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SUBNORMALITY AND k–HYPONORMALITY OF TOEPLITZ OPERATORS: A BRIEF SURVEY AND OPEN QUESTIONS

RAÚL E. CURTO
Department of Mathematics, University of Iowa
Iowa City, IA 52242, U.S.A.
E-mail:curto@math.uiowa.edu

WOO YOUNG LEE
Department of Mathematics, Seoul National University
Seoul 151-742, Korea
E-mail: wylee@math.snu.ac.kr

The present note concerns subnormality and k–hyponormality of Toeplitz operators. We begin with a brief survey of research related to P.R. Halmos's Problem 5 (cf. [Ha1],[Ha2]):

(Prob 5) Is every subnormal Toeplitz operator either normal or analytic ?

As we know, (Prob 5) was answered in the negative by C. Cowen and J. Long [CoL]; directly connected with it is the following problem:

(0.1) Which Toeplitz operators are subnormal ?

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal

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on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. Recall that the Hilbert space $L^2(\mathbf{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbf{Z}$, and that the Hardy space $H^2(\mathbf{T})$ is the closed linear span of $\{e_n : n = 0, 1, \cdots\}$. An element $f \in L^2(\mathbf{T})$ is said to be analytic if $f \in H^2(\mathbf{T})$, and co-analytic if $f \in L^2(\mathbf{T}) \ominus H^2(\mathbf{T})$. If P denotes the orthogonal projection from $L^2(\mathbf{T})$ to $H^2(\mathbf{T})$, then for every $\varphi \in L^\infty(\mathbf{T})$ the operators T_φ and H_φ on $H^2(\mathbf{T})$ defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi(g) := (I - P)(\varphi g) \qquad (g \in H^2(\mathbf{T}))$$

are called the *Toeplitz operator* and the *Hankel operator*, respectively, with symbol φ.

(Prob 5) has been answered in the affirmative for *trigonometric* Toeplitz operators [ItW], and for *quasinormal* Toeplitz operators [AIW]. In 1976, M. Abrahamse [Abr] gave a general sufficient condition for the answer to (Prob 5) to be affirmative.

THEOREM 1 ([Abr]). *If*

(i) T_φ *is hyponormal;*

(ii) φ *or* $\bar{\varphi}$ *is of bounded type (i.e.,* φ *or* $\bar{\varphi}$ *is a quotient of two analytic functions);*

(iii) $\ker[T_\varphi^*, T_\varphi]$ *is invariant for* T_φ,

then T_φ *is normal or analytic.*

Since $\ker[T^*, T]$ is invariant for every subnormal operator T, Theorem 1 answers (Prob 5) affirmatively when φ or $\bar{\varphi}$ is of bounded type. Also, in [Abr], Abrahamse proposed the following question, as a strategy to answer (Prob 5):

(Abr) Is the Bergman shift unitarily equivalent to a Toeplitz operator ?

To study this question, recall that given a bounded sequence of positive numbers α : $\alpha_0, \alpha_1, \cdots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbf{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bergman shift is a weighted shift W_α with weights $\alpha := \left\{\sqrt{\frac{n}{n+1}}\right\}_{n=1}^\infty$; it is well known that the Bergman shift is subnormal. In 1983, S. Sun [Sun] showed that if a Toeplitz operator T_φ is unitarily equivalent to a hyponormal weighted shift W_α with weight sequence α, then α must be of the form

$$\alpha = \left\{(1 - \beta^{2n+2})^{\frac{1}{2}} ||T_\varphi||\right\}_{n=0}^\infty \tag{1.1}$$

for some β ($0 < \beta < 1$), thus answering (Abr) in the negative. Cowen and Long [CoL] showed that a unilateral weighted shift with weight sequence of the form (1.1) must be subnormal (see also [Fa2]). Consequently, we have:

THEOREM 2 ([Sun], [Cow2]). *Every hyponormal Toeplitz operator which is unitarily equivalent to a weighted shift must be subnormal.*

Finally, in 1984 Cowen and Long [CoL] constructed a symbol φ for which T_φ is unitarily equivalent to a weighted shift with weight sequence (1.1). This helped answer (Prob 5) in the negative.

THEOREM 3 ([CoL],[Cow2]). *Let $0 < \alpha < 1$ and let ψ be a conformal map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. If $\varphi := (1-\alpha^2)^{-1}(\psi + \alpha\bar{\psi})$, then T_φ is a weighted shift with weight sequence $\alpha_n = (1-\alpha^{2n+2})^{-\frac{1}{2}}$. Therefore, T_φ is subnormal but neither normal nor analytic.*

In view of Theorem 3, it is worth turning one's attention to hyponormality of Toeplitz operators, which has been studied by M. Abrahamse [Abr], C. Cowen [Cow1],[Cow2], P. Fan [Fa1], C. Gu [Gu], T. Ito and T. Wong [ItW], T. Nakazi and K. Takahashi [NaT], D. Yu [Yu], K. Zhu [Zhu], D. Farenick, the authors, and their collaborators (cf. [FaL1],[FaL2],[CuL1],[HKL],[KiL]). An elegant theorem of C. Cowen [Cow3] characterizes the hyponormality of a Toeplitz operator T_φ on $H^2(\mathbf{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbf{T})$. K. Zhu [Zhu] reformulated Cowen's criterion and then showed that the hyponormality of T_φ with polynomial symbols φ can be decided by a method based on the classical interpolation theorem of I. Schur [Sch]. Here, we shall use a variant of Cowen's theorem [Cow3] that was first proposed by Nakazi and Takahashi [NaT].

COWEN'S THEOREM . *Suppose $\varphi \in L^\infty(\mathbf{T})$ is arbitrary and write*

$$\mathcal{E}(\varphi) = \{k \in H^\infty(\mathbf{T}) : ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty(\mathbf{T})\}.$$

Then T_φ is hyponormal if and only if the set $\mathcal{E}(\varphi)$ is nonempty.

On the other hand, the Bram–Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j}(T^i x_j, T^j x_i) \ge 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([Bra],[Con, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k} \\ T & T^*T & \dots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \dots & T^{*k}T^k \end{pmatrix} \ge 0 \qquad (\text{all } k \ge 1). \tag{3.1}$$

Condition (3.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (3.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (3.1) for all k. If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B, and if we define T to be k–*hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{3.2}$$

is positive, or equivalently, the $(k+1) \times (k+1)$ operator matrix in (3.1) is positive (via the operator version of Choleski's Algorithm), then the Bram–Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k–hyponormal for every $k \ge 1$ ([CMX]).

Recall now ([Ath],[Cu2],[CMX]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k–hyponormal* if

$$LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^{k} \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbf{C}^k \right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e.,

$$(M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}) \geq 0 \quad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbf{C} \quad ([\text{CMX}]).$$

If $k = 2$ then T is said to be *quadratically hyponormal*. Similarly, T is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbf{C}[z]$. It is known that k–hyponormal $\Rightarrow$ weakly k–hyponormal, but the converse is not true in general.

It is now natural to try to understand the gap between k–hyponormality and subnormality for Toeplitz operators. As a first inquiry in this line of thought we pose the following ([CuL1]):

Question A. *Is every 2–hyponormal Toeplitz operator subnormal ?*

In [CuL1], the following was shown:

Theorem 4 ([CuL1]). *Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic. Hence, in particular, every 2-hyponormal trigonometric Toeplitz operator is subnormal.*

It is well known ([Cu1],[Cu2]) that, for weighted shifts, there are gaps between hyponormality and quadratic hyponormality, and between quadratic hyponormality and 2–hyponormality. Note that Theorem 4 says more: every *quadratically* hyponormal trigonometric Toeplitz operator is subnormal. Thus Theorem 4 shows that there is a big gap between hyponormality and quadratic hyponormality for Toeplitz operators. For example, if

$$\varphi(z) \equiv \sum_{n=-m}^{N} a_n z^n \quad (m < N)$$

is such that T_φ is hyponormal, then by Theorem 4 T_φ is never quadratically hyponormal, since T_φ is neither analytic nor normal. (Recall that if such a T_φ is normal then $m = N$ (cf. [FaL1]).)

In view of Theorem 4, the following question arises naturally:

Question B. *Is every quadratically hyponormal Toeplitz operator* 2*–hyponormal ?*

An affirmative answer to Question B would show that there exists no gap between quadratic hyponormality and 2–hyponormality for Toeplitz operators. A negative answer would give rise to a challenging problem: *Characterize non–*2*–hyponormal quadratically hyponormal Toeplitz operators; more generally, characterize non–k–hyponormal weakly k–hyponormal Toeplitz operators.*

We can extend Theorem 4. First we observe:

PROPOSITION 5 ([CuL2]). *If $T \in \mathcal{L}(\mathcal{H})$ is 2–hyponormal then*

$$T\big(\ker[T^*,T]\big) \subseteq \ker[T^*,T]. \tag{5.1}$$

PROOF. Suppose that $[T^*,T]f = 0$. Since T is 2–hyponormal, it follows from (3.2) that (cf. [CMX, Lemma 1.4])

$$|([T^{*2},T]g,f)|^2 \le ([T^*,T]f,f)([T^{*2},T^2]g,g) \quad \text{for all } g \in \mathcal{H}.$$

By assumption, we have that for all $g \in \mathcal{H}$, $0 = ([T^{*2},T]g,f) = (g,[T^{*2},T]^*f)$, so that $[T^{*2},T]^*f = 0$, i.e., $T^*T^2f = T^2T^*f$. Therefore,

$$[T^*,T]Tf = (T^*T^2 - TT^*T)f = (T^2T^* - TT^*T)f = T[T^*,T]f = 0,$$

which proves (5.1). ∎

COROLLARY 6. *If T_φ is 2–hyponormal and if φ or $\bar\varphi$ is of bounded type then T_φ is normal or analytic, so that T_φ is subnormal.*

PROOF. This follows at once from Theorem 1 and Proposition 5. ∎

COROLLARY 7. *If T_φ is a 2–hyponormal operator such that $\mathcal{E}(\varphi)$ contains at least two elements then T_φ is normal or analytic, so that T_φ is subnormal.*

PROOF. This follows from Corollary 6 and the fact, shown in [NaT, Proposition 8], that if $\mathcal{E}(\varphi)$ contains at least two elements then φ is of bounded type. ∎

From Corollaries 6 and 7, we can see that if T_φ is 2–hyponormal but not subnormal then φ is not of bounded type and $\mathcal{E}(\varphi)$ consists of exactly one element.

From Corollary 6 we can see that if T_φ is a 2–hyponormal operator such that φ or $\bar\varphi$ is of bounded type then T_φ has a nontrivial invariant subspace. The following question arises naturally:

QUESTION C. *Does every 2–hyponormal Toeplitz operator have a nontrivial invariant subspace ? More generally, does every 2–hyponormal operator have a nontrivial invariant subspace ?*

It is well known ([Bro]) that if T is a hyponormal operator such that $R(\sigma(T)) \neq C(\sigma(T))$ then T has a nontrivial invariant subspace. But it remains still open whether every hyponormal operator with $R(\sigma(T)) = C(\sigma(T))$ (i.e., with a *thin* spectrum) has a nontrivial invariant subspace. Recall that $T \in \mathcal{L}(\mathcal{H})$ is called a *von Neumann operator* if $\sigma(T)$ is a spectral set for T; as shown by J. Agler [Agl], every von Neumann operator has a nontrivial invariant subspace. Recently, B. Prunaru [Pru] established that polynomially hyponormal operators also possess the same property. The following is a sub-question of Question C.

QUESTION D. *Is every 2–hyponormal operator with thin spectrum a von Neumann operator ?*

Recall that $\varphi \in L^\infty(\mathbf{T})$ is called *almost analytic* if $z^n\varphi$ is analytic for some positive n and is called *almost coanalytic* if $\bar\varphi$ is almost analytic. Observe that if φ is real–valued and almost analytic, then φ is constant. It is easy to check that if φ is almost coanalytic and T_φ is hyponormal then φ must be a trigonometric polynomial. But this is not the case for almost analytic functions φ. To see this, we reformulate Cowen's Theorem.

Suppose $\varphi \in L^\infty(\mathbf{T})$ is of the form $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $k(z) = \sum_{n=0}^{\infty} c_n z^n$ is in $H^2(\mathbf{T})$. Then $\varphi - k\,\bar\varphi \in H^\infty$ has a solution $k \in H^\infty$ if and only if

$$\begin{pmatrix} \bar a_1 & \bar a_2 & \bar a_3 & \dots & \bar a_n & \dots \\ \bar a_2 & \bar a_3 & \dots & \bar a_n & \dots & \\ \bar a_3 & \dots & \dots & \dots & & \\ \vdots & \bar a_n & \dots & & & \\ \bar a_n & \dots & & & & \\ \vdots & & & & & \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{-1} \\ a_{-2} \\ a_{-3} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}, \tag{7.1}$$

that is, $H_{\bar\varphi} k = H_\varphi e_0$, where $e_0 = (1, 0, 0, \cdots)$. Thus, by Cowen's Theorem, T_φ is hyponormal if and only if there exists a solution $k \in H^\infty(\mathbf{T})$ of the equation (7.1) such that $||k||_\infty \le 1$.

Now suppose $\varphi \in L^\infty(\mathbf{T})$ is a function of the form

$$\varphi(z) = \frac{1}{6}\, z^{-1} + \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} z^n .$$

Then $k(z) = \sum_{n=0}^{\infty} c_n z^n$ satisfies $\varphi - k\,\bar\varphi \in H^\infty$ if and only if

$$\begin{pmatrix} 0 & \frac12 & \frac14 & \frac18 & \dots \\ \frac12 & \frac14 & \frac18 & \dots & \\ \frac14 & \frac18 & \dots & & \\ \frac18 & \dots & & & \\ \vdots & & & & \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac16 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}.$$

A straightforward calculation shows that

$$k(z) = -\frac{1}{6} + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} z^n$$

satisfies (7.1). Also, it is easy to see that $k(z) = \frac13 \frac{z - \frac12}{1 - \frac12 z}$, so $||k||_\infty = \frac13$. Therefore T_φ is hyponormal (cf. [CuL1, Example 2.3]).

However we have:

THEOREM 8. *If T_φ is 2–hyponormal with non–analytic almost analytic symbol φ then φ must be a trigonometric polynomial.*

PROOF. Since almost analytic functions are of bounded type it follows from Corollary 6 that if T_φ is 2–hyponormal with non–analytic almost analytic symbol φ then T_φ must be normal. Since by the Brown–Halmos Theorem [BrH], every normal Toeplitz operator is a rotation and a translation of a hermitian Toeplitz operator, it follows that φ must be a trigonometric polynomial. ∎

Although the existence of a non–subnormal polynomially hyponormal weighted shift was established in [CuP1] and [CuP2], it is still an open question whether the implication "polynomially hyponormal ⇒ subnormal" can be disproved with a Toeplitz operator.

QUESTION E. *Does there exist a Toeplitz operator which is polynomially hyponormal but not subnormal ?*

It is well known that T is a von Neumann operator if and only if $q(T)$ is normaloid (i.e., norm equals spectral radius) for every rational function q with poles outside $\sigma(T)$. Thus if T is *rationally* hyponormal, i.e., $q(T)$ is hyponormal for every rational function q with poles outside $\sigma(T)$, then T is a von Neumann operator. Thus the following question arises naturally:

QUESTION F. *Does there exist a polynomially hyponormal operator which is not a von Neumann operator ? And within the class of Toeplitz operators ?*

An affirmative answer to Question F guarantees the existence of polynomially hyponormal operators which are not rationally hyponormal (and hence not subnormal). Within the class of trigonometric Toeplitz operators we have, by Theorem 4, that if T_φ is polynomially hyponormal then T_φ is a von Neumann operator.

In [CuL2] it was shown that every pure 2–hyponormal operator with rank-one self–commutator is a linear function of the unilateral shift. On the other hand, J.E. McCarthy and L. Yang [McCY] have classified all rationally cyclic subnormal operators with finite rank self–commutators. However, it is still open which are the pure subnormal operators with finite rank self–commutator. Related to this, we formulate the following:

QUESTION G. *If T_φ is a 2–hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that T_φ is analytic ? If the answer is affirmative, is φ either an analytic polynomial or a linear function of a finite Blaschke product ?*

We shall give a partial positive answer to Question G. To do this we recall Theorem 15 in [NaT], which states that if T_φ is subnormal and $\varphi = q\bar{\varphi}$, where q is a finite Blaschke product, then T_φ is normal or analytic. A careful examination of the proof of that theorem reveals that it uses the subnormality assumption only for the fact that $\ker [T_\varphi^*, T_\varphi]$ is invariant under T_φ. Thus in view of Proposition 5, the theorem is still valid for "2–hyponormal" in place of "subnormal". We thus have:

LEMMA 9. *If T_φ is 2–hyponormal and $\varphi = q\bar{\varphi}$, where q is a finite Blaschke product, then T_φ is normal or analytic.*

We now give a partial answer to Question G.

THEOREM 10. *Suppose* $\log|\varphi|$ *is not integrable. If T_φ is a 2–hyponormal operator with nonzero finite rank self–commutator then T_φ is analytic.*

PROOF. If T_φ is hyponormal such that $\log|\varphi|$ is not integrable then, by an argument of [NaT, Theorem 4], $\varphi = q\bar{\varphi}$ for some inner function q. Also if T_φ has a finite rank self–commutator then, by [NaT, Theorem 10], there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$. If $q \neq b$, so that $\mathcal{E}(\varphi)$ contains at least two elements, then by Corollary 7, T_φ is normal or analytic. If instead $q = b$ then, by Lemma 9, T_φ is also normal or analytic. ∎

Theorem 10 reduces Question G to the class of Toeplitz operators such that $\log|\varphi|$ is integrable. If $\log|\varphi|$ is integrable then there exists an outer function e such that $|\varphi| = |e|$. Thus we may write $\varphi = ue$, where u is a unimodular function. Since by the Douglas–Rudin

Theorem (cf. [Gar, p.192]), every unimodular function can be approximated by quotients of inner functions, it follows that if $\log|\varphi|$ is integrable then φ can be approximated by functions of bounded type. Therefore if we could obtain a sequence ψ_n converging to φ such that T_{ψ_n} is 2–hyponormal with finite rank self–commutator for each n, then we would answer Question G affirmatively. On the other hand, if T_φ attains its norm, then by a result of Brown and Douglas [BrD] φ is of the form $\varphi = \lambda\frac{\psi}{\theta}$ with $\lambda > 0$ and ψ, θ inner. Thus φ is of bounded type. Therefore, by Corollary 7, if T_φ is 2–hyponormal and attains its norm then T_φ is normal or analytic. However we have not been able to decide that if T_φ is a 2–hyponormal operator with finite rank self–commutator then T_φ attains its norm.

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