SUBNORMALITY AND \( k \)-HYPONORMALITY
OF TOEPLITZ OPERATORS:
A BRIEF SURVEY AND OPEN QUESTIONS

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The present note concerns subnormality and \( k \)-hyponormality of Toeplitz operators. We begin with a brief survey of research related to P.R. Halmos’s Problem 5 (cf. [Ha1],[Ha2]):

(Prob 5) Is every subnormal Toeplitz operator either normal or analytic?

As we know, (Prob 5) was answered in the negative by C. Cowen and J. Long [CoL]; directly connected with it was the following problem:

(0.1) Which Toeplitz operators are subnormal?

Let \( \mathcal{H} \) and \( \mathcal{K} \) be complex Hilbert spaces, let \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) be the set of bounded linear operators from \( \mathcal{H} \) to \( \mathcal{K} \) and write \( \mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H}) \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is said to be normal if \( T^*T = TT^* \), hyponormal if \( T^*T \geq TT^* \), and subnormal if \( T = N|_H \), where \( N \) is normal.
on some Hilbert space \( K \supseteq H \). If \( T \) is subnormal then \( T \) is also hyponormal. Recall that the Hilbert space \( L^2(T) \) has a canonical orthonormal basis given by the trigonometric functions \( e_n(z) = z^n \), for all \( n \in \mathbb{Z} \), and that the Hardy space \( H^2(T) \) is the closed linear span of \( \{ e_n : n = 0, 1, \cdots \} \). An element \( f \in L^2(T) \) is said to be analytic if \( f \in H^2(T) \), and co-analytic if \( f \in L^2(T) \ominus H^2(T) \). If \( P \) denotes the orthogonal projection from \( L^2(T) \) to \( H^2(T) \), then for every \( \varphi \in L^\infty(T) \) the operators \( T_\varphi \) and \( H_\varphi \) on \( H^2(T) \) defined by

\[
T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi(g) := (I - P)(\varphi g) \quad (g \in H^2(T))
\]

are called the Toeplitz operator and the Hankel operator, respectively, with symbol \( \varphi \).

(Prob 5) has been answered in the affirmative for trigonometric Toeplitz operators [ItW], and for quasinormal Toeplitz operators [AIW]. In 1976, M. Abrahamse [Abr] gave a general sufficient condition for the answer to (Prob 5) to be affirmative.

**Theorem 1 ([Abr]).** If

(i) \( T_\varphi \) is hyponormal;

(ii) \( \varphi \) or \( \bar{\varphi} \) is of bounded type (i.e., \( \varphi \) or \( \bar{\varphi} \) is a quotient of two analytic functions);

(iii) \( \ker[T_\varphi^*,T_\varphi] \) is invariant for \( T_\varphi \),

then \( T_\varphi \) is normal or analytic.

Since \( \ker[T^*,T] \) is invariant for every subnormal operator \( T \), Theorem 1 answers (Prob 5) affirmatively when \( \varphi \) or \( \bar{\varphi} \) is of bounded type. Also, in [Abr], Abrahamse proposed the following question, as a strategy to answer (Prob 5):

(Abr) Is the Bergman shift unitarily equivalent to a Toeplitz operator?

To study this question, recall that given a bounded sequence of positive numbers \( \alpha : \alpha_0, \alpha_1, \cdots \) (called weights), the (unilateral) weighted shift \( W_\alpha \) associated with \( \alpha \) is the operator on \( \ell^2(\mathbb{Z}_+) \) defined by \( W_\alpha e_n := \alpha_n e_{n+1} \) for all \( n \geq 0 \), where \( \{ e_n \}_{n=0}^\infty \) is the canonical orthonormal basis for \( \ell^2 \). It is straightforward to check that \( W_\alpha \) can never be normal, and that \( W_\alpha \) is hyponormal if and only if \( \alpha_n \leq \alpha_{n+1} \) for all \( n \geq 0 \). The Bergman shift is a weighted shift \( W_\alpha \) with weight sequence \( \alpha := \{ \sqrt{n/n+1} \}_{n=1}^\infty \); it is well known that the Bergman shift is subnormal. In 1983, S. Sun [Sun] showed that if a Toeplitz operator \( T_\varphi \) is unitarily equivalent to a hyponormal weighted shift \( W_\alpha \) with weight sequence \( \alpha \), then \( \alpha \) must be of the form

\[
(1.1) \quad \alpha = \left\{ (1 - \beta^{2n+2})^{n}||T_\varphi|| \right\}_{n=0}^\infty
\]

for some \( \beta \) (0 < \( \beta < 1 \)), thus answering (Abr) in the negative. Cowen and Long [CoL] showed that a unilateral weighted shift with weight sequence of the form (1.1) must be subnormal (see also [Fa2]). Consequently, we have:

**Theorem 2 ([Sun], [Cow2]).** Every hyponormal Toeplitz operator which is unitarily equivalent to a weighted shift must be subnormal.

Finally, in 1984 Cowen and Long [CoL] constructed a symbol \( \varphi \) for which \( T_\varphi \) is unitarily equivalent to a weighted shift with weight sequence (1.1). This helped answer (Prob 5) in the negative.
Subnormality and $k$-Hypornormality

Theorem 3 ([CoL],[Cow2]). Let $0 < \alpha < 1$ and let $\psi$ be a conformal map of the unit disk onto the interior of the ellipse with vertices $\pm (1+\alpha)i$ and passing through $\pm (1-\alpha)$. If $\varphi := (1-\alpha^2)^{-\frac{1}{2}} (\psi + \alpha \bar{\psi})$, then $T_\varphi$ is a weighted shift with weight sequence $\alpha_n = (1-\alpha^{2n+2})^{-\frac{1}{2}}$. Therefore, $T_\varphi$ is subnormal but neither normal nor analytic.

In view of Theorem 3, it is worth turning one’s attention to hyponormality of Toeplitz operators, which has been studied by M. Abrahamse [Abr], C. Cowen [Cow1],[Cow2], P. Fan [Fa1], C. Gu [Gu], T. Ito and T. Wong [ItW], T. Nakazi and K. Takahashi [NaT], D. Yu [Yu], K. Zhu [Zhu], D. Farenick, the authors, and their collaborators (cf. [FaL1],[FaL2],[CuL1],[HKL],[KiL]). An elegant theorem of C. Cowen [Cow3] characterizes the hyponormality of a Toeplitz operator $T_\varphi$ on $H^2$ by properties of the symbol $\varphi \in L^\infty(T)$. K. Zhu [Zhu] reformulated Cowen’s criterion and then showed that the hyponormality of $T_\varphi$ with polynomial symbols $\varphi$ can be decided by a method based on the classical interpolation theorem of I. Schur [Sch]. Here, we shall use a variant of Cowen’s theorem [Cow3] that was first proposed by Nakazi and Takahashi [NaT].

Cowen’s Theorem. Suppose $\varphi \in L^\infty(T)$ is arbitrary and write
\[ \mathcal{E}(\varphi) = \{ k \in H^\infty(T) : ||k||_\infty \leq 1 \text{ and } \varphi - k\varphi \in H^\infty(T) \}. \]
Then $T_\varphi$ is hyponormal if and only if the set $\mathcal{E}(\varphi)$ is nonempty.

On the other hand, the Bram–Halmos criterion for subnormality states that an operator $T$ is subnormal if and only if
\[ \sum_{i,j} (T^i x_j, T^j x_i) \geq 0 \]
for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([Bra],[Con, II.1.9]). It is easy to see that this is equivalent to the following positivity test:
\begin{equation}
\begin{pmatrix}
I & T & \cdots & T^k \\
T & T^*T & \cdots & T^{k}kT \\
\vdots & \vdots & \ddots & \vdots \\
T^k & T^*T^k & \cdots & T^kT^k
\end{pmatrix} \geq 0 \quad (\text{all } k \geq 1).
\end{equation}
Condition (3.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (3.1) for $k = 1$ is equivalent to the hyponormality of $T$, while subnormality requires the validity of (3.1) for all $k$. If we denote by $[A, B] := AB - BA$ the commutator of two operators $A$ and $B$, and if we define $T$ to be $k$-hyponormal whenever the $k \times k$ operator matrix
\begin{equation}
M_k(T) := ([T^i, T^j])_{i,j=1}^{k}
\end{equation}
is positive, or equivalently, the $(k+1) \times (k+1)$ operator matrix in (3.1) is positive (via the operator version of Choleski’s Algorithm), then the Bram–Halmos criterion can be rephrased as saying that $T$ is subnormal if and only if $T$ is $k$-hyponormal for every $k \geq 1$ ([CMX]).
Recall now ([Ath],[Cu2],[CMX]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be \textit{weakly $k$–hyponormal} if
\[
LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^{k} \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}
\]
consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is \textit{weakly positive}, i.e.,
\[
(M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}) \geq 0 \quad \text{for} \quad x \in \mathcal{H} \quad \text{and} \quad \lambda_0, \cdots, \lambda_k \in \mathbb{C} \quad ([CMX]).
\]
If $k = 2$ then $T$ is said to be \textit{quadratically hyponormal}. Similarly, $T$ is said to be \textit{polynomially hyponormal} if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that $k$–hyponormal $\Rightarrow$ weakly $k$–hyponormal, but the converse is not true in general.

It is now natural to try to understand the gap between $k$–hyponormality and subnormality for Toeplitz operators. As a first inquiry in this line of thought we pose the following ([CuL1]):

\textbf{QUESTION A. Is every 2–hyponormal Toeplitz operator subnormal ?}

In [CuL1], the following was shown:

\textbf{THEOREM 4 ([CuL1]).} Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic. Hence, in particular, every 2-hyponormal trigonometric Toeplitz operator is subnormal.

It is well known ([Cu1],[Cu2]) that, for weighted shifts, there are gaps between hyponormality and quadratic hyponormality, and between quadratic hyponormality and 2–hyponormality. Note that Theorem 4 says more: every \textit{quadratically} hyponormal trigonometric Toeplitz operator is subnormal. Thus Theorem 4 shows that there is a big gap between hyponormality and quadratic hyponormality for Toeplitz operators. For example, if
\[
\varphi(z) = \sum_{n=-m}^{N} a_n z^n \quad (m < N)
\]
is such that $T_\varphi$ is hyponormal, then by Theorem 4 $T_\varphi$ is never quadratically hyponormal, since $T_\varphi$ is neither analytic nor normal. (Recall that if such a $T_\varphi$ is normal then $m = N$ (cf. [FaL1]).)

In view of Theorem 4, the following question arises naturally:

\textbf{QUESTION B. Is every quadratically hyponormal Toeplitz operator 2–hyponormal ?}

An affirmative answer to Question B would show that there exists no gap between quadratic hyponormality and 2–hyponormality for Toeplitz operators. A negative answer would give rise to a challenging problem: \textit{Characterize non–2–hyponormal quadratically hyponormal Toeplitz operators; more generally, characterize non–$k$–hyponormal weakly $k$–hyponormal Toeplitz operators.}

We can extend Theorem 4. First we observe:
Proposition 5 ([CuL2]). If $T \in L(H)$ is 2-hyponormal then

$T(\ker [T^*, T]) \subseteq \ker [T^*, T]$.

**Proof.** Suppose that $[T^*, T] f = 0$. Since $T$ is 2-hyponormal, it follows from (3.2) that (cf. [CMX, Lemma 1.4])

$$|([T^*, T] g, f)| \leq ([T^*, T] f, f) ([T^2, T^2] g, g) \quad \text{for all} \quad g \in H.$$ 

By assumption, we have that for all $g \in H$, $0 = ([T^2, T] f, f) = (g, [T^2, T] f)$, so that $[T^2, T] f = 0$, i.e., $T^* T^2 f = T^2 T^* f$. Therefore,

$$[T^*, T] T f = (T^* T^2 - T T^* T) f = (T^2 T^* - T T^* T) f = T[T^*, T] f = 0,$$

which proves (5.1).

Corollary 6. If $T_\varphi$ is 2-hyponormal and if $\varphi$ or $\bar{\varphi}$ is of bounded type then $T_\varphi$ is normal or analytic, so that $T_\varphi$ is subnormal.

**Proof.** This follows at once from Theorem 1 and Proposition 5.

Corollary 7. If $T_\varphi$ is a 2-hyponormal operator such that $E(\varphi)$ contains at least two elements then $T_\varphi$ is normal or analytic, so that $T_\varphi$ is subnormal.

**Proof.** This follows from Corollary 6 and the fact, shown in [NaT, Proposition 8], that if $E(\varphi)$ contains at least two elements then $\varphi$ is of bounded type.

From Corollaries 6 and 7, we can see that if $T_\varphi$ is 2-hyponormal but not subnormal then $\varphi$ is not of bounded type and $E(\varphi)$ consists of exactly one element.

From Corollary 6 we can see that if $T_\varphi$ is a 2-hyponormal operator such that $\varphi$ or $\bar{\varphi}$ is of bounded type then $T_\varphi$ has a nontrivial invariant subspace. The following question arises naturally:

**Question C.** Does every 2-hyponormal Toeplitz operator have a nontrivial invariant subspace? More generally, does every 2-hyponormal operator have a nontrivial invariant subspace?

It is well known ([Bro]) that if $T$ is a hyponormal operator such that $R(\sigma(T)) \neq C(\sigma(T))$ then $T$ has a nontrivial invariant subspace. But it remains still open whether every hyponormal operator with $R(\sigma(T)) = C(\sigma(T))$ (i.e., with a thin spectrum) has a nontrivial invariant subspace. Recall that $T \in L(H)$ is called a von Neumann operator if $\sigma(T)$ is a spectral set for $T$; as shown by J. Agler [Agl], every von Neumann operator has a nontrivial invariant subspace. Recently, B. Prunaru [Pru] established that polynomially hyponormal operators also possess the same property. The following is a sub-question of Question C.

**Question D.** Is every 2-hyponormal operator with thin spectrum a von Neumann operator?

Recall that $\varphi \in L^\infty(T)$ is called almost analytic if $z^n \varphi$ is analytic for some positive $n$ and is called almost coanalytic if $\bar{\varphi}$ is almost analytic. Observe that if $\varphi$ is real-valued and almost analytic, then $\varphi$ is constant. It is easy to check that if $\varphi$ is almost coanalytic and $T_\varphi$ is hyponormal then $\varphi$ must be a trigonometric polynomial. But this is not the case for almost analytic functions $\varphi$. To see this, we reformulate Cowen’s Theorem.
Suppose \( \varphi \in L^\infty(T) \) is of the form 
\[
\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n
\]
and \( k(z) = \sum_{n=0}^{\infty} c_n z^n \) is in \( H^2(T) \). Then \( \varphi - k \bar{\varphi} \in H^\infty \) has a solution \( k \in H^\infty \) if and only if
\[
\begin{bmatrix}
\bar{a}_1 & \bar{a}_2 & \bar{a}_3 & \ldots & \bar{a}_n & \ldots \\
\bar{a}_2 & \bar{a}_3 & \ldots & \bar{a}_n & \ldots \\
\bar{a}_3 & \ldots & \ldots \\
\vdots & \bar{a}_n & \ldots \\
\bar{a}_n & \ldots \\
\vdots
\end{bmatrix}
\begin{bmatrix}
0 \\
1/2 \\
1/4 \\
\vdots \\
1/8
\end{bmatrix}
= \begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix},
\]
that is, \( H_\varphi k = H_\varphi e_0 \), where \( e_0 = (1, 0, 0, \ldots) \). Thus, by Cowen’s Theorem, \( T_\varphi \) is hyponormal if and only if there exists a solution \( k \in H^\infty(T) \) of the equation (7.1) such that \( ||k||_\infty \leq 1 \).

Now suppose \( \varphi \in L^\infty(T) \) is a function of the form
\[
\varphi(z) = \frac{1}{6} z^{-1} + \sum_{n=2}^{\infty} \frac{1}{2^n-1} z^n.
\]
Then \( k(z) = \sum_{n=0}^{\infty} c_n z^n \) satisfies \( \varphi - k \bar{\varphi} \in H^\infty \) if and only if
\[
\begin{bmatrix}
0 & 1/2 & 1/4 & 1/8 & \ldots \\
1/2 & 1/4 & 1/8 & \ldots \\
1/4 & 1/8 & \ldots \\
1/8 & \ldots \\
\vdots
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
1/6 \\
0 \\
0 \\
\vdots
\end{bmatrix},
\]
A straightforward calculation shows that
\[
k(z) = -\frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{2^n+1} z^n
\]
satisfies (7.1). Also, it is easy to see that \( k(z) = \frac{1}{3} \frac{z^{-1/2}}{1-\frac{1}{2} z} \), so \( ||k||_\infty = \frac{1}{3} \). Therefore \( T_\varphi \) is hyponormal (cf. [CuL1, Example 2.3]).

However we have:

**Theorem 8.** *If* \( T_\varphi \) *is 2–hyponormal with non–analytic almost analytic symbol \( \varphi \) *then* \( \varphi \) *must be a trigonometric polynomial.*

**Proof.** Since almost analytic functions are of bounded type it follows from Corollary 6 that if \( T_\varphi \) is 2–hyponormal with non–analytic almost analytic symbol \( \varphi \) then \( T_\varphi \) must be normal. Since by the Brown–Halmos Theorem [BrH], every normal Toeplitz operator is a rotation and a translation of a hermitian Toeplitz operator, it follows that \( \varphi \) must be a trigonometric polynomial.

Although the existence of a non–subnormal polynomially hyponormal weighted shift was established in [CuP1] and [CuP2], it is still an open question whether the implication “polynomially hyponormal \( \Rightarrow \) subnormal” can be disproved with a Toeplitz operator.
QUESTION E. Does there exist a Toeplitz operator which is polynomially hyponormal but not subnormal?

It is well known that $T$ is a von Neumann operator if and only if $q(T)$ is normaloid (i.e., norm equals spectral radius) for every rational function $q$ with poles outside $\sigma(T)$. Thus if $T$ is \textit{rationally} hyponormal, i.e., $q(T)$ is hyponormal for every rational function $q$ with poles outside $\sigma(T)$, then $T$ is a von Neumann operator. Thus the following question arises naturally:

QUESTION F. Does there exist a polynomially hyponormal operator which is not a von Neumann operator? And within the class of Toeplitz operators?

An affirmative answer to Question F guarantees the existence of polynomially hyponormal operators which are not rationally hyponormal (and hence not subnormal). Within the class of trigonometric Toeplitz operators we have, by Theorem 4, that if $T_\varphi$ is polynomially hyponormal then $T_\varphi$ is a von Neumann operator.

In [CuL2] it was shown that every pure 2-hyponormal operator with rank-one self-commutator is a linear function of the unilateral shift. On the other hand, J.E. McCarthy and L. Yang [McCY] have classified all rationally cyclic subnormal operators with finite rank self-commutators. However, it is still open which are the pure subnormal operators with finite rank self-commutator. Related to this, we formulate the following:

QUESTION G. If $T_\varphi$ is a 2-hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that $T_\varphi$ is analytic? If the answer is affirmative, is $\varphi$ either an analytic polynomial or a linear function of a finite Blaschke product?

We shall give a partial positive answer to Question G. To do this we recall Theorem 15 in [NaT], which states that if $T_\varphi$ is subnormal and $\varphi = q\bar{\varphi}$, where $q$ is a finite Blaschke product, then $T_\varphi$ is normal or analytic. A careful examination of the proof of that theorem reveals that it uses the subnormality assumption only for the fact that $\ker [T_\varphi^*, T_\varphi]$ is invariant under $T_\varphi$. Thus in view of Proposition 5, the theorem is still valid for “2-hyponormal” in place of “subnormal”. We thus have:

Lemmas 9. If $T_\varphi$ is 2-hyponormal and $\varphi = q\bar{\varphi}$, where $q$ is a finite Blaschke product, then $T_\varphi$ is normal or analytic.

We now give a partial answer to Question G.

Theorem 10. Suppose $\log |\varphi|$ is not integrable. If $T_\varphi$ is a 2-hyponormal operator with nonzero finite rank self-commutator then $T_\varphi$ is analytic.

Proof. If $T_\varphi$ is hyponormal such that $\log |\varphi|$ is not integrable then, by an argument of [NaT, Theorem 4], $\varphi = q\bar{\varphi}$ for some inner function $q$. Also if $T_\varphi$ has a finite rank self-commutator then, by [NaT, Theorem 10], there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$. If $q \neq b$, so that $\mathcal{E}(\varphi)$ contains at least two elements, then by Corollary 7, $T_\varphi$ is normal or analytic. If instead $q = b$ then, by Lemma 9, $T_\varphi$ is also normal or analytic.

Theorem 10 reduces Question G to the class of Toeplitz operators such that $\log |\varphi|$ is integrable. If $\log |\varphi|$ is integrable then there exists an outer function $e$ such that $|\varphi| = |e|$. Thus we may write $\varphi = ue$, where $u$ is a unimodular function. Since by the Douglas–Rudin
Theorem (cf. [Gar, p.192]), every unimodular function can be approximated by quotients of inner functions, it follows that if \( \log|\varphi| \) is integrable then \( \varphi \) can be approximated by functions of bounded type. Therefore if we could obtain a sequence \( \psi_n \) converging to \( \varphi \) such that \( T_{\psi_n} \) is 2–hyponormal with finite rank self–commutator for each \( n \), then we would answer Question G affirmatively. On the other hand, if \( T_\varphi \) attains its norm, then by a result of Brown and Douglas [BrD] \( \varphi \) is of the form \( \varphi = \lambda \frac{\psi}{\theta} \) with \( \lambda > 0 \) and \( \psi, \theta \) inner. Thus \( \varphi \) is of bounded type. Therefore, by Corollary 7, if \( T_\varphi \) is 2–hyponormal and attains its norm then \( T_\varphi \) is normal or analytic. However we have not been able to decide that if \( T_\varphi \) is a 2–hyponormal operator with finite rank self–commutator then \( T_\varphi \) attains its norm.

References


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