TOWARDS A MODEL THEORY FOR 2–HYPONORMAL OPERATORS

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We introduce the notion of *weak subnormality*, which generalizes subnormality in the sense that for the extension $\widehat{T} \in \mathcal{L}(\mathcal{K})$ of $T \in \mathcal{L}(\mathcal{H})$ we only require that $\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f$ hold for $f \in \mathcal{H}$; in this case we call $\widehat{T}$ a partially normal extension of T. After establishing some basic results about weak subnormality (including those dealing with the notion of minimal partially normal extension), we proceed to characterize weak subnormality for weighted shifts and to prove that 2-hyponormal weighted shifts are weakly subnormal. Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ be a weight sequence and let W_α denote the associated unilateral weighted shift on $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+)$. If W_α is 2-hyponormal then W_α is weakly subnormal. Moreover, there exists a partially normal extension $\widehat{W}_\alpha$ on $\mathcal{K} := \mathcal{H} \oplus \mathcal{H}$ such that (i) $\widehat{W}_\alpha$ is hyponormal; (ii) $\sigma(\widehat{W}_\alpha) = \sigma(W_\alpha)$; and (iii) $||\widehat{W}_\alpha|| = ||W_\alpha||$. In particular, if α is strictly increasing then $\widehat{W}_\alpha$ can be obtained as

$$\widehat{W}_\alpha = \begin{pmatrix} W_\alpha & [W_\alpha^*, W_\alpha]^{\frac{1}{2}} \\ 0 & W_\beta \end{pmatrix} \qquad \text{on } \ \mathcal{K} := \mathcal{H} \oplus \mathcal{H},$$

where W_β is a weighted shift whose weight sequence $\{\beta_n\}_{n=0}^\infty$ is given by

$$\beta_n := \alpha_n \sqrt{\frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2}} \quad (n = 0, 1, \cdots ; \ \alpha_{-1} := 0).$$

In this case, $\widehat{W}_\alpha$ is a minimal partially normal extension of W_α. In addition, if W_α is 3-hyponormal then $\widehat{W}_\alpha$ can be chosen to be weakly subnormal. This allows us to shed new light on Stampfli's geometric construction of the minimal normal extension of a subnormal weighted shift. Our methods also yield two additional results: (i) the square of a weakly subnormal operator whose minimal partially normal extension is always hyponormal, and (ii) a 2-hyponormal operator with rank-one self-commutator is necessarily subnormal. Finally, we investigate the connections of weak subnormality and 2–hyponormality with Agler's model theory.

1 Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_\mathcal{H}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Thus the operator T is subnormal if and only if there exist

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operators A and B such that $\widehat{T} := \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ is normal, i.e.,

$$\tag{1.1} \begin{cases} [T^*, T] := T^*T - TT^* = AA^* \\ A^*T = BA^* \\ [B^*, B] + A^*A = 0. \end{cases}$$

We now introduce:

Definition 1.1. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly subnormal* if there exist operators $A \in \mathcal{L}(\mathcal{H}', \mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H}')$ such that the first two conditions in (1.1) hold: $[T^*, T] = AA^*$ and $A^*T = BA^*$. The operator $\widehat{T}$ is said to be a *partially normal extension* of T.

Clearly,

$$\tag{1.2} \text{subnormal} \implies \text{weakly subnormal} \implies \text{hyponormal}.$$

The converses of both implications in (1.2) are not true in general; see Examples 4.1 and 4.4. Moreover, we can easily see that the following statements are equivalent for $T \in \mathcal{L}(\mathcal{H})$:

(i) T is weakly subnormal;
(ii) There is an extension $\widehat{T}$ of T such that $\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f$ for all $f \in \mathcal{H}$;
(iii) There is an extension $\widehat{T}$ of T such that $\mathcal{H} \subseteq \ker [\widehat{T}^*, \widehat{T}]$.

Weakly subnormal operators possess the following invariance properties:

(i) (Unitary equivalence) if T is weakly subnormal with a partially normal extension $\begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$ then for every unitary U, $\begin{pmatrix} U^*TU & U^*A \\ 0 & B \end{pmatrix}$ $(= \begin{pmatrix} U^* & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} T & A \\ 0 & B \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix})$ is a partially normal extension of U^*TU, i.e., U^*TU is also weakly subnormal.
(ii) (Translation) if $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal then $T - \lambda$ is also weakly subnormal for every $\lambda \in \mathbb{C}$: indeed if T has a partially normal extension $\widehat{T}$ then $\widehat{T - \lambda} := \widehat{T} - \lambda$ satisfies the properties in Definition 1.1.
(iii) (Restriction) if $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal and if $\mathfrak{M} \in \operatorname{Lat} T$ then $T|_{\mathfrak{M}}$ is also weakly subnormal because for a partially normal extension $\widehat{T}$ of T, $\widehat{T|_{\mathfrak{M}}} := \widehat{T}$ still satisfies the required properties.

An alternative description of subnormality is given by the Bram-Halmos criterion, which states that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([**Bra**],[**Con**, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\tag{1.3} \begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1).$$

Condition (1.3) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (1.3) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1.3) for all k. Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{1.4}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (1.4) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (1.3); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([**CMX**]). Now it is natural to ask whether k-hyponormal operators admit an extension $\widehat{T}$ with one or more of the properties listed in (1.1).

In this paper we characterize weak subnormality for weighted shifts and establish that 2-hyponormal weighted shifts are weakly subnormal operators possessing partially normal extensions which are hyponormal.

Theorem 1.2. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ be a weight sequence. If W_α is a 2-hyponormal weighted shift on $\mathcal{H} \equiv \ell^2(\mathbb{Z}_+)$, then W_α is weakly subnormal. Moreover, there exists a partially normal extension $\widehat{W}_\alpha$ on $\mathcal{K} := \mathcal{H} \oplus \mathcal{H}$ such that*

(i) *$\widehat{W}_\alpha$ is hyponormal;*
(ii) *$\sigma(\widehat{W}_\alpha) = \sigma(W_\alpha)$; and*
(iii) *$||\widehat{W}_\alpha|| = ||W_\alpha||$.*

In particular, if α is strictly increasing then $\widehat{W}_\alpha$ can be obtained as

$$\widehat{W}_\alpha = \begin{pmatrix} W_\alpha & [W_\alpha^*, W_\alpha]^{\frac{1}{2}} \\ 0 & W_\beta \end{pmatrix} \qquad \text{on } \mathcal{K} := \mathcal{H} \oplus \mathcal{H}, \tag{1.5}$$

where W_β is a weighted shift whose weight sequence $\{\beta_n\}_{n=0}^\infty$ given by

$$\beta_n = \alpha_n \sqrt{\frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2}} \quad (n = 0, 1, \cdots; \ \alpha_{-1} := 0).$$

In this case, $\widehat{W}_\alpha$ is a minimal partially normal extension of W_α. In addition, if W_α is 3-hyponormal then $\widehat{W}_\alpha$ can be chosen to be weakly subnormal.

Recall ([**Ath**],[**CMX**],[**CoS**]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS((T, T^2, \cdots, T^k)) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e., ([**CMX**])

$$(1.6) \qquad (M_k(T)\begin{pmatrix}\lambda_0 x\\ \vdots \\ \lambda_k x\end{pmatrix}, \begin{pmatrix}\lambda_0 x\\ \vdots \\ \lambda_k x\end{pmatrix}) \geq 0 \qquad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C}.$$

If $k = 2$ then T is said to be *quadratically hyponormal*. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general. The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([**Cu1**],[**Cu2**],[**CF1**], [**CF2**],[**CF3**],[**CL1**],[**CMX**], [**DPY**],[**McCP**]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle: in fact, even subnormality for Toeplitz operators has not yet been characterized (cf.[**Ha1**], [**Cow**], [**CoL**]]). For weighted shifts, positive results appear in [**Cu1**] and [**CF3**], although no concrete example of a weighted shift which is polynomially hyponormal but not subnormal has yet been found (the existence of such weighted shifts was established in [**CP1**] and [**CP2**]).

In Section 2 we provide a characterization of weak subnormality for weighted shifts and some basic results needed for proving Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.2. In Section 4 we consider connections with subnormality, and in Section 5 connections with Agler's model theory.

2 Some Basic Results about Weak Subnormality

How does one find partially normal extensions of weakly subnormal operators? Since weakly subnormal operators are hyponormal, one possible solution of the equation $AA^* = [T^*, T]$ is $A := [T^*, T]^{\frac{1}{2}}$. Indeed this is the case.

Lemma 2.1. *If $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal then T has a partially normal extension $\widehat{T}$ on $\mathcal{K}$ of the form*

$$(2.1.1) \qquad \widehat{T} = \begin{pmatrix} T & [T^*, T]^{\frac{1}{2}} \\ 0 & B \end{pmatrix} \quad \textit{on } \ \mathcal{K} := \mathcal{H} \oplus \mathcal{H}.$$

The proof of Lemma 2.1 will make use of the following unpublished result of the first-named author; for an alternative proof, see [**DrMcC**, par. after (2.6)].

Lemma 2.2. *If $T \in \mathcal{L}(\mathcal{H})$ is 2-hyponormal then*

$$(2.2.1) \qquad T\big(ker[T^*, T]\big) \subseteq ker[T^*, T].$$

Proof. Suppose that $[T^*, T]f = 0$. If T is 2-hyponormal, it follows from (1.3) that (cf.[**CMX**, Lemma 1.4])

$$|([T^{*2}, T]g, f)|^2 \leq ([T^*, T]f, f)([T^{*2}, T^2]g, g) \quad \text{for all } g \in \mathcal{H}.$$

By assumption, we have that for all $g \in \mathcal{H}$, $0 = ([T^{*2}, T]g, f) = (g, [T^{*2}, T]^* f)$, so that $[T^{*2}, T]^* f = 0$, i.e., $T^* T^2 f = T^2 T^* f$. Therefore,

$$[T^*, T]Tf = (T^*T^2 - TT^*T)f = (T^2T^* - TT^*T)f = T[T^*, T]f = 0,$$

which proves (2.2.1). $\square$

Corollary 2.3. *If T is weakly subnormal then T also satisfies the property (2.2.1).*

Proof. By definition, there exist operators A and B such that $[T^*, T] = AA^*$ and $A^*T = BA^*$. If $[T^*, T]f = 0$ then $AA^*f = 0$ and hence $A^*f = 0$. Therefore

$$[T^*, T]Tf = AA^*Tf = ABA^*f = 0,$$

as desired. $\square$

Proof of Lemma 2.1. Suppose that T is weakly subnormal. Then there exists a partially normal extension $\widetilde{T}$ on $\mathcal{K}' := \mathcal{H} \oplus \mathcal{H}'$ such that $\widetilde{T} = \left(\begin{smallmatrix} T & A^* \\ 0 & B' \end{smallmatrix}\right)$. By weak subnormality we have $[T^*, T] = A^*A$ and $AT = B'A$. We thus have that $|A| = [T^*, T]^{\frac{1}{2}}$. Suppose $A = U\,|A|$ is the polar decomposition of A, where U is a partial isometry with $\mathrm{cl}\big(\mathrm{ran}\,|A|\big)$ $(\subseteq \mathcal{H})$ as its initial space and $\mathrm{cl}\big(\mathrm{ran}\,A\big)$ $(\subseteq \mathcal{H}')$ as its final space. Since by Corollary 2.3, $T\big(\ker\,[T^*, T]\big) \subseteq \ker\,[T^*, T]$, we can write T as

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} : \ker|A| \oplus \mathrm{cl}\big(\mathrm{ran}\,|A|\big) \longrightarrow \ker|A| \oplus \mathrm{cl}\big(\mathrm{ran}\,|A|\big).$$

Also $|A|$ can be decomposed as

$$|A| = \begin{pmatrix} 0 & 0 \\ 0 & |A|_0 \end{pmatrix} : \ker|A| \oplus \mathrm{cl}\big(\mathrm{ran}\,|A|\big) \longrightarrow \ker|A| \oplus \mathrm{cl}\big(\mathrm{ran}\,|A|\big),$$

where $|A|_0$ denotes the restriction of $|A|$ to $\mathrm{cl}\big(\mathrm{ran}\,|A|\big)$. Write U_0 for the restriction of U to $\mathrm{cl}\big(\mathrm{ran}\,|A|\big)$. Then $U_0 : \mathrm{cl}\big(\mathrm{ran}\,|A|\big) \to \mathrm{cl}\big(\mathrm{ran}\,A\big)$ is an isometrical isomorphism, so U can be decomposed as

$$U = \begin{pmatrix} 0 & 0 \\ 0 & U_0 \end{pmatrix} : \ker|A| \oplus \mathrm{cl}\big(\mathrm{ran}\,|A|\big) \longrightarrow \mathrm{cl}\big(\mathrm{ran}\,A\big)^{\perp} \oplus \mathrm{cl}\big(\mathrm{ran}\,A\big).$$

Let P be the projection of $\mathcal{H}'$ onto $\mathrm{cl}\big(\mathrm{ran}\,A\big)$. Define an operator $B : \mathcal{H} \to \mathcal{H}$ by

$$B := \begin{pmatrix} 0 & 0 \\ 0 & U_0^{-1}PB'PU_0 \end{pmatrix} : \ker|A| \oplus \mathrm{cl}\big(\mathrm{ran}\,|A|\big) \longrightarrow \ker|A| \oplus \mathrm{cl}\big(\mathrm{ran}\,|A|\big).$$

Evidently, B is bounded. Also

$$|A|\,T = \begin{pmatrix} 0 & 0 \\ 0 & |A|_0 T_3 \end{pmatrix} \quad \text{and} \quad B\,|A| = \begin{pmatrix} 0 & 0 \\ 0 & U_0^{-1}PB'PU_0|A|_0 \end{pmatrix}. \tag{2.1.2}$$

But since $AT = B'A$, and hence $U\,|A|T = B'U\,|A|$, it follows that if we write

$$B' := \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} : \operatorname{cl}\big(\operatorname{ran} A\big)^{\perp} \oplus \operatorname{cl}\big(\operatorname{ran} A\big) \longrightarrow \operatorname{cl}\big(\operatorname{ran} A\big)^{\perp} \oplus \operatorname{cl}\big(\operatorname{ran} A\big),$$

then

$$\begin{pmatrix} 0 & 0 \\ 0 & U_0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & |A|_0 \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & U_0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & |A|_0 \end{pmatrix},$$

or

$$\begin{pmatrix} 0 & 0 \\ 0 & U_0|A|_0 T_3 \end{pmatrix} = \begin{pmatrix} 0 & B_2 U_0 |A|_0 \\ 0 & B_4 U_0 |A|_0 \end{pmatrix}.$$

Observe that $B_4 = PB'P$ and so $U_0|A|_0T_3 = PB'PU_0|A|_0$, i.e.,

$$|A|_0 T_3 = U_0^{-1} PB'PU_0|A|_0, \tag{2.1.3}$$

which together with (2.1.2) implies $|A|\,T = B\,|A|$. Therefore $\widehat{T} = \begin{pmatrix} T & |A| \\ 0 & B \end{pmatrix} : \mathcal{H}\oplus\mathcal{H} \to \mathcal{H}\oplus\mathcal{H}$ is a partially normal extension of T, which proves the lemma. $\square$

Definition 2.4. Let T be a weakly subnormal operator on $\mathcal{H}$ and let $\widehat{T}$ be a partially normal extension of T on $\mathcal{K}$. We shall say that $\widehat{T}$ is a *minimal partially normal extension* of T if $\mathcal{K}$ has no proper subspace containing $\mathcal{H}$ to which the restriction of $\widehat{T}$ is also a partially normal extension of T.

Lemma 2.5. *Let T be a weakly subnormal operator on $\mathcal{H}$ and let $\widehat{T}$ be a partially normal extension of T on $\mathcal{K}$. Then $\widehat{T}$ is a minimal partially normal extension of T if and only if*

$$\mathcal{K} = \bigvee \{\widehat{T}^{*n}h : \ h \in \mathcal{H}, \ n = 0, 1\}. \tag{2.5.1}$$

Proof. Write $\mathcal{L} \equiv \bigvee\{\widehat{T}^{*n}h : \ h \in \mathcal{H}, \ n = 0, 1\}$. Evidently, $\mathcal{L}$ contains $\mathcal{H}$ and $\widehat{T}\mathcal{L} \subseteq \mathcal{L}$, which follows from the fact that if $\widehat{T}$ is a partially normal extension of T then $\widehat{T}\widehat{T}^*h = \widehat{T}^*\widehat{T}h$ $(h \in \mathcal{H})$. We now must show that

(i) the restriction, $\widehat{T}_{\mathcal{L}}$, of $\widehat{T}$ to $\mathcal{L}$ is also a partially normal extension of T;
(ii) $\widehat{T}_{\mathcal{L}}$ is minimal.

Proof of (i): For $f \in \mathcal{H}$, and if $P_{\mathcal{L}}$ denotes the projection of $\mathcal{K}$ onto $\mathcal{L}$,

$$\widehat{T}_{\mathcal{L}}^*\widehat{T}_{\mathcal{L}} f = P_{\mathcal{L}}\widehat{T}^*\widehat{T}f = P_{\mathcal{L}}\widehat{T}\widehat{T}^*f = \widehat{T}\widehat{T}^*f$$

(because $\widehat{T}^*f \in \mathcal{L}$ and $\widehat{T}\mathcal{L} \subseteq \mathcal{L}$); on the other hand,

$$\widehat{T}_{\mathcal{L}}\widehat{T}_{\mathcal{L}}^* f = \widehat{T}P_{\mathcal{L}}\widehat{T}^*f = \widehat{T}\widehat{T}^*f$$

(because $\widehat{T}^*f \in \mathcal{L}$). Therefore, $\mathcal{H} \subseteq \ker\,[\widehat{T}_{\mathcal{L}}^*, \widehat{T}_{\mathcal{L}}]$, showing that $\widehat{T}_{\mathcal{L}}$ is a partially normal extension of T.

Proof of (ii): Let $\mathfrak{M} \subseteq \mathcal{K}$ be such that $\widehat{T}\mathfrak{M} \subseteq \mathfrak{M}$ and $\widehat{T}_{\mathfrak{M}} := \widehat{T}|_{\mathfrak{M}}$ is a partially normal extension of T. We shall prove that $\mathcal{L} \subseteq \mathfrak{M}$. Since $\mathcal{H} \subseteq \ker[\widehat{T}_{\mathfrak{M}}^*, \widehat{T}_{\mathfrak{M}}]$, for $f \in \mathcal{H}$ we have

$$P_{\mathfrak{M}}\widehat{T}^*\widehat{T}f = \widehat{T}P_{\mathfrak{M}}\widehat{T}^*f. \tag{2.5.2}$$

We claim that $\widehat{T}^*f = P_{\mathfrak{M}}(\widehat{T}^*f)$, which will show that $\widehat{T}^*f \in \mathfrak{M}$ (all $f \in \mathcal{H}$), or $\mathcal{L} \subseteq \mathfrak{M}$. Now,

$$\begin{aligned}
(\widehat{T}^*f,\ P_{\mathfrak{M}}\widehat{T}^*f)_{\mathcal{K}} &= (f,\ \widehat{T}P_{\mathfrak{M}}\widehat{T}^*f)_{\mathcal{K}} \\
&= (f,\ P_{\mathfrak{M}}\widehat{T}^*\widehat{T}f)_{\mathcal{K}} \quad \text{(by (2.5.2))} \\
&= (f,\ \widehat{T}^*\widehat{T}f)_{\mathcal{K}} \quad \text{(since } f \in \mathcal{H} \subseteq \mathfrak{M}) \\
&= (f,\ \widehat{T}\widehat{T}^*f)_{\mathcal{K}} \quad \text{(because } \widehat{T} \text{ is partially normal)} \\
&= (\widehat{T}^*f,\ \widehat{T}^*f)_{\mathcal{K}}.
\end{aligned}$$

Therefore, $\widehat{T}^*f = P_{\mathfrak{M}}\widehat{T}^*f$, as desired. This concludes the proof. □

It is well known (cf. [**Con**, Proposition II.2.4]) that if T is a subnormal operator on $\mathcal{H}$ and N is a normal extension of T then N is a minimal normal extension of T if and only if

$$\mathcal{K} = \bigvee\{\widehat{T}^{*n}h :\ h \in \mathcal{H},\ n \geq 0\}.$$

Thus if T is a subnormal operator then T may have a partially normal extension different from a normal extension. For, consider the unilateral (unweighted) shift U_+ acting on $\ell^2(\mathbb{Z}_+)$. Then m.n.e. $(U_+) = U$, the bilateral shift acting on $\ell^2(\mathbb{Z})$, with orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$. It is easy to verify that m.p.n.e. $(U_+) = U|_{\mathcal{L}}$, where $\mathcal{L} :=< e_{-1} > \oplus\ \ell^2(\mathbb{Z}_+)$.

On the other hand, it is well known that a minimal normal extension of a subnormal operator is unique. By comparison, a minimal partially normal extension of a weakly subnormal operator is also unique.

Lemma 2.6. *For $k = 1, 2$, let T_k be a weakly subnormal operator on $\mathcal{H}_k$ and let $\widehat{T}_k$ be a minimal partially normal extension of T_k on $\mathcal{K}_k$. If T_1 and T_2 are unitarily equivalent then so are $\widehat{T}_1$ and $\widehat{T}_2$.*

Proof. Suppose that $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary operator such that $UT_1 = T_2U$. Define V on $\mathcal{K}_1$ by

$$V(\widehat{T}_1^{*n}h) := \widehat{T}_2^{*n}Uh \quad (h \in \mathcal{H}_1,\ n = 0, 1).$$

Note that $V|_{\mathcal{H}_1} = U$. Observe that if $f, g \in \mathcal{H}$ then

$$\begin{aligned}
||Uf + \widehat{T}_2^* Ug||^2 &= (Uf + \widehat{T}_2^* Ug,\ Uf + \widehat{T}_2^* Ug) \\
&= (Uf, Uf) + (Ug, \widehat{T}_2 Uf) + (\widehat{T}_2 Uf, Ug) + (\widehat{T}_2 Ug, \widehat{T}_2 Ug) \\
&\qquad\qquad (\text{because } \widehat{T}_2 \widehat{T}_2^* h = \widehat{T}_2^* \widehat{T}_2 h \text{ for all } h \in \mathcal{H}_2) \\
&= (Uf, Uf) + (Ug, U\widehat{T}_1 f) + (U\widehat{T}_1 f, Ug) + (U\widehat{T}_1 g, U\widehat{T}_1 g) \\
&\qquad\qquad (\text{because } UT_1 = T_2 U) \\
&= (f, f) + (g, \widehat{T}_1 f) + (\widehat{T}_1 f, g) + (\widehat{T}_1 g, \widehat{T}_1 g) \quad (\text{because } U \text{ is isometry}) \\
&= ||f + \widehat{T}^* g||^2.
\end{aligned}$$

Thus

$$V\left[f + \widehat{T}_1^* g\right] = Uf + \widehat{T}_2^* Ug$$

is well–defined between dense subsets of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Thus V extends to a unitary operator from $\mathcal{K}_1$ onto $\mathcal{K}_2$. Moreover we have $V\widehat{T}_1 = \widehat{T}_2 V$, which follows from the following observation: for every $h \in \mathcal{H}_1$,

$$V\widehat{T}_1 h = U\widehat{T}_1 h = \widehat{T}_2 Uh = \widehat{T}_2 Vh$$

and

$$V\widehat{T}_1(\widehat{T}_1^* h) = V\widehat{T}_1^* \widehat{T}_1 h = \widehat{T}_2^* U(\widehat{T}_1 h) = \widehat{T}_2^* \widehat{T}_2 Uh = \widehat{T}_2 \widehat{T}_2^* Uh = \widehat{T}_2 V(\widehat{T}_1^* h).$$

This completes the proof. $\square$

Corollary 2.7. *If T is a weakly subnormal operator and $\widehat{T}^{(1)}$ and $\widehat{T}^{(2)}$ are minimal partially normal extensions of T, then $\widehat{T}^{(1)}$ and $\widehat{T}^{(2)}$ are unitarily equivalent.*

Lemma 2.8. *Let $T \in \mathcal{L}(\mathcal{H})$ be weakly subnormal and write $T \equiv \left(\begin{smallmatrix} T_1 & T_2 \\ 0 & T_3 \end{smallmatrix}\right)$ on $\ker[T^*, T] \oplus \mathrm{cl}(\mathrm{ran}\,[T^*, T])$. If $\widehat{T} = \left(\begin{smallmatrix} T & [T^*, T]^{\frac{1}{2}} \\ 0 & B \end{smallmatrix}\right)$ on $\mathcal{H} \oplus \mathcal{H}$ is a partially normal extension of T then the minimal partially normal extension of T can be obtained as*

$$\widehat{T} := \begin{pmatrix} T_1 & T_2 & 0 \\ 0 & T_3 & [T^*, T]_0^{\frac{1}{2}} \\ 0 & 0 & PBP \end{pmatrix} : \begin{pmatrix} \ker[T^*, T] \\ \mathrm{cl}(\mathrm{ran}\,[T^*, T]) \\ \mathrm{cl}\big(\mathrm{ran}\,[T^*, T]\big) \end{pmatrix} \longrightarrow \begin{pmatrix} \ker[T^*, T] \\ \mathrm{cl}(\mathrm{ran}\,[T^*, T]) \\ \mathrm{cl}\big(\mathrm{ran}\,[T^*, T]\big) \end{pmatrix}, \tag{2.8.1}$$

where $[T^*, T]_0^{\frac{1}{2}}$ denotes the restriction of $[T^*, T]^{\frac{1}{2}}$ to $\mathrm{cl}\big(\mathrm{ran}\,[T^*, T]\big)$ and P is the projection of $\mathcal{H}$ onto $\mathrm{cl}\big(\mathrm{ran}\,[T^*, T]\big)$.

Proof. After a careful examination of the proof of Lemma 2.1 (with $\widehat{T}$ in place of $\widetilde{T}$), and since $|A| = A = [T^*, T]^{\frac{1}{2}}$, one can show that $\widehat{T}$ in (2.8.1) is also a partially normal extension of T. Furthermore, $\bigvee\{\widehat{T}^{*n} h :\ h \in \mathcal{H},\ n = 0, 1\} = \mathcal{H} \oplus \mathrm{cl}\big(\mathrm{ran}\,[T^*, T]\big)$, which implies that $\widehat{T}$ is minimal. This proves the lemma $\square$

3 Characterization of Weak Subnormality for Weighted Shifts

Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the *(unilateral) weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \ge 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \le \alpha_{n+1}$ for all $n \ge 0$.

We now have:

Theorem 3.1 (Weak Subnormality of Weighted Shifts). *If $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ is a strictly increasing weight sequence then*

$$W_\alpha \textit{ is weakly subnormal} \Longleftrightarrow \limsup \frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2} \ < \ \infty \tag{3.1.1}$$

$(n = 0, 1, \cdots ; \alpha_{-1} := 0)$.

Proof. Suppose that W_α is weakly subnormal. In view of Lemma 2.1, W_α has a partially normal extension $\widehat{W}_\alpha$ of the form $\widehat{W}_\alpha = \begin{pmatrix} W_\alpha & [W_\alpha^*, W_\alpha]^{\frac{1}{2}} \\ 0 & B \end{pmatrix}$ on $\ell_2 \oplus \ell_2$, where $[W_\alpha^*, W_\alpha]^{\frac{1}{2}} W_\alpha = B\,[W_\alpha^*, W_\alpha]^{\frac{1}{2}}$. Since $[W_\alpha^*, W_\alpha]^{\frac{1}{2}}$ is diagonal it follows that B is a unilateral weighted shift. Write $B := W_\beta$, where $\beta \equiv \{\beta_n\}_{n=0}^\infty$. To determine W_β, it suffices to check the $(n+1, n)$–entries:

$$\left([W_\alpha^*, W_\alpha]^{\frac{1}{2}} W_\alpha e_n,\ e_{n+1}\right) = \left(W_\beta [W_\alpha^*, W_\alpha]^{\frac{1}{2}} e_n,\ e_{n+1}\right),$$

which implies

$$\alpha_n \sqrt{\alpha_{n+1}^2 - \alpha_n^2} = \beta_n \sqrt{\alpha_n^2 - \alpha_{n-1}^2} \quad (n = 0, 1, \cdots ;\ \alpha_{-1} := 0). \tag{3.1.2}$$

Thus $\widehat{W}_\alpha$ can be obtained as

$$\widehat{W}_\alpha = \begin{pmatrix} W_\alpha & [W_\alpha^*, W_\alpha]^{\frac{1}{2}} \\ 0 & W_\beta \end{pmatrix} \quad \text{on } \mathcal{K} := \mathcal{H} \oplus \mathcal{H}, \tag{3.1.3}$$

where W_β is a unilateral weighted shift whose weight sequence $\{\beta_n\}_{n=0}^\infty$ is given by

$$\beta_n = \alpha_n \sqrt{\frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2}} \quad (n = 0, 1, \cdots ;\ \alpha_{-1} := 0).$$

But since W_β is bounded it follows that $\limsup \beta_n < \infty$ and hence $\limsup \frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2} < \infty$. This proves the forward implication in (3.1.1). The backward implication follows at once from the observation that $\widehat{W}_\alpha$ in (3.1.3) is a partially normal extension of W_α. $\square$

If W_α is a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, then the *moments* of W_α are usually defined by $\beta_0 := 1,\ \beta_{n+1} := \alpha_n \beta_n\ (n \ge 0)$ [**Shi**]; however, we prefer to reserve this term for the sequence $\gamma_n := \beta_n^2\ (n \ge 0)$. A criterion for k-hyponormality can be given in terms of these moments ([**Cu1**, Theorem 4]): if we build a $(k+1)\times(k+1)$ Hankel matrix $A(n;k)$ by

$$A(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \quad (n \geq 0), \tag{3.1.4}$$

then

$$W_\alpha \text{ is } k\text{-hyponormal} \iff A(n;k) \geq 0 \quad (n \geq 0). \tag{3.1.5}$$

In particular, for α strictly increasing, W_α is 2-hyponormal if and only if

$$\det \begin{pmatrix} \gamma_n & \gamma_{n+1} & \gamma_{n+2} \\ \gamma_{n+1} & \gamma_{n+2} & \gamma_{n+3} \\ \gamma_{n+2} & \gamma_{n+3} & \gamma_{n+4} \end{pmatrix} \geq 0 \quad (n \geq 0). \tag{3.1.6}$$

We let

$$p_n := u_n\, v_{n+1} - w_n \qquad (n \geq 0),$$

where

$$\begin{cases} u_n := \alpha_n^2 - \alpha_{n-1}^2 \\ v_n := \alpha_n^2 \alpha_{n+1}^2 - \alpha_{n-1}^2 \alpha_{n-2}^2 \\ w_n := \alpha_n^2 (\alpha_{n+1}^2 - \alpha_{n-1}^2)^2. \end{cases}$$

Lemma 3.2. *If $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ is a strictly increasing weight sequence then the following statements are equivalent:*

(i) *W_α is 2-hyponormal;*

(ii) *$\alpha_{n+1}^2 (u_{n+1} + u_{n+2})^2 \leq u_{n+1} v_{n+2} \qquad (n \geq 0)$;*

(iii) *$\frac{\alpha_n^2}{\alpha_{n+2}^2} \frac{u_{n+2}}{u_{n+3}} \leq \frac{u_{n+1}}{u_{n+2}} \qquad (n \geq 0)$;*

(iv) *$p_n \geq 0 \quad (n \geq 0)$.*

Proof. This follows from a straightforward calculation. □

J. Stampfli [**Sta**] showed that for subnormal weighted shifts W_α a *propagation* phenomenon occurs which forces the flatness of W_α whenever two equal weights are present. Later, A. Joshi proved in [**Jos**] that the shift with weights $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = \cdots = b$, $0 < a < b$, is *not* quadratically hyponormal, and P. Fan [**Fan**] established that for $a = 1$, $b = 2$, and $0 < s < \sqrt{5}/5$, $W_\alpha + s\,W_\alpha^2$ is *not* hyponormal. On the other hand, it was shown in [**Cu1**, Theorem 2] that a hyponormal weighted shift with *three* equal weights cannot be quadratically hyponormal without being flat: *If W_α is quadratically hyponormal and $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal.* Furthermore, in [**Cu1**, Proposition 11] it was shown that, in the presence of quadratic hyponormality, two consecutive pairs of equal weights again force flatness, thereby subnormality.

Lemma 3.3 (Propagation). *Let W_α be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$.*

(i) ([**Sta**, Theorem 6]) *Let W_α be subnormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.*

(ii) ([**Cu1**, Corollary 6]) *Let W_α be 2-hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat.*

(iii) ([**Cu1**, Proposition 11]) *Let W_α be quadratically hyponormal. If $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \geq 0$, then α is flat.*

(iv) ([**Cho**, Theorem 1]) *Let W_α be quadratically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then α is flat.*

We denote by $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ the recursive weighted shift whose weights are calculated according to the recursive relation

$$\alpha_{n+1}^2 = \varphi_1 + \varphi_0 \frac{1}{\alpha_n^2},$$

where

$$\varphi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2} \quad \text{and} \quad \varphi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}.$$

It is well-known that $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ is subnormal with 2–atomic Berger measure ([**CF2**]). We also denote by $W_{x_1,\cdots,x_n,(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ the weighted shift whose weight sequence consists of the weights $x_1, \cdots, x_n$ followed by the weight sequence of $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$.

To derive a flatness condition on p_n for 3-hyponormal operators, we recall ([**CF2**, Proposition 2.8]) the outer propagation of positive matrices: Let $C \in M_n(\mathbb{C})$ ($n \geq 2$) be a positive matrix, and suppose that

$$C = \begin{pmatrix} R & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & S \\ * & * \end{pmatrix}, \tag{3.3.1}$$

where $R, S \in M_{n-1}(\mathbb{C})$. Then $\operatorname{rank}(S) \leq \operatorname{rank}(R)$, so in particular, $\det R = 0$ implies $\det S = 0$.

Lemma 3.4. *Let W_α be a 3-hyponormal weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$. If $p_{n_0} = 0$ for some $n_0 \geq 0$ then W_α is subnormal. More concretely, if $n_0 \geq 1$ is the first integer such that $p_{n_0} = 0$ then $W_\alpha = W_{\alpha_0,\cdots,\alpha_{n_0-2},(\alpha_{n_0-1},\alpha_{n_0},\alpha_{n_0+1})^\wedge}$.*

Proof. If $p_0 = 0$ then $\alpha_1 = \alpha_2$. By Lemma 3.3 (ii), $\alpha_1 = \alpha_2 = \cdots$, which implies $p_n = 0$ for all $n \geq 0$, so evidently W_α is subnormal. We now assume $n_0 \geq 1$ and $p_{n_0} = 0$. A straightforward calculation shows that

$$\det A(n_0 - 1; 2) = \begin{cases} \alpha_0^4\alpha_1^2\, p_1 & (n_0 = 1) \\ (\alpha_0^2 \cdots \alpha_{n_0-2}^2)^3 \alpha_{n_0-1}^2 \alpha_{n_0}^2\, p_{n_0} & (n_0 \geq 2), \end{cases} \tag{3.4.1}$$

which implies $\det A(n_0 - 1; 2) = 0$. Since W_α is 3-hyponormal we have $A(n_0 - 1; 3) \geq 0$ and

$$A(n_0 - 1; 3) = \begin{pmatrix} A(n_0 - 1; 2) & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & A(n_0; 2) \\ * & * \end{pmatrix}.$$

It thus follows from (3.3.1) that $\det A(n_0;2) = 0$ which by (3.4.1) implies $p_{n_0+1} = 0$. Repeating this argument shows that $p_n = 0$ for all $n \geq n_0$. In turn, this implies $W_\alpha|_{<e_{n_0-1},\cdots>} = W_{(\alpha_{n_0-1},\alpha_{n_0},\alpha_{n_0+1})^\wedge}$. Thus we have $W_\alpha = W_{\alpha_0,\cdots,\alpha_{n_0-2},(\alpha_{n_0-1},\alpha_{n_0},\alpha_{n_0+1})^\wedge}$, which is subnormal by [**CJL**, Theorem 1.3]: if $\alpha: x_n,\cdots,x_1,(\alpha_0,\cdots,\alpha_k)^\wedge$ then

$$W_\alpha \text{ is subnormal } \iff \begin{cases} W_\alpha \text{ is } ([\frac{k+1}{2}]+1)\text{-hyponormal} & (n=1) \\ W_\alpha \text{ is } ([\frac{k+1}{2}]+2)\text{-hyponormal} & (n>1). \end{cases}$$

This completes the proof. □

We now show that the *consecutive differences* of weights for 2-hyponormal weighted shifts must satisfy a rigid condition.

Lemma 3.5. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ be a strictly increasing weight sequence. If W_α is 2-hyponormal then the sequence of quotients*

$$\Theta_n := \frac{u_{n+1}}{u_{n+2}} \qquad (n \geq 0) \tag{3.5.1}$$

is bounded away from 0 and from ∞. More precisely,

$$1 \leq \Theta_n \leq \frac{u_1}{u_2}\left(\frac{||W_\alpha||^2}{\alpha_0\alpha_1}\right)^2 \quad \textit{for sufficiently large } n. \tag{3.5.2}$$

In particular, $\{u_n\}_{n=0}^\infty$ is eventually decreasing.

Proof. Suppose W_α is 2-hyponormal. By iterating the inequality in Lemma 3.2 (iii), we obtain

$$\frac{u_{n+1}}{u_{n+2}} \leq \frac{u_1}{u_2}\cdot\frac{\alpha_n^2\alpha_{n+1}^2}{\alpha_0^2\alpha_1^2} \leq \frac{u_1}{u_2}\left(\frac{||W_\alpha||^2}{\alpha_0\alpha_1}\right)^2 \quad (n \geq 2),$$

so that the sequence $\{\frac{u_{n+1}}{u_{n+2}}\}_{n=0}^\infty$ is bounded. We must now show that $\frac{u_{n+2}}{u_{n+1}} \leq 1$ for sufficiently large n. To do this observe that for every $\epsilon > 0$, there exists $N_1 \in \mathbb{Z}_+$ such that

$$\frac{||W_\alpha||^2}{\alpha_n\alpha_{n+1}} < \sqrt{1+\epsilon} \qquad \text{for } n \geq N_1. \tag{3.5.3}$$

Suppose that there exists $N_2 > N_1$ such that

$$\frac{u_{N_2+2}}{u_{N_2+1}} > 1+\epsilon. \tag{3.5.4}$$

By iterating again the inequality in Lemma 3.2 (iii), we have

$$\frac{u_{n+1}}{u_{n+2}} \leq \frac{u_{N_2+1}}{u_{N_2+2}}\left(\frac{\alpha_n\alpha_{n+1}}{\alpha_{N_2}\alpha_{N_2+1}}\right)^2 < \frac{1}{1+\epsilon}\left(\frac{||W_\alpha||^2}{\alpha_{N_2}\alpha_{N_2+1}}\right)^2 < \frac{1}{1+\epsilon}\cdot(1+\epsilon) = 1$$

for $n \geq N_2$, which contradicts the fact that $u_{n+1} \to 0$ $(n \to \infty)$. This completes the proof. □

Remark 3.6. Note that Lemma 3.5 says that if W_α is 2-hyponormal then the sequence $\{u_n\}_{n=0}^\infty$ eventually decreases, and it does so very *slowly*. To exemplify this, consider the following weight sequences:

(i) $\alpha_n := (\sum_{k=0}^n \delta_n)^{\frac{1}{2}}$ $(n \ge 0)$, where

$$\{\delta_n\}_{n=0}^\infty : \ \frac{1}{2}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2^5}, \frac{1}{2^4}, \frac{1}{2^7}, \frac{1}{2^6}, \cdots$$

and

(ii)

$$\alpha'_n := \left(1 + \sum_{k=0}^n \frac{1}{2^{2^k}}\right)^{\frac{1}{2}}.$$

Then W_α and $W_{\alpha'}$ are both hyponormal but not 2-hyponormal because (i) $\{u_n\}_{n=0}^\infty$ is not eventually decreasing; (ii) $\Theta_n(\alpha')$ is not bounded (i.e., $\{u'_n\}_{n=0}^\infty$ is decreasing too *fast*).

We now exhibit a gap between 2–hyponormality and weak subnormality for weighted shifts.

Example 3.7. *There exists a weakly subnormal weighted shift (whose weight sequence is strictly increasing) which is not* 2*–hyponormal.*

For, let W_α be the weighted shift whose weight sequence is given by Remark 3.6 (i). Then W_α is not 2–hyponormal. However by Theorem 3.1, W_α is weakly subnormal because

$$\limsup \frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2} = \limsup \frac{\delta_{n+1}}{\delta_n} = 2.$$

□

We pause to state an unexpected consequence of Lemma 3.5. First, recall that a compact operator $T \in \mathcal{L}(\mathcal{H})$ is *trace-class* if

$$\sum_{n=0}^\infty s_n(T) \ < \ \infty,$$

where $s_n(T)$ is the n-th s-number corresponding to T. Thus if T is a trace-class operator with $s_n \equiv s_n(T) > 0$ $(n \ge 0)$, then evidently $\limsup \frac{s_{n+1}}{s_n} \le 1$. Of course, we need not expect that $\{\frac{s_n}{s_{n+1}}\}_{n=0}^\infty$ be bounded. We shall say that a trace-class operator $T \in \mathcal{L}(\mathcal{H})$ has the *ratio property* if $\{\frac{s_n}{s_{n+1}}\}_{n=0}^\infty$ is bounded.

Corollary 3.8. *Let $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ be a strictly increasing weight sequence. If W_α is 2-hyponormal then its self-commutator has the ratio property.*

Proof. Straightforward from Lemma 3.5 and the well-known fact that the self-commutator of a hyponormal weighted shift is trace-class. □

In general, if $\widehat{T} = T|_{\mathcal{H}}$ then the spectrum of $\widehat{T}$, $\sigma(\widehat{T})$, may contain strictly the spectrum of T, $\sigma(T)$; in fact, the passage from $\sigma\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ to $\sigma(A)\cup\sigma(B)$ is the filling in of some holes of $\sigma\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, which happen to be subsets of $\sigma(A)\cap\sigma(B)$ (cf. [**HLL**, Corollary 7]). Write $\sigma_l(\cdot)$ and $\sigma_r(\cdot)$ for the left- and the right- spectrum, respectively. From Rosenblum's corollary [**LuR**, Theorem 4] we know that if $\sigma_r(A)\cap\sigma_l(B)=\emptyset$ then $\sigma\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma(A)\cup\sigma(B)$. But it may happen that $\sigma\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma(A)\cup\sigma(B)$ even though $\sigma_r(A)\cap\sigma_l(B)\neq\emptyset$. The following lemma is an example of such a case.

Lemma 3.9. *Let W_α and W_β be weighted shifts. Then*

$$\sigma\begin{pmatrix} W_\alpha & C \\ 0 & W_\beta \end{pmatrix} = \sigma(W_\alpha)\cup\sigma(W_\beta) \quad \textit{for every } C\in\mathcal{L}(\ell^2). \tag{3.9.1}$$

Proof. Recall ([**HLL**, Corollary 10]) that if $A\in\mathcal{L}(\mathcal{H})$ and $B\in\mathcal{L}(\mathcal{K})$ then
(3.9.2)

$$\Big[\sigma(A)\setminus\sigma_l(A)\Big]\cap\Big[\sigma(B)\setminus\sigma_r(B)\Big]=\emptyset \Longrightarrow \sigma\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma(A)\cup\sigma(B) \quad \text{for every } C\in\mathcal{L}(\mathcal{K},\mathcal{H}).$$

Since every weighted shift W with positive weights has no eigenvalues ([**Shi**, Theorem 8]) and hence $\sigma(W)\setminus\sigma_r(W)=\emptyset$, the result immediately follows from (3.9.2). $\square$

4 Proof of Theorem 1.2

It is well-known that if T is subnormal then it has a normal extension $\widehat{T}$ such that $\sigma(\widehat{T})\subseteq\sigma(T)$ and $||\widehat{T}|| = ||T||$, namely its *minimal* normal extension. By comparison, our main theorem shows that every 2-hyponormal weighted shift T has a minimal partially normal extension $\widehat{T}$ such that $\sigma(\widehat{T})=\sigma(T)$ and $||\widehat{T}||=||T||$. We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $T\equiv W_\alpha$. If $\alpha_n=\alpha_{n+1}$ for some $n\geq 0$ then by Lemma 3.3 (ii) α is flat, so that T is subnormal. Thus in this case, the result is evident. Suppose that α is strictly increasing. Then the first assertion follows at once from Theorem 3.1 and Lemma 3.5. Furthermore, as in (3.1.3) we can choose a partially normal extension $\widehat{T}$ of T as

$$\widehat{T} = \begin{pmatrix} T & [T^*,T]^{\frac{1}{2}} \\ 0 & S \end{pmatrix} \qquad \text{on } \mathcal{K}:=\mathcal{H}\oplus\mathcal{H}, \tag{4.1}$$

where S is a weighted shift whose weight sequence $\{\beta_n\}_{n=0}^\infty$ is given by

$$\beta_n = \alpha_n\sqrt{\frac{u_{n+1}}{u_n}} \quad (n\geq 0).$$

We shall now prove that $\widehat{T}$ is hyponormal. Since

$$[\widehat{T}^*,\widehat{T}] = \begin{pmatrix} 0 & 0 \\ 0 & [S^*,S]+[T^*,T] \end{pmatrix},$$

we need to show that $[S^*, S] + [T^*, T] \geq 0$. Observe that $[S^*, S] + [T^*, T]$ is a diagonal operator whose diagonals $(d_n)_{n=0}^{\infty}$ are given by

$$\begin{aligned} d_0 &= \alpha_1^2; \quad d_1 = \beta_1^2; \\ d_n &= u_n + \alpha_n^2 \frac{u_{n+1}}{u_n} - \alpha_{n-1}^2 \frac{u_n}{u_{n-1}} \quad (n \geq 2) \\ &= \frac{u_n^2 u_{n-1} + \alpha_n^2 u_{n+1} u_{n-1} - \alpha_{n-1}^2 u_n^2}{u_n u_{n-1}} \quad (n \geq 2), \end{aligned}$$

which implies that

$$d_n = \frac{p_{n-1}}{u_n\, u_{n-1}} \quad (n \geq 2).$$

We thus have

$$[S^*, S] + [T^*, T] = \begin{pmatrix} \alpha_1^2 & & & & \\ & \beta_1^2 & & 0 & \\ & & \frac{p_1}{u_1 u_2} & & \\ & 0 & & \frac{p_2}{u_2 u_3} & \\ & & & & \ddots \end{pmatrix}. \tag{4.2}$$

Since T is 2-hyponormal it follows from Lemma 3.2 that $p_n \geq 0$ for every $n \geq 0$, so that writing C for the square root of the (positive) diagonal matrix in (4.2), we have

$$[\widehat{T}^*, \widehat{T}] = 0_{\mathcal{H}} \bigoplus C^2, \tag{4.3}$$

which proves that $\widehat{T}$ is hyponormal. For spectral equality, note that S is a weighted shift. Since by Lemma 3.5,

$$\frac{u_{n+1}}{u_n} \leq 1 \quad \text{for sufficiently large } n,$$

it follows that (cf. [**Ha2**, Solution 91])

$$\begin{aligned} r(S) &= \lim_k \sup_n \left| \prod_{i=0}^{k-1} \sqrt{\alpha_{n+i}^2 \frac{u_{n+i+1}}{u_{n+i}}} \right|^{\frac{1}{k}} \\ &\leq \lim_k \sup_n \left| \prod_{i=0}^{k-1} \alpha_{n+i} \right|^{\frac{1}{k}} \\ &= r(T), \end{aligned}$$

where $r(\cdot)$ denotes the spectral radius. Now recall that the spectrum of a weighted shift W_α is the disc $|z| \leq r(W_\alpha)$ (cf. [**Shi**, Theorem 4]); it follows that $\sigma(S) \subseteq \sigma(T)$. Therefore, by Lemma 3.9, we have $\sigma(\widehat{T}) = \sigma(S) \cup \sigma(T) = \sigma(T)$. For norm equality, use the hyponormality of $\widehat{T}$ and T to see that $||\widehat{T}|| = r(\widehat{T}) = r(T) = ||T||$, where $r(\cdot)$ denotes the spectral radius.

The minimality of $\widehat{T}$ follows from Lemma 2.5 since

$$\{\widehat{T}^{*n}h : \ h \in \mathcal{H}, \ n = 0, 1\} = \mathcal{H} \oplus \mathcal{H}.$$

Towards the weak subnormality of $\widehat{T}$, suppose T is 3-hyponormal and write

$$[T^*, T]^{\frac{1}{2}} =: D \equiv \operatorname{diag}(u_j)_{j=0}^{\infty} \quad \text{and} \quad \beta_j := \alpha_j \sqrt{\frac{u_{j+1}}{u_j}} \ \ (j \geq 0).$$

Then (4.1) can be rewritten as

$$\widehat{T} = \begin{pmatrix} T & D \\ 0 & W_\beta \end{pmatrix}, \quad \text{where} \ \ \beta := \{\beta_j\}_{j=0}^{\infty}. \tag{4.4}$$

Recall that C is a diagonal operator with diagonal entries $c_0 = \alpha_1 \neq 0$, $c_1 = \beta_1 \neq 0$ and $c_j = \sqrt{\frac{p_{j-1}}{u_{j-1}u_j}}$ $(j \geq 2)$. If $p_j = 0$ for some $j \geq 2$ then by Lemma 3.4, T is subnormal; thus its partially normal extension can be chosen as a normal operator and therefore the proof is complete. Therefore suppose $p_j \neq 0$ for all $j \geq 2$ and hence $c_j \neq 0$ for all $j \geq 0$. Looking at (4.4), with $\widehat{T}$ in place of T, we define

$$\widehat{T}^{(2)} := \begin{pmatrix} \widehat{T} & [\widehat{T}^*, \widehat{T}]^{\frac{1}{2}} \\ 0 & 0_{\mathcal{H}} \oplus W_{\beta^{(2)}} \end{pmatrix},$$

where $W_{\beta^{(2)}}$ is the weighted shift with weight sequence $\{\beta_j^{(2)}\}_{j=0}^{\infty}$ given by

$$\beta_j^{(2)} := \begin{cases} \beta_j \frac{c_{j+1}}{c_j} & \text{if } c_j \neq 0 \\ 0 & \text{if } c_j = 0. \end{cases}$$

Thus we have

$$\widehat{T}^{(2)} \ \cong \ \begin{pmatrix} T & D & 0 \\ 0 & W_\beta & C \\ 0 & 0 & W_{\beta^{(2)}} \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \\ \mathcal{H} \end{matrix} \bigoplus 0_{\mathcal{H}}.$$

We claim that $\widehat{T}^{(2)}$ is a partially normal extension of $\widehat{T}$. A straightforward calculation shows that

$$CW_\beta = \begin{pmatrix} 0 & & & & \\ c_1\beta_0 & 0 & & & \\ & c_2\beta_1 & 0 & & \\ & & c_3\beta_2 & 0 & \\ & & & \ddots & \ddots \end{pmatrix} = W_{\beta^{(2)}}C,$$

so that

$$\begin{aligned} [\widehat{T}^{(2)*}, \widehat{T}^{(2)}] &\cong \begin{pmatrix} [T^*, T] - D^2 & T^*D - DW_\beta^* & 0 \\ DT - W_\beta D & D^2 + [W_\beta^*, W_\beta] - C^2 & W_\beta^* C - CW_{\beta^{(2)}}^* \\ 0 & CW_\beta - W_{\beta^{(2)}}C & C^2 + [W_{\beta^{(2)}}^*, W_{\beta^{(2)}}] \end{pmatrix} \bigoplus 0_{\mathcal{H}} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C^2 + [W_{\beta^{(2)}}^*, W_{\beta^{(2)}}] \end{pmatrix} \bigoplus 0_{\mathcal{H}}, \end{aligned}$$

which implies that $\widehat{T}$ is weakly subnormal. This completes the proof of Theorem 1.2. $\square$

5 Connections with Subnormality

§5-1. Stampfli's Normal Extension

In the proof of Theorem 1.2 we observed that if T is subnormal then $C^2+[W^*_{\beta^{(2)}}, W_{\beta^{(2)}}] \geq 0$ (cf. [**Sta**, Proof of Theorem 4]), showing that $\widehat{T}^{(2)}$ is hyponormal. We can repeat that argument to obtain Stampfli's normal extension $\widehat{W}_\alpha^{(\infty)}$ of the subnormal weighted shift W_α. This says that if W_α is subnormal then the partially normal extension $\widehat{W}_\alpha$ in (1.5) is also subnormal. Also, note that $W_{\beta^{(2)}}$ may be a finite rank operator, as briefly observed in the proof of Theorem 1.2. To illustrate this recall the recursively generated weighted shift $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$. In this case we have $p_n = 0$ $(n = 1, 2, \cdots)$. Thus if $\widehat{W}$ is the corresponding partially normal extension of $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ then from (4.2) we can see that

$$[\widehat{W}^*, \widehat{W}] = 0 \ \oplus \begin{pmatrix} \alpha_1^2 & 0 \\ 0 & \beta_1^2 \end{pmatrix}.$$

Thus, using the above process, the Stampfli's (minimal) normal extension of $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ is obtained in the form

$$\left(\begin{array}{ccccc|ccccc|cc|cccccc}
0 &&&&& \sqrt{u_0} &&&&& && &&&&& \\
\alpha_0 & 0 &&&& & \sqrt{u_1} &&&& && &&&&& \\
& \alpha_1 & 0 &&& && \sqrt{u_2} &&& && &&&&& \\
&& \alpha_2 & 0 && &&& \ddots && && &&&&& \\
&&& \ddots & \ddots & &&&& \ddots & && &&&&& \\
\hline
&&&&& 0 &&&& & \alpha_1 & 0 & &&&&& \\
&&&&& \beta_0 & 0 &&& & 0 & \beta_1 & &&&&& \\
&&&&& & \beta_1 & 0 && & 0 & 0 & &&&&& \\
&&&&& && \beta_2 & 0 & & \vdots & \vdots & &&&&& \\
&&&&& &&& \ddots & \ddots & \vdots & \vdots & &&&&& \\
\hline
&&&&& &&&&& 0 & 0 & \alpha_2 & 0 &&&& \\
&&&&& &&&&& \sqrt{u_2} & 0 & 0 & \sqrt{\beta_1^2-u_2} &&&& \\
\hline
&&&&& &&&&& && 0 & 0 & * & 0 && \\
&&&&& &&&&& && * & 0 & 0 & * && \\
&&&&& &&&&& && && 0 & 0 & * & 0 \\
&&&&& &&&&& && && \ddots & \ddots & \ddots & \ddots
\end{array}\right)$$

(cf. [**Sta**, p. 374]).

Example 5.1. *There exists a weighted shift which is weakly subnormal but not subnormal.*

Proof. In view of Theorem 1.2, it suffices to show that there exists a weighted shift which is 2-hyponormal but not subnormal. Such examples abound; e.g., consider the weighted shift whose weights are given by

$$\alpha_0 := x, \quad \alpha_n := \sqrt{\frac{n+1}{n+2}} \ \ (n \geq 1) \quad \text{(cf. [**Cu1**, Proposition 7])}.$$

For $\sqrt{\frac{1}{2}} < x \leq \frac{3}{4}$, W_α is 2-hyponormal, but not subnormal. $\square$

Theorem 1.2 says that every 2-hyponormal weighted shift has an economical "norm- and spectrum-preserving" partially normal extension; moreover, the discussion preceding Example 5.1 shows that 2-hyponormality is a useful notion for the study of subnormality.

We now formulate a natural question:

Question 5.2. *Is every 2-hyponormal operator weakly subnormal?*

Here is a partial answer.

Theorem 5.3. *If $T \in \mathcal{L}(\mathcal{H})$ is 2-hyponormal then T has a linear (not necessarily bounded) extension $\widehat{T}$ on $\mathcal{H} \oplus \mathcal{H}$ satisfying the equality $\widehat{T}^*\widehat{T}f = \widehat{T}\widehat{T}^*f$ for all $f \in \mathcal{H}$. More precisely,*

$$\widehat{T} := \begin{pmatrix} T & [T^*, T]^{\frac{1}{2}} \\ 0 & \widetilde{S} \end{pmatrix} : \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H}, \tag{5.3.1}$$

where $\widetilde{S} : ker[T^*, T] \oplus ran[T^*, T] \longrightarrow \mathcal{H}$ is defined by

$$\widetilde{S}f := \begin{cases} [T^*, T]^{\frac{1}{2}}Tg & \text{if } f = [T^*, T]^{\frac{1}{2}}g \text{ with } g \in ran\,[T^*, T] \\ 0 & \text{if } f \in ker\,[T^*, T]. \end{cases} \tag{5.3.2}$$

Moreover if $[T^*, T]$ has closed range (e.g., if $[T^*, T]$ is finite rank) then T is weakly subnormal.

Proof. Put $A := [T^*, T]^{\frac{1}{2}}$. We look for $\widetilde{S}$ such that $AT = \widetilde{S}A$, which naturally leads to define $\widetilde{S}(Ag) := ATg$ ($g \in \mathcal{H}$), and $\widetilde{S}f := 0$ for $f \perp \operatorname{ran} A$. To guarantee that $\widetilde{S}$ is well-defined, we need $Ag = 0 \Longrightarrow ATg = 0$ (all $g \in \mathcal{H}$), i.e., $T(\ker A) \subseteq \ker A$, which holds by Lemma 2.2. Thus $\widetilde{S}$ is well-defined. Note that

$$[\widehat{T}^*, \widehat{T}] = \begin{pmatrix} 0 & T^*A - A\widetilde{S}^* \\ AT - \widetilde{S}A & [\widetilde{S}^*, \widetilde{S}] + A^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & [\widetilde{S}^*, \widetilde{S}] + A^2 \end{pmatrix}. \tag{5.3.3}$$

We have thus established the first assertion. For the second assertion, a matricial argument works well. Take $A := [T^*, T]^{\frac{1}{2}}$. Since $T(\ker A) \subseteq \ker A$ and $\operatorname{ran} A$ is closed, we can write T as

$$T = \begin{pmatrix} T_0 & R \\ 0 & V \end{pmatrix} \begin{matrix} \ker A \\ \operatorname{ran} A \end{matrix}. \tag{5.3.4}$$

Also A can be decomposed as

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix} \begin{matrix} \ker A \\ \operatorname{ran} A \end{matrix}. \tag{5.3.5}$$

Note that A_0 is invertible. Thus if we define

$$B := \begin{pmatrix} 0 & 0 \\ 0 & A_0 V A_0^{-1} \end{pmatrix}, \tag{5.3.6}$$

then $A^*T = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A_0 V \end{smallmatrix}\right) = BA^*$ and evidently, $[T^*, T] = AA^*$. This completes the proof. $\square$

Towards an affirmative answer to Question 5.2 we must find a partially normal extension $\widehat{T}$. As a candidate one might suggest, in view of (5.3.1),

$$\widehat{T} = \begin{pmatrix} T & [T^*, T]^{\frac{1}{2}} \\ 0 & S \end{pmatrix},$$

where S is a continuous linear extension of $\widetilde{S}$ in Theorem 5.3. The key missing step is to show that S is bounded.

§5-2. Outer Propagation for Weighted Shifts

Do there exist hyponormal operators which are not weakly subnormal? To answer this question, we first establish that weakly subnormal weighted shifts possess a propagation property.

Theorem 5.4 (Outer Propagation of Weak Subnormality). *Let $T \equiv W_\alpha$ be a weighted shift with weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$. Assume that T is weakly subnormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$ then $\alpha_{n+k} = \alpha_n$ for all $k \geq 1$. In particular, if for some n_0, $\alpha_0 < \cdots < \alpha_{n_0}$ and $\alpha_{n_0+k} = \alpha_{n_0}$ for all $k \geq 1$ then T is weakly subnormal.*

Proof. The first assertion follows at once from (3.1.2). Towards the second assertion, observe that if W_β is a weighted shift whose weight sequence $\beta \equiv \{\beta_n\}_{n=0}^\infty$ is given by

$$\beta_n = \begin{cases} \alpha_n \sqrt{\frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 - \alpha_{n-1}^2}} & \text{if } n < n_0 \\ 0 & \text{if } n \geq n_0, \end{cases}$$

then $\widehat{T} = \left(\begin{smallmatrix} W_\alpha & [W_\alpha^*, W_\alpha]^{\frac{1}{2}} \\ 0 & W_\beta \end{smallmatrix}\right)$ is a partially normal extension of T. This completes the proof. $\square$

Example 5.5. *There exists a quadratically hyponormal weighted shift which is not weakly subnormal.*

For, let

$$\alpha_0 = \alpha_1 = \sqrt{\frac{2}{3}}, \quad \alpha_n = \sqrt{\frac{n+1}{n+2}}\ (n \geq 2) \quad (\text{cf. } [\mathbf{Cu1}, \text{ Proposition } 7]);$$

then W_α is quadratically hyponormal but not weakly subnormal (by Theorem 5.4). $\square$

Example 5.6. *There exists a weakly subnormal weighted shift which is not quadratically hyponormal.*

For, let W_α be a weighted shift with weight sequence $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$, where $\alpha_0 < \alpha_1 < \alpha_2 = \alpha_3 = \cdots$. Then, by Theorem 5.4 W_α is weakly subnormal while it is not quadratically hyponormal by Lemma 3.3 (iv). $\square$

We now show an additional property that must be satisfied by a weakly subnormal operator whose partially normal extension is hyponormal. First, recall that T^2 need not be hyponormal when T is just hyponormal (cf. [**Ha2**, Solution 209]).

Theorem 5.7. *Let $T \in \mathcal{L}(\mathcal{H})$ be a weakly subnormal operator whose minimal partially normal extension is hyponormal (e.g., a weighted shift; see Theorem 1.2). Then T^2 is hyponormal.*

Proof. Let $\widehat{T} :=$ m.p.n.e. (T), and let $f \in \mathcal{H}$. Then we have

$$\begin{aligned}
||T^2 f|| = ||\widehat{T}^2 f|| &= ||\widehat{T}(\widehat{T} f)|| \\
&\geq ||\widehat{T}^*(\widehat{T} f)|| \qquad \text{(by hyponormality of } \widehat{T}) \\
&= ||\widehat{T}\widehat{T}^* f|| \qquad \text{(because } \widehat{T} \text{ is partially normal)} \\
&\geq ||\widehat{T}^{*2} f|| \qquad \text{(again by hyponormality of } \widehat{T}) \\
&\geq ||T^{*2} f|| \\
&= ||(T^2)^* f||,
\end{aligned}$$

which implies that T^2 is hyponormal. $\square$

§5.3. The Case of Finite-Rank Self-Commutator

The self-commutator of an operator plays an important role in the study of subnormality (cf. [**McCY**]). On the other hand, weak subnormality gives useful information on self-commutator, i.e., if $T \in \mathcal{L}(\mathcal{H})$ is weakly subnormal then there exist operators A and B satisfying

$$[T^*, T] = AA^* \quad \text{and} \quad A^*T = BA^*.$$

Subnormal operators with finite rank self-commutators have been extensively studied ([**Ale**], [**McCY**], [**Mor**], [**OTT**], [**Xi1**], [**Xi2**]). In particular, B. Morrel [**Mor**] showed that a pure subnormal operator with rank-one self-commutator is unitarily equivalent to a linear function of the unilateral shift. Morrel essentially showed (also see [**Con**, p.162]) that if

(i) T is hyponormal;
(ii) $[T^*, T]$ is rank-one; and
(iii) $\ker [T^*, T]$ is invariant for T,

then $T - \beta$ is quasinormal for some $\beta \in \mathbb{C}$. Now remember that every pure quasinormal operator is unitarily equivalent to $U \otimes A$, where U is the unilateral shift and A is a positive operator with trivial kernel. Thus if $[T^*, T]$ is of rank-one (and hence so is $[(T-\beta)^*, (T-$

$\beta)]$), we must have $A \cong \lambda\ (\neq 0) \in \mathbb{C}$, so that $T-\beta \cong \alpha U$, or $T \cong \alpha U + \beta$. Now, by the above considerations, (1.2), Lemma 2.2, and Corollary 2.3, we can see that every pure weakly subnormal (or 2-hyponormal) operator with rank-one self-commutator is unitarily equivalent to a linear function of the unilateral shift.

Theorem 5.8. *Every weakly subnormal or 2-hyponormal operator T with rank-one self-commutator is subnormal. In addition, if T is pure then T is unitarily equivalent to a linear function of the unilateral shift.*

What can we say about weakly subnormal operators with finite rank self-commutator? The following example illustrates that we need not expect that they be subnormal in general.

Example 5.9. Consider a weighted shift W_α with weight sequence

$$\alpha: \sqrt{\frac{1}{2}}, \sqrt{\frac{8}{5}}, (\sqrt{3}, \sqrt{\frac{10}{3}}, \sqrt{\frac{17}{5}})^\wedge. \tag{5.9.1}$$

Then W_α is 2-hyponormal but not subnormal (cf. [**CL2**]). Observe that α is a two-step extension of a recursively generated weight sequence, so $p_n = 0$ for all $n \geq 3$. A straightforward calculation shows that $p_1 = 6/125$ and $p_2 = 16/125$. Using the notation in the proof of Theorem 1.2, we define

$$\widehat{W}_\alpha := \begin{pmatrix} W_\alpha & D \\ 0 & W_\beta \end{pmatrix}, \tag{5.9.2}$$

where $D := [W_\alpha^*, W_\alpha]^{\frac{1}{2}}$. Then $\widehat{W}_\alpha$ is a partially normal extension of W_α, and a straightforward calculation using (4.3) gives

$$[\widehat{W}_\alpha^*, \widehat{W}_\alpha]^{\frac{1}{2}} = 0_{\mathcal{H}} \bigoplus \begin{pmatrix} \sqrt{8/5} & & & \\ & \sqrt{112/55} & & \\ & & \sqrt{12/385} & \\ & & & \sqrt{48/175} \end{pmatrix} \oplus 0_\infty,$$

which shows that $\widehat{W}_\alpha$ has a rank-four self-commutator. We now claim that $\widehat{W}_\alpha$ is weakly subnormal. To this end, we need to construct a partially normal extension of $\widehat{W}_\alpha$. The argument is similar to that in the proof of Theorem 1.2. Consider the extension

$$\widehat{W}_\alpha^{(2)} := \begin{pmatrix} W_\alpha & D & 0 \\ 0 & W_\beta & C \\ 0 & 0 & R \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \\ \mathcal{H}, \end{matrix}$$

where

$$C := [\widehat{W}_\alpha^*, \widehat{W}_\alpha]^{\frac{1}{2}} \ominus 0_{\mathcal{H}} \quad \text{and} \quad R := \begin{pmatrix} 0 & & & \\ \sqrt{7/5} & 0 & & \\ & \sqrt{12/385} & 0 & \\ & & \sqrt{44/7} & 0 \end{pmatrix} \oplus 0_\infty.$$

Then we have

$$CW_\beta = \begin{pmatrix} 0 & & & \\ \sqrt{56/25} & 0 & & \\ & \sqrt{192/3025} & 0 & \\ & & \sqrt{48/245} & 0 \end{pmatrix} \oplus 0_\infty = RC.$$

A straightforward calculation shows that

$$\begin{aligned}
[\widehat{W}_\alpha^{(2)*}, \widehat{W}_\alpha^{(2)}] &= \begin{pmatrix} [W_\alpha^*, W_\alpha] - D^2 & W_\alpha^* D - D W_\beta^* & 0 \\ D W_\alpha - W_\beta D & D^2 + [W_\beta^*, W_\beta] - C^2 & W_\beta^* C - C R^* \\ 0 & C W_\beta - RC & C^2 + [R^*, R] \end{pmatrix} \\
&= 0_{\mathcal{H}\oplus\mathcal{H}} \bigoplus \begin{pmatrix} 3 & & & \\ & 257/385 & & \\ & & 44/7 & \\ & & & -1052/175 \end{pmatrix} \oplus 0_\infty,
\end{aligned}$$

which implies that $\mathcal{H} \oplus \mathcal{H} \subseteq \ker [\widehat{W}_\alpha^{(2)*}, \widehat{W}_\alpha^{(2)}]$, so $\widehat{W}_\alpha$ is weakly subnormal. However $\widehat{W}_\alpha$ is not subnormal, otherwise W_α would be subnormal. $\square$

6 Connections with Agler's Model Theory

Recently, M. Dritschel and S. McCullough [**DrMcC**] have developed a model theory for hyponormal contractions in the context of the Agler's abstract model theory [**Agl**]. The purpose is to find a small, representative subcollection of a given family of operators, a so–called *model*, with the property that any member of the family extends to a member of the subcollection. Following Agler [**Agl**], a *family* $\mathcal{F}$ is a bounded collection of Hilbert space operators which is closed with respect to arbitrary direct sums, restrictions to invariant subspaces, and unital $*$-representations. There are many examples of such families: subnormal contractions, contractions, isometries, etc. The *extremals* $\operatorname{ext}\mathcal{F}$ of $\mathcal{F}$ are those operators T in $\mathcal{F}$ whose only extensions in $\mathcal{F}$ are obtained by adding a direct summand to T. The extremals have a role in finding the smallest possible model for $\mathcal{F}$, the *boundary* $\partial\mathcal{F}$ of $\mathcal{F}$. In [**Agl**, Propositions 5.9 and 5.10] it was shown that the extremals belong to every model, and that every element of $\mathcal{F}$ lifts to an element of $\operatorname{ext}\mathcal{F}$. In [**DrMcC**] it was proved that if T is a contractive n-hyponormal operator and if

$$\operatorname{ran}(T^{*k}A) \cap \operatorname{ran} A = \{0\} \tag{6.1}$$

and

$$\ker T^{*k} \cap \operatorname{ran} A = \{0\} \tag{6.2}$$

for some $1 \le k \le n$, where $[T^*, T] = AA^*$, then T is extremal. The following corollary shows that if T is weakly subnormal then conditions (6.1) and (6.2) force T to be normal.

Theorem 6.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a weakly subnormal operator satisfying (6.1) and (6.2) for some $1 \le k \le n$. Then T must be normal, and therefore T is extremal for the collection $\mathcal{F}_{ws}$ of contractive weakly subnormal operators.*

Proof. Suppose T is weakly subnormal. Then there exists a partially normal extension $\widehat{T}$ of T such that

$$\widehat{T} = \begin{pmatrix} T & A \\ 0 & B \end{pmatrix} \quad \text{with } [T^*, T] = AA^* \text{ and } T^*A = AB^*.$$

Thus by induction, $T^{*k}A = AB^{*k}$, so $\operatorname{ran} T^{*k}A \subseteq \operatorname{ran} A$ for $1 \le k \le n$. Thus $\mathfrak{M}_k := \operatorname{ran}(T^{*k}A) \cap \operatorname{ran} A = \operatorname{ran}(T^{*k}A)$. By (6.1) we have that $\mathfrak{M}_k = \{0\}$, i.e., $T^{*k}A = 0$ for some $1 \le k \le n$. Let $f \in \mathcal{H}$, and let $g := Af$. We have

$$\begin{aligned} T^{*k}g = T^{*k}Af = 0 &\Longrightarrow g \in \ker T^{*k} \cap \operatorname{ran} A \\ &\Longrightarrow g = 0 \quad \text{(by (6.2))} \\ &\Longrightarrow Af = 0. \end{aligned}$$

It follows that $A = 0$, which implies that T is normal. The extremality of normal operators for $\mathcal{F}_{ws}$ follows by looking at self-commutators. □

Corollary 6.2. *Let T be a contractive 2-hyponormal operator with closed range self-commutator. Assume that T satisfies (6.1) and (6.2). Then T must be normal, and therefore T is extremal for $\mathfrak{h}_2$, the family of 2–hyponormal contractions.*

A natural question arises: *Is every 2-hyponormal operator satisfying (6.1) and (6.2) normal?* If the answer is negative then in view of Theorem 6.1 we can conclude that there exists a 2-hyponormal operator which is not weakly subnormal; this would answer Question 5.2 in the negative.

Finally, we examine five additional questions.

Question 6.3. *Does every 2-hyponormal operator have a partially normal extension which is also 2-hyponormal?*

Let us suppose that the answer is affirmative. Let $\mathcal{N}$, $\mathcal{S}$, and $\mathfrak{h}_2$ denote the collections of normal, subnormal, and 2-hyponormal contractions, respectively. We now claim that if every element of $\mathfrak{h}_2$ has a partially normal extension in $\mathfrak{h}_2$, then $\operatorname{ext} \mathfrak{h}_2 = \mathcal{N}$. The inclusion $\mathcal{N} \subseteq \operatorname{ext} \mathfrak{h}_2$ is evident, and was mentioned in Corollary 6.2. For the converse, suppose $T \in \operatorname{ext} \mathfrak{h}_2$. By our assumption T has a partially normal extension $\widehat{T}$ which is 2-hyponormal:

$$\widehat{T} = \begin{pmatrix} T & A \\ 0 & S \end{pmatrix} \in \mathfrak{h}_2.$$

By extremality, we have $A = 0$, so weak subnormality forces T to be normal. Therefore $\operatorname{ext} \mathfrak{h}_2 = \mathcal{N}$. By [**Agl**, Proposition 5.10], every element in $\mathfrak{h}_2$ would then have a normal extension, and hence $\mathfrak{h}_2 = \mathcal{S}$, which leads to a contradiction because we know that there are non–subnormal 2-hyponormal operators. We have thus obtained the following result, which answers Question 6.3 in the negative.

Proposition 6.4. *There exists a 2–hyponormal operator T which either does not have a partially normal extension, or such that* m.p.n.e. (T) *is not 2–hyponormal.*

Question 6.5. *Does the collection $\mathcal{F}_{ws}$ of weakly subnormal contractions form a family ?*

Note that $\mathcal{F}_{ws}$ is closed with respect to (i) restrictions to invariant subspaces (c.f. basic facts below Definition 1.1); (ii) unital $*$-representations (evident from the definition); and (iii) *finite* direct sums, by the following observation: if T_1 and T_2 have partially normal extensions $\left(\begin{smallmatrix} T_1 & A \\ 0 & B \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} T_2 & C \\ 0 & D \end{smallmatrix}\right)$, then

$$\begin{pmatrix} T_1 & 0 & A & 0 \\ 0 & T_2 & 0 & C \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & D \end{pmatrix}$$

is a partially normal extension of $T_1 \oplus T_2$. But it is not clear whether $\mathcal{F}_{ws}$ is closed with respect to *arbitrary* direct sums.

Question 6.6. *Is $\mathcal{F}_{ws}$ sot-closed ?*

Remember that $\mathcal{S}$ is *sot*-closed (in fact, $\mathcal{S} =$ *sot*-cl$\,\mathcal{N}$) and that the collection $\mathfrak{h}_k$ of k-hyponormal contractions is also *sot*-closed for each $k \geq 1$ (cf. [**CL2**]). In view of Theorem 1.2, we anticipate that every 2-hyponormal operator is weakly subnormal, so we conjecture that

$$\mathfrak{h}_2 \subseteq \mathcal{F}_{ws} \subseteq \mathfrak{h}_1.$$

Thus an affirmative answer to Question 6.6 would probably exhibit a *sot*-closed collection of operators between $\mathfrak{h}_2$ and $\mathfrak{h}_1$. More generally, we have:

Question 6.7. *Is there a sot-closed collection of operators between $\mathfrak{h}_k$ and $\mathfrak{h}_{k+1}$ for each $k \geq 1$?*

On the other hand, if $\mathcal{F}_{ws}$ were *not sot*-closed, we would ask:

Question 6.8. *Is every hyponormal operator a sot-limit of a sequence of weakly subnormal operators, i.e., $\mathfrak{h}_1 =$ sot-cl$\,\mathcal{F}_{ws}$?*

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