \ddots & \ddots & \\ & & & \ddots & D_{n-1}^{\frac{1}{2}} \\ 0 & & & & A_n \end{pmatrix} = \widehat{A_0}^{(n)};$$

(ii) $||A_0|| = ||\widehat{A_0}^{(n)}||$ for all $n \geq 1$ (see [CJP, Corollary 3.3]).

Thus we can see that $||A_n||, ||D_{n-1}^{\frac{1}{2}}|| \leq ||A_0||$ for all $n \geq 1$. Therefore evidently, N is bounded. For the minimality of N we must show that

$$\bigoplus_{i=0}^{\infty} \mathcal{H}_i = \bigvee \{N^{*k} h_0 : \ h_0 \in \mathcal{H}_0, \ k \geq 0\}. \tag{4.1}$$

To show this it will suffice to prove that

$$\bigoplus_{i=0}^{n} \mathcal{H}_i = \bigvee \left\{ (\widehat{A_0}^{(n)})^{*k} h_0 : \ h_0 \in \mathcal{H}_0, \ k = 0, 1, \cdots, n \right\} \quad \text{for all } n \geq 1. \tag{4.2}$$

We use induction on n. Since $\widehat{A_0} = \text{m.p.n.e.}\,(A_0)$ it follows from (2.1) that

$$\mathcal{H}_0 \oplus \mathcal{H}_1 = \bigvee \{\widehat{A_0}^{*k} h_0 : \ h_0 \in \mathcal{H}_0, \ k = 0, 1\},$$

which proves (4.2) for $n = 1$. We assume that (4.2) holds for $n = m$. Note that

$$\left(\widehat{A_0}^{(m+1)}\right)^{*(m+1)} = \begin{pmatrix} * & * \\ D_m^{\frac{1}{2}} D_{m-1}^{\frac{1}{2}} \cdots D_0^{\frac{1}{2}} & * \end{pmatrix} : \begin{pmatrix} \mathcal{H}_0 \\ \oplus_{i=1}^{m+1} \mathcal{H}_i \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_0 \\ \oplus_{i=1}^{m+1} \mathcal{H}_i \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} &\bigvee \left\{ \left(\widehat{A_0}^{(m+1)}\right)^{*k} h_0 : \ h_0 \in \mathcal{H}_0, \ k = 0, 1, \cdots, m+1 \right\} \\ &= \bigvee \left\{ \left\{ \left(\widehat{A_0}^{(m+1)}\right)^{*k} h_0 : \ h_0 \in \mathcal{H}_0, \ k = 0, 1, \cdots, m \right\}, \ \left(\widehat{A_0}^{(m+1)}\right)^{*(m+1)} (\mathcal{H}_0) \right\} \\ &= \bigvee \left\{ \bigoplus_{i=0}^{m} \mathcal{H}_i, \ \left(\widehat{A_0}^{(m+1)}\right)^{*(m+1)} (\mathcal{H}_0) \right\} \\ &= \bigvee \left\{ \left(\bigoplus_{i=0}^{m} \mathcal{H}_i \right) \bigoplus D_m^{\frac{1}{2}} D_{m-1}^{\frac{1}{2}} \cdots D_0^{\frac{1}{2}} (\mathcal{H}_0) \right\} = \bigoplus_{i=0}^{m+1} \mathcal{H}_i, \end{aligned}$$

which proves (4.2) for $n = m + 1$. This completes the proof. □

It is interesting to note that Corollary 4 is a strengthened version of Spitkovskii's criterion in the sense that Spitkovskii's criterion requires the uniform boundedness of $\{A_n\}$ and $\{D_n\}$ for subnormality of A_0, while our criterion drops this condition. In fact, the uniform boundedness condition follows automatically from the construction of the minimal partially normal extension via weak subnormality as we saw in the proof of Corollary 4. This is an advantage of using weak subnormality.

2. Gaps between k-hyponormality and Subnormality for Toeplitz Operators

Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \cdots\}$. An element $f \in L^2(\mathbb{T})$ is said to be analytic if $f \in H^2(\mathbb{T})$, and co-analytic if $f \in L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. If P denotes the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$ the operator T_φ on $H^2(\mathbb{T})$ defined by

$$T_\varphi g := P(\varphi g) \quad (g \in H^2(\mathbb{T}))$$

is called the *Toeplitz operator* with symbol φ.

It was recently shown in [CLL] that there exists a gap between 2-hyponormality and subnormality for Toeplitz operators. Subsequently, C. Gu [Gu] gave a more general result: there exists a gap between k-hyponormality and $(k+1)$-hyponormality for Toeplitz operators for each $k \geq 1$.

Theorem 5 ([Gu, Theorem 3.5]). *Let $0 < \alpha < 1$ and let ψ be the conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. Let $\varphi = \psi + \lambda\bar{\psi}$ and let T_φ be the corresponding Toeplitz operator on H^2. Then T_φ is k-hyponormal if and only if λ is in the circle $\left|z - \frac{\alpha(1-\alpha^{2j})}{1-\alpha^{2j+2}}\right| = \frac{\alpha^j(1-\alpha^2)}{1-\alpha^{2j+2}}$ for $j = 0, 1, \cdots, k-2$ or in the closed disk $\left|z - \frac{\alpha(1-\alpha^{2(k-1)})}{1-\alpha^{2k}}\right| \leq \frac{\alpha^{k-1}(1-\alpha^2)}{1-\alpha^{2k}}$.*

Gu's proof of Theorem 5 relies on intricate and explicit computations using a special case of Smul'jan's Theorem [Smu]. We here give a simple proof using our criterion (Theorem 3). This illustrates that our criterion on k-hyponormality is more effective than the positivity conditions (0.3) or (0.4).

For $0 < \alpha < 1$, let $T \equiv W_\beta$ be the weighted shift with weight sequence $\beta = \{\beta_n\}_{n=0}^\infty$, where (cf. [Cow2, Proposition 9])

$$\beta_n := (\sum_{j=0}^{n} \alpha^{2j})^{\frac{1}{2}} \quad \text{for } n = 0, 1, \cdots. \tag{5.1}$$

Let D be the diagonal operator, $D = \operatorname{diag}(\alpha^n)$, and let $S_\lambda \equiv T + \lambda T^*$ $(\lambda \in \mathbb{C})$. Then we have that

$$[T^*, T] = D^2 = \operatorname{diag}(\alpha^{2n}) \quad \text{and} \quad [S_\lambda^*, S_\lambda] = (1 - |\lambda|^2)[T^*, T] = (1 - |\lambda|^2)D^2.$$

Define

$$A_l := \alpha^l T + \frac{\lambda}{\alpha^l} T^* \quad (l = 0, \pm 1, \pm 2, \cdots).$$

It follows that $A_0 = S_\lambda$ and

$$DA_l = A_{l+1}D \quad \text{and} \quad A_l^* D = DA_{l+1}^* \quad (l = 0, \pm 1, \pm 2, \cdots). \tag{5.2}$$

The following theorem is the essence of C. Gu's argument [Gu], which consists of a complicated computation. By contrast our proof is shorter and more transparent.

Theorem 6 ([Gu, Theorem 3.3]). *Let $0 < \alpha < 1$ and $T \equiv W_\beta$ be the weighted shift with weight sequence $\beta = \{\beta_n\}_{n=0}^\infty$, where*

$$\beta_n = (\sum_{j=0}^{n} \alpha^{2j})^{\frac{1}{2}} \quad \textit{for } n = 0, 1, \cdots.$$

Then $A_0 := T + \lambda T^*$ is k-hyponormal if and only if $|\lambda| \le \alpha^{k-1}$ or $|\lambda| = \alpha^j$ for some $j = 0, 1, \cdots, k-2$.

Proof. Observe that

$$\begin{aligned} [A_l^*, A_l] &= [\alpha^l T^* + \frac{\overline{\lambda}}{\alpha^l} T,\ \alpha^l T + \frac{\lambda}{\alpha^l} T^*] \\ &= \alpha^{2l}[T^*, T] - \frac{|\lambda|^2}{\alpha^{2l}}[T^*, T] = \left(\alpha^{2l} - \frac{|\lambda|^2}{\alpha^{2l}}\right) D^2. \end{aligned} \tag{6.1}$$

Since $\operatorname{Ker} D = \{0\}$ and $DA_n = A_{n+1}D$, it follows that $\mathcal{H}_n = \mathcal{H}$ for all n; if we use A_l for the operator A_n in Theorem 3 then we have, by (6.1) and the definition of D_j, that

$$\begin{aligned} D_j &= D_{j-1} + [A_j^*, A_j] = D_{j-2} + [A_{j-1}^*, A_{j-1}] + [A_j^*, A_j] = \cdots \\ &= [A_0^*, A_0] + [A_1^*, A_1] + \cdots + [A_j^*, A_j] = (1 - |\lambda|^2)D^2 + \cdots + \left(\alpha^{2j} - \frac{|\lambda|^2}{\alpha^{2j}}\right) D^2 \\ &= \left(\frac{1 - \alpha^{2(j+1)}}{1 - \alpha^2}\right)\left(1 - \frac{|\lambda|^2}{\alpha^{2j}}\right) D^2. \end{aligned}$$

By Theorem 3, A_0 is k-hyponormal if and only if $D_{k-1} \ge 0$ or $D_j = 0$ for some j such that $0 \le j \le k-2$ (in this case A_0 is subnormal). Note that $D_j = 0$ if and only if $|\lambda| = \alpha^j$. On the other hand, if $D_j > 0$ for $j = 0, 1, \cdots, k-2$, then

$$D_{k-1} = \left(\frac{1 - \alpha^{2k}}{1 - \alpha^2}\right)\left(1 - \frac{|\lambda|^2}{\alpha^{2(k-1)}}\right) D^2 \ge 0$$

if and only if $|\lambda| \le \alpha^{k-1}$. Therefore A_0 is k-hyponormal if and only if $|\lambda| \le \alpha^{k-1}$ or $|\lambda| = \alpha^j$ for some j, $j = 0, 1, \cdots, k-2$. □

We are ready for:

Proof of Theorem 5. It was shown in [CoL] that $T_{\psi + \alpha\bar{\psi}}$ is unitarily equivalent to $(1-\alpha^2)^{\frac{3}{2}} T$, where T is the weighted shift in Theorem 6. Thus T_ψ is unitarily equivalent to $(1 - \alpha^2)^{\frac{1}{2}}(T - \alpha T^*)$, so T_φ is unitarily equivalent to

$$(1 - \alpha^2)^{\frac{1}{2}}(1 - \lambda\,\alpha)(T + \frac{\lambda - \alpha}{1 - \lambda\,\alpha} T^*) \quad \text{(cf. [Cow1, Theorem 2.4])}.$$

Applying Theorem 6 with $\frac{\lambda-\alpha}{1-\lambda\,\alpha}$ in place of λ, we have that for $k = 0, 1, 2, \cdots$,

$$\begin{aligned}\left|\frac{\lambda-\alpha}{1-\lambda\,\alpha}\right| \le \alpha^k &\Longleftrightarrow |\lambda-\alpha|^2 \le \alpha^{2k}|1-\lambda\,\alpha|^2 \\ &\Longleftrightarrow |\lambda|^2 - \frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}(\lambda+\bar{\lambda}) + \frac{\alpha^2-\alpha^{2k}}{1-\alpha^{2k+2}} \le 0 \\ &\Longleftrightarrow \left|\lambda - \frac{\alpha(1-\alpha^{2k})}{1-\alpha^{2k+2}}\right| \le \frac{\alpha^k(1-\alpha^2)}{1-\alpha^{2k+2}}.\end{aligned}$$

This completes the proof. □

3. k-Hyponormality of Weighted Shifts

If A_0 is a weighted shift then by Theorem 3 and the remarks following Theorem 1 we know that the following are equivalent:

(i) A_0 is k-hyponormal;
(ii) conditions (I), (II), (III), given right after (0.5), hold for all n, $0 \le n \le k-1$.

In [Cu1, Theorem 4], it was shown that the k-hyponormality for a weighted shift W_α with $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ can be tested by the positivity of a $(k+1)\times(k+1)$ Hankel matrix $A(n;k)$ built in terms of the *moments* γ_n of W_α, where

$$\beta_0 := 1, \quad \beta_{n+1} := \alpha_n\beta_n \ (n \ge 0), \quad \text{and} \quad \gamma_n := \beta_n^2 \ (n \ge 0)$$

and

$$A(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \ge 0).$$

Thus we have:

Corollary 7 (k-Hyponormality of Weighted Shifts). *Let A_0 be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$. Then the following are equivalent:*

(i) *A_0 is k-hyponormal;*
(ii) *$A(n;k) \ge 0$ for all $n \ge 0$;*
(iii) *conditions* (I), (II) *and* (III) *hold for all n, $0 \le n \le k-1$.*

Sometimes the equivalence (i) ⇔ (iii) is more helpful than the equivalence (i) ⇔ (ii). The following example illustrates this fact.

Example 8. *For $x > 0$, let T_x be the weighted shift whose weight sequence is given by*

$$\alpha_0 := x, \quad \alpha_n = \sqrt{\frac{n+1}{n+2}} \quad (n \ge 1).$$

Then T_x is k-hyponormal if and only if $x \le \frac{k+1}{\sqrt{2k(k+2)}}$. In particular, T_x is subnormal if and only if $x \le \sqrt{\frac{1}{2}}$; that is, the Bergman shift is extremal amongst all subnormal weighted shifts of the form T_x.

Proof. We use the equivalence (i) ⇔ (iii) in Corollary 7. Write $a_j^{(0)} := \alpha_j$ $(j \ge 0)$. First, observe that

$$T_x \text{ is } k\text{-hyponormal} \iff (b_j^{(k)})^2 := (a_j^{(k-1)})^2 - (a_{j-1}^{(k-1)})^2 + (b_j^{(k-1)})^2 \ge 0 \text{ for all } j \ge 0.$$

Since T_x has, beginning with $n = 1$, a Bergman tail, it follows that

$$T_x \text{ is } k\text{-hyponormal} \iff (b_k^{(k)})^2 := (a_k^{(k-1)})^2 - (a_{k-1}^{(k-1)})^2 + (b_k^{(k-1)})^2 \geq 0.$$

We now claim that

$$(b_k^{(k)})^2 = \frac{k}{2(2k+1)} \cdot \frac{(k+1)^2 - 2k(k+2)x^2}{k^2 - 2(k-1)(k+1)x^2}. \tag{8.1}$$

Note that for $j \geq k+1$

$$(b_j^{(k)})^2 = \frac{k^2}{(j+k)(j+k+1)} \quad \text{and} \quad (a_j^{(k)})^2 = \frac{(j+1)^2}{(j+k+1)(j+k+2)}. \tag{8.2}$$

For (8.1) we use induction on k. If $k = 1$ then $(b_1^{(1)})^2 = (a_1^{(0)})^2 - (a_0^{(0)})^2 = \frac{2}{3} - x^2$, so (8.1) holds for $k = 1$. Suppose that (8.1) holds for k. Then by (8.2),

$$\begin{aligned}
(b_{k+1}^{(k+1)})^2 &= (a_{k+1}^{(k)})^2 - (a_k^{(k)})^2 + (b_{k+1}^{(k)})^2 \\
&= (a_{k+1}^{(k)})^2 - (a_k^{(k-1)})^2 \frac{(b_{k+1}^{(k)})^2}{(b_k^{(k)})^2} + (b_{k+1}^{(k)})^2 \\
&= (a_{k+1}^{(k)})^2 - (b_{k+1}^{(k)})^2 \left(\frac{(a_k^{(k-1)})^2 - (b_k^{(k)})^2}{(b_k^{(k)})^2} \right) \\
&= (a_{k+1}^{(k)})^2 - (b_{k+1}^{(k)})^2 \left(\frac{2(2k+1)x^2}{k^2\{(k+1)^2 - 2k(k+2)x^2\}} \right) \\
&= \frac{(k+2)^2}{(2k+2)(2k+3)} - \frac{k^2}{(2k+1)(2k+2)} \cdot \frac{2(2k+1)x^2}{k^2\{(k+1)^2 - 2k(k+2)x^2\}} \\
&= \frac{(k+1)\{(k+2)^2 - 2(k+1)(k+3)x^2\}}{2(2k+3)\{(k+1)^2 - 2k(k+2)x^2\}},
\end{aligned}$$

which proves (8.1). Therefore T_x is k-hyponormal $\iff x \leq \frac{k+1}{\sqrt{2k(k+2)}}$. □

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