WEAK SUBNORMALITY OF OPERATORS

By

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Abstract. We consider the gap between weak subnormality and 2–hyponormality for Toeplitz operators. In addition, we study the spectrum of the minimal partially normal extension of a weakly subnormal operator, and the inverse of an invertible weakly subnormal operator.

Introduction. In [15], the notion of weak subnormality of an operator was introduced as a generalization of subnormality, with an aim at providing a model for 2–hyponormal operators. Weak subnormality was conceived as a notion at least as strong as hyponormality, and as a tool to understand the gap between hyponormality and 2–hyponormality; however, it remains open whether every 2–hyponormal operator is weakly subnormal. In this paper we explore weak subnormality of operators.

In Section 1, we consider the gap between weak subnormality and 2–hyponormality for Toeplitz operators. In Section 2, we consider the passage from the spectrum of a weakly subnormal operator whose self-commutator has closed range to the spectrum of its minimal partially normal extension. In Section 3, we provide an example of an invertible subnormal operator whose inverse is neither 2–hyponormal nor weakly subnormal.

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Thus the operator T is subnormal if and only if there exist operators A and B such that $\widehat{T} := \left(\begin{smallmatrix} T & A \\ 0 & B \end{smallmatrix}\right)$ is normal, i.e.,

$$\left\{ \begin{array}{l} [T^*,T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*,B] + A^*A = 0. \end{array} \right. \tag{0.1}$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly subnormal* if there exist operators $A \in \mathcal{L}(\mathcal{H}',\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}')$ such that the first two conditions in (0.1) hold: $[T^*,T] = AA^*$ and $A^*T = BA^*$, or equivalently, there is an extension $\widehat{T}$ of T such that

$$\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f \quad \text{for all } f \in \mathcal{H}. \tag{0.2}$$

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The operator $\widehat{T}$ is said to be a *partially normal extension* (briefly, p.n.e.) of T. Note that the condition (0.2) implies $||\widehat{T}f|| = ||\widehat{T}^*f||$ for all $f \in \mathcal{H}$, and that if (0.2) holds for all $f \in \mathcal{H} \oplus \mathcal{H}'$, then $\widehat{T}$ becomes normal, so T is subnormal. We also say that $\widehat{T} \in \mathcal{L}(\mathcal{K})$ is a *minimal partially normal extension* (briefly, m.p.n.e.) of a weakly subnormal operator T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a partially normal extension of T. It is known ([15, Lemma 2.5 and Corollary 2.7]) that $\widehat{T} = \text{m.p.n.e.}(T)$ if and only if $\mathcal{K} = \bigvee\{\widehat{T}^{*n}h : \ h \in \mathcal{H}, \ n = 0, 1\}$ and that m.p.n.e.(T) is unique. Clearly,

$$\text{subnormal} \implies \text{weakly subnormal} \implies \text{hyponormal.} \tag{0.3}$$

However the converse of both implications in (0.3) are not true in general (cf. [15, Examples 4.1 and 4.4]). On the other hand, it is easy to see that weak subnormality is invariant under unitary equivalence, translation, and restriction.

An alternative description of subnormality is given by the Bram–Halmos criterion, which states that an operator T is subnormal if and only if $\sum_{i,j}(T^ix_j, T^jx_i) \geq 0$ for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([2],[4, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1). \tag{0.4}$$

Condition (0.4) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (0.4) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (0.4) for all k. If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B, and if we define T to be *k–hyponormal* whenever the $k \times k$ operator matrix $M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k$ is positive, or equivalently, the $(k+1) \times (k+1)$ operator matrix in (0.4) is positive (via the operator version of Choleski's Algorithm), then the Bram–Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k–hyponormal for every $k \geq 1$ ([18]). Recall ([1],[18],[5]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS((T, T^2, \cdots, T^k)) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e.,

$$(M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}) \geq 0 \text{ for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C} \qquad ([18]).$$

If $k = 2$ then T is said to be *quadratically hyponormal*. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general. The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([8],[9],[10],[11],[12],[14],[17],[18],[19],[21]). The study of this gap has been successful for weighted shifts.

1. The Gap between Weak Subnormality and 2–hyponormality for Toeplitz Operators. Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \cdots\}$. If P denotes the orthogonal projection from $L^2(\mathbb{T})$ to $H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$ the operator T_φ defined by $T_\varphi g := P(\varphi g)$ is called the *Toeplitz operator* with symbol φ. The gap between subnormality and 2–hyponormality for Toeplitz operators has been considered in [14], [15], [16], and [13]. We would now like to consider whether there is a gap between 2–hyponormality and weak subnormality for Toeplitz operators. We answer this in the affirmative. We first establish a theorem which provides an example of an operator for which hyponormality and weak subnormality coincide.

Theorem 1.1. *For $0 < \alpha < 1$, let $T \equiv W_\beta$ be the weighted shift with weight sequence*

$$\beta_n := \left(\sum_{j=0}^{n} \alpha^{2j}\right)^{\frac{1}{2}}. \tag{1.1.1}$$

If $S_\lambda := T + \lambda T^*$ $(\lambda \in \mathbb{C})$, then

(i) *S_λ is hyponormal if and only if $|\lambda| \le 1$;*
(ii) *S_λ is 2–hyponormal if and only if $|\lambda| = 1$ or $|\lambda| \le \alpha$;*
(iii) *S_λ is weakly subnormal if and only if $|\lambda| \le 1$;*
(iv) *S_λ is weakly subnormal with hyponormal m.p.n.e.(T) if and only if $|\lambda| = 1$ or $|\lambda| \le \alpha$.*

Proof. (i) is known [6, Theorem 2.3], and (ii) appears in [13, Theorem 5]. Thus it suffices to focus on assertions (iii) and (iv). Since $T + \lambda T^*$ is unitarily equivalent to $e^{\frac{i\theta}{2}}(T + |\lambda|\, T^*)$ with $|\lambda| = \lambda\, e^{-i\theta}$, it follows that $T + \lambda T^*$ is weakly subnormal if and only if $T + |\lambda|\, T^*$ is weakly subnormal. Thus we can assume $\lambda \ge 0$. If $\lambda = 0, 1$ then evidently S_λ is weakly subnormal because T is subnormal (cf. [7]). Thus we assume $0 < \lambda < 1$. Then a straightforward calculation shows that $[S_\lambda^*, S_\lambda] = (1 - \lambda^2)\, D$, where $D := \operatorname{diag}(\alpha^{2k})_{k=0}^\infty$. Put $A_\lambda := [S_\lambda^*, S_\lambda]^{\frac{1}{2}}$ and define $B_\lambda := \alpha\, T + \frac{\lambda}{\alpha}\, T^*$. Then we have $A_\lambda^* S_\lambda = B_\lambda A_\lambda^*$. This implies that

$$\widehat{S}_\lambda := \begin{pmatrix} S_\lambda & A_\lambda \\ 0 & B_\lambda \end{pmatrix} \tag{1.1.2}$$

is a partially normal extension of S_λ, which proves (iii). Towards (iv), observe that $\widehat{S}_\lambda$ in (1.1.2) is the minimal partially normal extension of S_λ because

$$\bigvee \{\widehat{S}_\lambda^{*n} h : \ h \in \mathcal{H}, \ n = 0, 1\} = \ell_2 \oplus \ell_2.$$

For the hyponormality of $\widehat{S}_\lambda$ we compute

$$\begin{aligned}
[\widehat{S}_\lambda^*, \widehat{S}_\lambda] &= 0 \oplus ([B_\lambda^*, B_\lambda] + A_\lambda^* A_\lambda) \\
&= 0 \oplus \left((\alpha^2 - \frac{\lambda^2}{\alpha^2}) D + (1 - \lambda^2) D \right) \\
&= 0 \oplus \left((1 + \alpha^2) - (1 + \frac{1}{\alpha^2})\lambda^2 \right) D.
\end{aligned}$$

Thus $\widehat{S}_\lambda$ is hyponormal if and only if $(1 + \frac{1}{\alpha^2})\lambda^2 \le 1 + \alpha^2$, or $\lambda \le \alpha$. This proves (iv). □

We now have:

Corollary 1.2. *Let $0 < \alpha < 1$ and let ψ be the conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. Let $\varphi_\lambda = \psi + \lambda\bar{\psi}$ and let T_{φ_λ} be the corresponding Toeplitz operator on H^2. Then*

(i) *T_{φ_λ} is hyponormal if and only if λ is in the closed unit disk $|\lambda| \le 1$.*
(ii) *T_{φ_λ} is 2–hyponormal if and only if λ is in the unit circle $|\lambda| = 1$ or in the closed disk $\left|\lambda - \frac{\alpha}{1+\alpha^2}\right| \le \frac{\alpha}{1+\alpha^2}$.*
(iii) *T_{φ_λ} is weakly subnormal if and only if λ is in the closed unit disk $|\lambda| \le 1$.*

Proof. (i) is known [6, Theorem 2.4] and (ii) appears in [13, Theorem 6]. For (iii), recall that T_{φ_α} is unitarily equivalent to $(1-\alpha^2)^{\frac{1}{2}}T$ ([7]), where T is the weighted shift in Theorem 1.1. Thus T_ψ is unitarily equivalent to $(1-\alpha^2)^{\frac{1}{2}}(T - \alpha T^*)$, so T_{φ_λ} is unitarily equivalent to

$$(1-\alpha^2)^{\frac{1}{2}}(1-\lambda\alpha)\left(T + \frac{\lambda-\alpha}{1-\lambda\alpha}T^*\right).$$

Thus by Theorem 1.1, T_{φ_λ} is weakly subnormal if and only if $\left|\frac{\lambda-\alpha}{1-\lambda\alpha}\right| \le 1$, or $|\lambda| \le 1$. $\square$

One might guess that the minimal partially normal extension of a weakly subnormal operator T is always hyponormal. Theorem 1.1 shows, however, that this is not the case.

Question A. *Is every 2–hyponormal operator weakly subnormal? If so, does it follow that its minimal partially normal extension is hyponormal?*

Theorem 1.1 provides evidence that the answer to Question A may be affirmative.

We now give a strategy to answer Question A in the affirmative.
First of all we recall two lemmas.

Lemma 1.3 ([15, Lemma 2.2 and Corollary 2.3]). *If $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal or 2–hyponormal then $T\big(ker\,[T^*, T]\big) \subseteq ker\,[T^*, T]$.*

Lemma 1.4([15, Lemmas 2.1 and 2.8]). *If $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal then T has a partially normal extension $\widehat{T}$ on $\mathcal{K}$ of the form*

$$\widehat{T} = \begin{pmatrix} T & [T^*,T]^{\frac{1}{2}} \\ 0 & B \end{pmatrix} \quad on \ \ \mathcal{K} := \mathcal{H} \oplus \mathcal{H}. \tag{1.4.1}$$

Moreover, a minimal partially normal extension of $T \equiv \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} : \ker[T^*,T] \oplus \mathrm{cl}(\mathrm{ran}\,[T^*,T]) \to \ker[T^*,T] \oplus \mathrm{cl}(\mathrm{ran}\,[T^*,T])$ can be obtained as

$$\begin{pmatrix} T_1 & T_2 & 0 \\ 0 & T_3 & [T^*,T]_0^{\frac{1}{2}} \\ 0 & 0 & PBP \end{pmatrix} : \begin{pmatrix} \ker[T^*,T] \\ \mathrm{cl}(\mathrm{ran}\,[T^*,T]) \\ \mathrm{cl}\big(\mathrm{ran}\,[T^*,T]\big) \end{pmatrix} \longrightarrow \begin{pmatrix} \ker[T^*,T] \\ \mathrm{cl}(\mathrm{ran}\,[T^*,T]) \\ \mathrm{cl}\big(\mathrm{ran}\,[T^*,T]\big) \end{pmatrix}, \tag{1.4.2}$$

where $[T^*,T]_0^{\frac{1}{2}}$ denotes the restriction of $[T^*,T]^{\frac{1}{2}}$ to $\mathrm{cl}\big(\mathrm{ran}\,[T^*,T]\big)$ and P is the projection of $\mathcal{H}$ onto $\mathrm{cl}\big(\mathrm{ran}\,[T^*,T]\big)$.

Proposition 1.5. *Let $P, T \in \mathcal{L}(\mathcal{H})$, let P be positive and one-one, and let P^{-1} denote its linear inverse (possibly unbounded). Then*

$$P^2TP^{-2} \text{ bounded } \Longrightarrow PTP^{-1} \text{ bounded.} \tag{1.5.1}$$

Proof. Without loss of generality we may assume $||P|| = 1$, and that P does not have closed range. Let E be the spectral measure for P on the Borel subsets of $\sigma(P)$ such that $P = \int t\, dE(t)$. For each $n \in \mathbb{Z}_+$, define

$$G_n := \sigma(P) \ \cap \ (\frac{1}{n+1}, \frac{1}{n}].$$

Then $G_n \neq \emptyset$ for infinitely many values of n. There is no loss in simplifying the notation and assuming that $G_n \neq \emptyset$ for all $n \in \mathbb{Z}_+$. Write $\mathcal{H}_n := E(G_n)\mathcal{H}$ for each $n \in \mathbb{Z}_+$. Then $P_n := P|_{\mathcal{H}_n}$ is invertible for each $n \in \mathbb{Z}_+$. Thus since $\sigma(P) = \cup_{n=1}^{\infty} G_n \cup \{0\}$, it follows that $\sum_{n=1}^{\infty} E(G_n) = I$. Now assume that PTP^{-1} is not bounded. Thus there exists a sequence $\{e_n\}_n$ of unit vectors $e_n \in \mathcal{H}_n$ for each $n \in \mathbb{Z}_+$ such that $||PTP^{-1}e_n|| \to \infty$ as $n \to \infty$. Since $\sum_{n=1}^{\infty} E(G_n) = I$, we can see that $\mathcal{H} = \{\sum_{j=1}^{\infty} \alpha_j h_j : \ (\alpha_j)_j \in \ell_2, \ h_j \in \mathcal{H}_j, \ ||h_j|| = 1\}$. Thus we can write, for each $n \in \mathbb{Z}_+$,

$$TP^{-1}e_n \equiv \sum_{k=1}^{\infty} \alpha_k^{(n)} h_k^{(n)} \quad \text{where } h_k^{(n)} \in \mathcal{H}_k \text{ with } ||h_k^{(n)}|| = 1.$$

Then $||\sum_{k=1}^{\infty} \alpha_k^{(n)} P(h_k^{(n)})|| = ||PTP^{-1}e_n|| \to \infty$ as $n \to \infty$. Note that $\{P(h_k^{(n)})\}_k$ forms an orthogonal sequence. Therefore $||\sum_{k=1}^{\infty} \alpha_k^{(n)} P(h_k^{(n)})||^2 = \sum_{k=1}^{\infty} |\alpha_k^{(n)}|^2 \, ||P(h_k^{(n)})||^2 \to \infty$ as $n \to \infty$. Also note that $||P(h_k^{(n)})||^2 \leq \frac{1}{k^2}$. Thus $\sum_{k=1}^{\infty}(|\alpha_k^{(n)}|/k)^2 \to \infty$ as $n \to \infty$, so that $\sum_{k=1}^{\infty}(|\alpha_k^{(n)}|/(k+1))^2 \to \infty$ as $n \to \infty$. Since the $\mathcal{H}_n$'s are reducing subspace for P, we have $f_n := Pe_n \in \mathcal{H}_n$ for each $n \in \mathbb{Z}_+$. Thus

$$P^2TP^{-2}f_n = P^2TP^{-1}e_n = P(\sum_{k=1}^{\infty} \alpha_k^{(n)} P(h_k^{(n)})) = \sum_{k=1}^{\infty} \alpha_k^{(n)} P^2(h_k^{(n)}),$$

so

$$||P^2TP^{-2}f_n||^2 = \sum_{k=1}^{\infty} |\alpha_k^{(n)}|^2 ||P^2(h_k^{(n)})||^2 \geq \sum_{k=1}^{\infty} \frac{|\alpha_k^{(n)}|^2}{(k+1)^4}$$

since $\sigma(P|_{\mathcal{H}_k}) \subset (\frac{1}{k+1}, \frac{1}{k}]$. Observe $||f_n|| = ||Pe_n|| \leq \frac{1}{n}$ and hence $||nf_n|| \leq 1$. But

$$\begin{aligned} ||P^2TP^{-2}(nf_n)||^2 \geq \sum_{k=1}^{\infty} \frac{n^2 \, |\alpha_k^{(n)}|^2}{(k+1)^4} &= n^2 \sum_{k=1}^{\infty} \frac{|\alpha_k^{(n)}|^2}{(k+1)^4} \\ &\geq n^2 \sum_{k=1}^{n-1} \frac{|\alpha_k^{(n)}|^2}{(k+1)^4} \\ &\geq \sum_{k=1}^{n-1} \left(\frac{|\alpha_k^{(n)}|}{k+1} \right)^2 \to \infty \quad \text{as } n \to \infty, \end{aligned}$$

where the last assertion follows from the observation that

$$\sum_{k=1}^{n-1} |\alpha_k^{(n)}|^2 \, ||P(h_k^{(n)})||^2 \to \infty \quad \text{as } n \to \infty. \tag{1.5.2}$$

To see (1.5.2) we assume to the contrary that (1.5.2) does not hold. Then, passing to a subsequence, there exists $M > 0$ such that $\sum_{k=1}^{n} |\alpha_k^{(n)}|^2\, ||P(h_k^{(n)})||^2 \le M$ for each $n \in \mathbb{Z}_+$, and so $\sum_{k=n+1}^{\infty} |\alpha_k^{(n)}|^2\, ||P(h_k^{(n)})||^2 \to \infty$ as $n \to \infty$. But since $\sum_{k=1}^{\infty} |\alpha_k^{(n)}|^2 = ||TP^{-1}e_n||^2 \le (n+1)^2\, ||T||^2$, it follows that

$$\sum_{k=n+1}^{\infty} |\alpha_k^{(n)}|^2\, ||P(h_k^{(n)})||^2 \le \sum_{k=n+1}^{\infty} \frac{|\alpha_k^{(n)}|^2}{k^2} \le \sum_{k=n+1}^{\infty} \frac{|\alpha_k^{(n)}|^2}{(n+1)^2} \le ||T||^2,$$

which is a contradiction. This proves (1.5.2) and completes the proof of (1.5.1). $\square$

The converse of (1.5.1) is not true in general. For example, let T be the Hankel operator H_φ on $H^2(\mathbb{T})$ with symbol $\varphi(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n^2}$ and let P be the diagonal operator on $H^2(\mathbb{T})$ defined by $Pz^n = \frac{z^n}{n+1}$. Then a straightforward calculation shows that the first row of the matrix of P^2TP^{-2} is $(1, 1, 1, \cdots)$, which shows that P^2TP^{-2} is not bounded. On the other hand, the matrix of TP^{-1} is dominated by the matrix of Hankel operator T_ψ with symbol $\psi(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n}$, which shows that TP^{-1} is bounded and so is PTP^{-1}.

We would like to formulate:

Conjecture B. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $A := [T^*, T]^{\frac{1}{2}}$. Suppose P is the projection of $\mathcal{H}$ onto* $\mathrm{cl}\,(\mathrm{ran}\, A)$ *and write $A_0 := A|_{\mathrm{cl}\,(\mathrm{ran}\, A)}$. If T is 2–hyponormal then $A_0^2 PTPA_0^{-2}$ is bounded.*

If Conjecture B is true then the answer to Question A *would* be affirmative. The reason is as follows. If Conjecture B is true then by Proposition 1.5, $A_0PTPA_0^{-1}$ is also bounded. Thus if we write $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} \equiv \ker A \oplus \mathrm{cl}\,(\mathrm{ran}\, A)$ then since by (1.4.2), $A_0T_3 = PBPA_0$, i.e., $PBP = A_0PTPA_0^{-1}$, it follows that

$$\widehat{T} = \begin{pmatrix} T_1 & T_2 & 0 \\ 0 & T_3 & A_0 \\ 0 & 0 & A_0PTPA_0^{-1} \end{pmatrix} : \mathcal{H} \oplus \mathrm{cl}\,(\mathrm{ran}\, A) \longrightarrow \mathcal{H} \oplus \mathrm{cl}\,(\mathrm{ran}\, A) \tag{1.5.3}$$

is a partially normal extension of T.

The following is a more simplified conjecture:

Conjecture C. *If $[T^*, T]$ has the linear inverse $[T^*, T]^{-1}$ then*

$$T \text{ is 2–hyponormal} \implies [T^*, T]\, T\, [T^*, T]^{-1} \text{ is bounded.}$$

2. The Spectrum of m.p.n.e. (T) for T Weakly Subnormal. If $T \in \mathcal{L}(\mathcal{H})$ write $\sigma(T)$ for the spectrum of T. We also write $\partial\, \mathbf{K}$ and $\eta\, \mathbf{K}$ for the topological boundary and the polynomially convex hull of the compact set $\mathbf{K} \subseteq \mathbb{C}$, respectively. It is well-known that if T is a subnormal operator and $\widehat{T}$ is a minimal normal extension of T then $\sigma(\widehat{T}) \subseteq \sigma(T)$. It was also known ([15, Theorem 1.2]) that if T is a weakly subnormal unilateral weighted shift and $\widehat{T} = \text{m.n.p.e.}\,(T)$ then $\sigma(T) = \sigma(\widehat{T})$. However we do not know yet if this result holds for general weakly subnormal operators. In this section we consider the relationship between the spectrum of a weakly subnormal operator T and the spectrum of its minimal partially normal extension $\widehat{T}$. We let $A \triangle B := (A \setminus B) \cup (B \setminus A)$ denote the symmetric difference of the sets A and B.

Lemma 2.1 ([20, Theorem 2, Corollaries 6 and 7]. *Let*

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}.$$

Then M_C is invertible for some $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ if and only if

(i) *A is left invertible;*
(ii) *B is right invertible;*
(iii) $(\operatorname{ran} A)^\perp \cong \ker B$.

Moreover we have

$$\eta(\sigma(M_C)) = \eta(\sigma(A) \cup \sigma(B)) \quad \textit{for every } C \in \mathcal{L}(\mathcal{K}, \mathcal{H}).$$

In particular, the passage from $\sigma(M_C)$ to $\sigma(A) \cup \sigma(B)$ consists of filling in certain holes in $\sigma(M_C)$ which happen to be subsets of $\sigma(A) \cap \sigma(B)$.

We have:

Theorem 2.2. *If $T \in \mathcal{L}(\mathcal{H})$ is a weakly subnormal operator whose self-commutator has closed range and $\widehat{T} = m.n.p.e.\,(T)$ then $\eta\,\sigma(T) = \eta\,\sigma(\widehat{T})$.*

Proof. Since by assumption $\operatorname{ran}[T^*, T]$ is closed and $T(\ker[T^*, T]) \subseteq \ker[T^*, T]$, we can write

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} : \begin{pmatrix} \ker[T^*, T] \\ \operatorname{ran}[T^*, T] \end{pmatrix} \to \begin{pmatrix} \ker[T^*, T] \\ \operatorname{ran}[T^*, T] \end{pmatrix}.$$

Since minimal partially normal extensions of T are unitarily equivalent ([15, Corollary 2.7]) we can write $\widehat{T}$ as in (1.5.3):

$$\widehat{T} = \begin{pmatrix} T_1 & T_2 & 0 \\ 0 & T_3 & A \\ 0 & 0 & AT_3A^{-1} \end{pmatrix} : \begin{pmatrix} \ker[T^*, T] \\ \operatorname{ran}[T^*, T] \\ \operatorname{ran}[T^*, T] \end{pmatrix} \to \begin{pmatrix} \ker[T^*, T] \\ \operatorname{ran}[T^*, T] \\ \operatorname{ran}[T^*, T] \end{pmatrix}, \tag{2.2.1}$$

where $A := [T^*, T]_0^{\frac{1}{2}}$ and in particular A is invertible by assumption. Using Lemma 2.1 we obtain

$$\eta\,\sigma(\widehat{T}) = \eta(\sigma(T) \cup \sigma(AT_3A^{-1})) = \eta(\sigma(T) \cup \sigma(T_3)) = \eta\,\sigma(T).$$

□

The following theorem provides the passage from $\sigma(T)$ to $\sigma(\widehat{T})$ under the assumption that $[T^*, T]$ has closed range.

Theorem 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a weakly subnormal operator whose self-commutator has closed range and $\widehat{T} = m.p.n.e.\,(T)$. If we write*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} : \begin{pmatrix} \ker[T^*, T] \\ \operatorname{ran}[T^*, T] \end{pmatrix} \to \begin{pmatrix} \ker[T^*, T] \\ \operatorname{ran}[T^*, T] \end{pmatrix},$$

then $\sigma(T) \triangle \sigma(\widehat{T})$ is the union of certain holes in $\sigma(T)$ or $\sigma(\widehat{T})$, which happen to be subsets of $\sigma(T_1) \cap \sigma(T_3)$. In particular, if T has finite rank self-commutator then $\sigma(T) = \sigma(\widehat{T})$.

Proof. Write $\widehat{T}$ as in (2.2.1):

$$\widehat{T} = \begin{pmatrix} T_1 & T_2 & 0 \\ 0 & T_3 & A \\ 0 & 0 & AT_3A^{-1} \end{pmatrix}.$$

Evidently,

$$\sigma(\widehat{T}) \subseteq \sigma(T_1) \cup \sigma(T_3) \cup \sigma(AT_3A^{-1}) = \sigma(T_1) \cup \sigma(T_3). \tag{2.3.1}$$

By Theorem 2.2 we can see that

$$\partial\,(\sigma(T_1) \cup \sigma(T_3)) = \partial(\sigma(T) \cup \sigma(T_3)) = \partial(\sigma(T) \cup \sigma(AT_3A^{-1})) \subseteq \partial\,\sigma(\widehat{T}). \tag{2.3.2}$$

We now claim that

$$\sigma(T) \triangle \sigma(\widehat{T}) \subseteq \sigma(T_1) \cap \sigma(T_3). \tag{2.3.3}$$

Towards (2.3.3) we first prove that $\sigma(T) \setminus \sigma(\widehat{T}) \subseteq \sigma(T_1) \cap \sigma(T_3)$. Suppose that $\lambda \in \sigma(T) \setminus \sigma(\widehat{T})$. Assume to the contrary that $\lambda \notin \sigma(T_1) \cap \sigma(T_3)$. If $T_1 - \lambda$ is invertible then, since $\widehat{T} - \lambda$ is invertible, it follows from Lemma 2.1 that $\begin{pmatrix} T_3-\lambda & A \\ 0 & A(T_3-\lambda)A^{-1} \end{pmatrix}$ is invertible, and so $T_3 - \lambda$ is left invertible and $A(T_3 - \lambda)A^{-1}$ is right invertible, which implies that $T_3 - \lambda$ is invertible and hence so is $T - \lambda$, a contradiction. If instead $T_3 - \lambda$ is invertible then $\begin{pmatrix} T_3-\lambda & A \\ 0 & A(T_3-\lambda)A^{-1} \end{pmatrix}$ is also invertible, so that Lemma 2.1 again shows that $T_1 - \lambda$ is invertible and hence so is $T - \lambda$, a contradiction.

We next prove that $\sigma(\widehat{T}) \setminus \sigma(T) \subseteq \sigma(T_1) \cap \sigma(T_3)$. Suppose that $\lambda \in \sigma(\widehat{T}) \setminus \sigma(T)$. Assume that $\lambda \notin \sigma(T_1) \cap \sigma(T_3)$. Thus $T_1 - \lambda$ or $T_3 - \lambda$ is invertible. But since $T - \lambda$ is invertible it follows that $T_1 - \lambda$ and $T_3 - \lambda$ are both invertible, and hence so is $\widehat{T} - \lambda$, a contradiction. This proves (2.3.3).

On the other hand, Lemma 2.1 shows that the passage from $\sigma(T)$ to $\sigma(T_1) \cup \sigma(T_3)$ is the filling of certain holes in $\sigma(T)$, which happen to be subsets of $\sigma(T_1) \cap \sigma(T_3)$. Moreover, by (2.3.1), (2.3.2), and (2.3.3), the passage from $\sigma(\widehat{T})$ to $\sigma(T_1) \cup \sigma(T_3)$ is the filling of certain holes in $\sigma(T_1) \cup \sigma(T_3)$, which happen to be subsets of $\sigma(T_1) \cap \sigma(T_3)$. This proves the first assertion. The second assertion follows from the first together with the observation that if $\sigma(T_1) \cap \sigma(T_3)$ has no interior points then $\sigma(T) = \sigma(\widehat{T})$ and that if $\operatorname{rank}[T^*, T] < \infty$ then $\operatorname{ran}[T^*, T]$ is finite dimensional and hence T_3 is a finite dimensional operator. $\square$

In [15, Lemma 2.1] it was shown that if T is a 2–hyponormal operator whose self-commutator has closed range then T is weakly subnormal. Thus Theorem 2.2 and Theorem 2.3 also hold for 2–hyponormal operators.

In general we need not expect that if T is a weakly subnormal operator and $\widehat{T} = \text{m.p.n.e}\,(T)$ then $\sigma(\widehat{T}) \subseteq \sigma(T)$. To see this we use the bilateral weighted shift. Let $\{e_n\}_{n=-\infty}^{+\infty}$ denote the canonical orthonormal basis for $\ell^2(\mathbb{Z})$. For a bounded sequence of positive numbers $\alpha \equiv \{\alpha_n\}_{n=-\infty}^{+\infty}$, let U_α be the bilateral weighted shift on $\ell^2(\mathbb{Z})$ defined by $U_\alpha e_n := \alpha_n e_{n+1}$ $(-\infty < n < +\infty)$. Then we have:

Proposition 2.4 (Weak Subnormality of Bilateral Weighted Shifts). *Let U_α be the bilateral weighted shift with strictly increasing weight sequence $\alpha \equiv \{\alpha_n\}_{n=-\infty}^{+\infty}$. Then*

$$U_\alpha \text{ is weakly subnormal} \iff \sup_{n\in\mathbb{Z}} \left\{ \frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2} \right\} < \infty. \tag{2.4.1}$$

Proof. If U_α is weakly subnormal then by Lemma 1.4, U_α has a partially normal extension $\widehat{U}_\alpha$ of the form

$$\widehat{U}_\alpha = \begin{pmatrix} U_\alpha & [U_\alpha^*, U_\alpha]^{\frac{1}{2}} \\ 0 & B \end{pmatrix} \tag{2.4.2}$$

on $\ell_2 \oplus \ell_2$, where $[U_\alpha^*, U_\alpha]^{\frac{1}{2}} U_\alpha = B\,[U_\alpha^*, U_\alpha]^{\frac{1}{2}}$. Since $[U_\alpha^*, U_\alpha]^{\frac{1}{2}}$ is diagonal it follows that B is a bilateral weighted shift. Write $B := U_\beta$, where $\beta \equiv \{\beta_n\}_{n=-\infty}^{\infty}$. To determine U_β, it suffices to check the $(n+1, n)$–entries ($n \in \mathbb{Z}$):

$$\Big([U_\alpha^*, U_\alpha]^{\frac{1}{2}} U_\alpha e_n,\ e_{n+1}\Big) = \Big(U_\beta [U_\alpha^*, U_\alpha]^{\frac{1}{2}} e_n,\ e_{n+1}\Big),$$

which implies

$$\alpha_n \sqrt{\alpha_{n+1}^2 - \alpha_n^2} = \beta_n \sqrt{\alpha_n^2 - \alpha_{n-1}^2}, \tag{2.4.3}$$

so

$$\beta_n = \alpha_n \sqrt{\frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2}}.$$

But since U_β is bounded it follows that $\sup \beta_n < \infty$ and hence $\sup \frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2} < \infty$. This proves the forward implication in (2.4.1). The backward implication follows at once from the observation that $\widehat{U}_\alpha$ in (2.4.2) is a partially normal extension of U_α. $\square$

Example 2.5. Let U_α be the bilateral weighted shift with weight sequence given by

$$\alpha_n := \begin{cases} \left(\sum_{k=0}^{n} \delta_k\right)^{\frac{1}{2}} & (n \geq 0) \\ \left(\delta_0 - \sum_{k=1}^{-n} \delta_{-k}\right)^{\frac{1}{2}} & (n < 0), \end{cases}$$

where

$$\delta_0 = 2, \quad \delta_k = \frac{1}{2^k}\ (k \geq 1), \quad \text{and}$$

$$\{\delta_{-k}\}_{k=1}^{\infty} : \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^3}, \frac{1}{2^4}, \cdots, \frac{1}{2^9}, \frac{1}{2^4}, \frac{1}{2^5}, \cdots, \frac{1}{2^{11}}, \cdots\cdots.$$

Note that α is strictly increasing, and so U_α is hyponormal. A straightforward calculation shows that

$$\lim_{n\to\infty} \alpha_n = \sqrt{3} \quad \text{and} \quad \lim_{n\to-\infty} \alpha_n = \sqrt{\frac{13}{24}},$$

which implies that $\sigma(U_\alpha) = \{z \in \mathbb{C} : \sqrt{\frac{13}{24}} \leq |z| \leq \sqrt{3}\}$ (cf. [4, Theorem II.6.7]). Also note that $\delta_{n+1} = \alpha_{n+1}^2 - \alpha_n^2$ ($N \in \mathbb{Z}$) and that $\sup_{n\in\mathbb{Z}_+} \frac{\delta_{n+1}}{\delta_n} = \frac{1}{2}$ and $\sup_{n\in\mathbb{Z}_-} \frac{\delta_{n+1}}{\delta_n} = 2$, which together with Proposition 2.4 implies that U_α is weakly subnormal. On the other hand, if $\widehat{U_\alpha} = \text{m.p.n.e.}\,(U_\alpha)$ then, by the proof of Proposition 2.4, $\widehat{U_\alpha}$ is given by

$$\widehat{U_\alpha} = \begin{pmatrix} U_\alpha & [U_\alpha^*, U_\alpha]^{\frac{1}{2}} \\ 0 & U_\beta \end{pmatrix},$$

where U_β is the bilateral weighted shift with the weight sequence

$$\beta_n := \alpha_n \sqrt{\frac{\delta_{n+1}}{\delta_n}} \quad (n \in \mathbb{Z}).$$

Observe that the sequence $\left\{\frac{\delta_{n+1}}{\delta_n}\right\}$ contains a subsequence which converges to 0 and so ran U_β is not closed. Therefore U_β is not right invertible, and hence $0 \in \sigma(\widehat{U_\alpha})$. Therefore $\sigma(\widehat{U_\alpha}) \nsubseteq \sigma(U_\alpha)$.

3. The Inverse of an Invertible Weakly Subnormal Operator. It is well known that if T is an invertible hyponormal operator then T^{-1} is also hyponormal. One might ask if the inverse of an invertible weakly subnormal operator is weakly subnormal. However, there exist invertible subnormal operators whose inverses are neither 2-hyponormal nor weakly subnormal. In this section we provide such an example. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathfrak{N}_T := \ker[T^*, T]$.

Proposition 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an invertilbe operator such that $T\mathfrak{N}_T \subseteq \mathfrak{N}_T$. If $T^{-1}\mathfrak{N}_{T^{-1}} \subseteq \mathfrak{N}_{T^{-1}}$ then $T\mathfrak{N}_T = \mathfrak{N}_T$.*

Proof. Observe that

$$[T^{-1*}, T^{-1}] = (T^*T)^{-1}[T^*, T](TT^*)^{-1},$$

so

$$\mathfrak{N}_{T^{-1}} = TT^*\mathfrak{N}_T = T^*T\mathfrak{N}_T.$$

Then

$$\begin{aligned}
\mathfrak{N}_T &= (TT^*)^{-1}\mathfrak{N}_{T^{-1}} \\
&= T^{*-1}T^{-1}\mathfrak{N}_{T^{-1}} \\
&\subseteq T^{*-1}\mathfrak{N}_{T^{-1}} \\
&= T^{*-1}T^*T\mathfrak{N}_T \\
&= T\mathfrak{N}_T \subseteq \mathfrak{N}_T,
\end{aligned}$$

from which it follows that $T\mathfrak{N}_T = \mathfrak{N}_T$. □

Corollary 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator. If T and T^{-1} are both 2-hyponormal or weakly subnormal then $T(ker[T^*, T]) = ker[T^*, T]$.*

Proof. This follows from Proposition 3.1 and Lemma 1.3. □

We can now present the above mentioned example.

Example 3.3. Let θ be a nonconstant inner function and put $\varphi = \theta + 2$. Then T_φ is an invertible subnormal operator (consequently 2-hyponormal and weakly subnormal). Since $[T_\varphi^*, T_\varphi] = [T_\theta^*, T_\theta] = H_{\bar\theta}^* H_{\bar\theta}$, we have $\mathfrak{N}_{T_\varphi} \equiv \ker[T_\varphi^*, T_\varphi] = \ker H_{\bar\theta} = \theta H^2$. But $T_\varphi\mathfrak{N}_{T_\varphi} = T_{\theta+2}(\theta H^2) = \theta(\theta+2)H^2$. Thus if $T_\varphi\mathfrak{N}_{T_\varphi} = \mathfrak{N}_{T_\varphi}$ then $\theta(\theta+2) = c\,\theta$ for some constant c, and hence $\theta + 2 = c$, which contradicts the assumption that θ is nonconstant. Therefore we must have that $T_\varphi\mathfrak{N}_{T_\varphi} \neq \mathfrak{N}_{T_\varphi}$, which by Corollary 3.2, implies that T_φ^{-1} is neither 2-hyponormal nor weakly subnormal.

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